Supplement to "Uniform Inference for Characteristic Effects of Large Continuous-Time Linear Models"

Yuan Liao Xiye Yang*

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Abstract

This document contains simulations, additional empirical results, and all the proofs.

Contents

*Department of Economics, Rutgers University, 75 Hamilton St. New Brunswick, NJ 08901, USA. Email: yuan.liao@rutgers.edu; xiyeyang@econ.rutgers.edu

A Simulation Examples

A.1 Linear factor models

We conduct simple simulation studies on the estimated characteristic betas, to illustrate the issue of uniformity and the under/over coverages of the usual plug-in methods. The locally constant betas are generated from the following discrete time DGP:

$$\Delta_i^n \mathbf{Y} = \boldsymbol{\alpha} + (\mathbf{G} + \boldsymbol{\Gamma})\Delta_i^n \mathbf{F} + \Delta_i^n \mathbf{U}, \quad j = 1, ..., k_n \tag{A.1}$$

where the $p \times K$ matrices $\mathbf{G}$ and $\boldsymbol{\Gamma}$ are respectively given by $\mathbf{G} = 3\mathbf{X} + 1$, and $\boldsymbol{\Gamma} = \sqrt{w_\gamma}\boldsymbol{\Gamma}_0$. Here $\Delta_i^n \mathbf{U} \sim N(0, \mathbf{I})\sqrt{\Delta_n}$, $\Delta_i^n \mathbf{F} \sim N(0,1)\sqrt{\Delta_n}$, and $\alpha_m = \Delta_n$ for each $m \leq p$. The number of factors $K = 1$. The cross-sectional components of the $p \times 1$ vector $\mathbf{X}$ are generated independently from $N(0,1)$, and the components of $\boldsymbol{\Gamma}_0$ are generated independently from $N(0,1)$, Here w_γ is taken as a scalar value in the range $[0.001, 3]$, which determines the strength of $\boldsymbol{\Gamma}$. The goal is to study the coverage properties of g_1, the first component of $\mathbf{G}$, with various values of w_γ. We set $\Delta_n = (pk_n)^{-1}$. As described in the paper, we use the cross-sectional bootstrap to generate critical values for the estimated $\mathbf{g}_1$, and construct confidence intervals. The number of bootstrap replications is $B = 5000$.

We construct the confidence interval using three methods and compare the coverage probabilities:

(i) the bootstrap confidence interval: $\widehat{\mathbf{g}}_1 \pm q_{\tau,bootstrap}$

(ii) the "over-coverage" confidence interval: $\widehat{\mathbf{g}}_1 \pm 1.96\sqrt{\frac{1}{k_n p}\widehat{\mathbf{V}}_u + \frac{1}{p}\widehat{\mathbf{V}}_\gamma}$

(iii) the "under-coverage" confidence interval: $\widehat{\mathbf{g}}_1 \pm 1.96\sqrt{\frac{1}{k_n p}\widehat{\mathbf{V}}_u}$, where

$$\begin{aligned}\widehat{\mathbf{V}}_u &= \widehat{\mathbf{c}}_{FF,t}^{-1}\frac{1}{pk_n}\sum_{i=1}^{k_n}\boldsymbol{\phi}_{i,l}'(\frac{1}{p}\boldsymbol{\Phi}_i'\boldsymbol{\Phi}_i)^{-1}\boldsymbol{\Phi}_i' \operatorname{diag}\{\frac{1}{k_n\Delta_n}\sum_{j=1}^{k_n}\widehat{\Delta_i^n\mathbf{U}}\widehat{\Delta_i^n\mathbf{U}}'\}\boldsymbol{\Phi}_i(\frac{1}{p}\boldsymbol{\Phi}_i'\boldsymbol{\Phi}_i)^{-1}\boldsymbol{\phi}_{i,l}\\ \widehat{\mathbf{V}}_\gamma &= \frac{1}{p}\sum_{m=1}^{p}h_{ml}^2\widehat{\boldsymbol{\gamma}}_m\widehat{\boldsymbol{\gamma}}_m', \quad l=1, h_{ml} = \boldsymbol{\phi}_{i,l}'(\frac{1}{p}\boldsymbol{\Phi}_i'\boldsymbol{\Phi}_i)^{-1}\boldsymbol{\phi}_{i,m}'.\end{aligned}$$

Note that the "over-coverage" uses the naive plug-in method to estimate the asymptotic variance. This method can lead to over-coveraging probabilities when w_γ is near zero. In addition, the "under-coverage" is the benchmark method that ignores the variance component coming from $\boldsymbol{\Gamma}$, which is the common practice in the literature. Consequently, it is expected to produce substantial under-coveraging probabilities when $\mathbf{V}_\gamma$ is non-negligible.

Table 1 summarizes the coverage probabilities using 2000 replications under the 95% nominal coverage. The numerical findings are consistent with what our theory predicts: (1) The bootstrap confidence interval has good coverage probabilities, uniformly over w_γ. (2) The "over-coverage" method is severely conservative when w_γ is small, and becomes better as w_γ increases. (3) The "under-coverage" method has a fine coverage (but still with noticeable size distortions) when w_γ is near zero, but quickly has substantial under coverages as w_γ increases. (iv) The coverage probabilities for the bootstrap is satisfactory even if k_n is small. This is due to the fact that the idiosyncratic effects are mostly projected off when applying the cross-sectional regression step.

A.2 Idiosyncratic variance models

Next we simulate the idiosyncratic variance models as described in Example 4.2. The data are generated from a similar DGP of (A.1), but with heteroskedastic idiosyncratic variances of $\Delta_i^n\mathbf{U}$. Specifically, we generate $\Delta_i^n\mathbf{U}/\sqrt{\Delta_n}$ from a p-dimensional multivariate normal distribution, whose covariance matrix is diagonal with the l th diagonal entry as

$$c_{uu,l} = \boldsymbol{\beta}_l\mathbf{c}_{FF} + \bar{c}$$

Table 1: Coverage probabilities in linear factor model, nominal probability= 95%

			w_γ			
k_n	p		0.001	0.1	1	3
10	100	bootstrap	0.950	0.951	0.945	.945
		over-coverage	0.993	0.992	0.959	0.949
		under-coverage	0.943	0.919	0.455	0.257
10	300	bootstrap	0.951	0.942	0.952	0.938
		over-coverage	0.992	0.989	0.956	0.939
		under-coverage	0.947	0.918	0.435	0.259
100	100	bootstrap	0.945	0.946	0.944	0.942
		over-coverage	0.993	0.982	0.944	0.941
		under-coverage	0.928	0.822	0.151	0.106
100	300	bootstrap	0.947	0.948	0.954	0.948
		over-coverage	0.994	0.986	0.946	0.950
		under-coverage	0.939	0.841	0.156	0.089
200	100	bootstrap	0.943	0.947	0.949	0.952
		over-coverage	0.987	0.974	0.941	0.938
		under-coverage	0.927	0.738	0.115	0.065
200	300	bootstrap	0.948	0.946	0.949	0.953
		over-coverage	0.991	0.973	0.948	0.953
		under-coverage	0.929	0.756	0.111	0.069

Here w_γ measures the strength of $\mathbf{\Gamma}$. The "bootstrap" is the proposed cross-sectional bootstrap inference. The "over-coverage" uses the plug-in method to estimate the asymptotic variance. This method can lead to over-coveraging probabilities when w_γ is near zero. In addition, the "under-coverage" is the benchmark method that ignores the variance component coming from $\mathbf{\Gamma}$, which is the common practice in the literature.

where $\mathbf{c}_{FF} = 1$ is the variance of $\Delta_i^n \mathbf{F}/\sqrt{\Delta_n}$. Here $\boldsymbol{\beta} = (\boldsymbol{\beta}_1, ..., \boldsymbol{\beta}_p)' = \mathbf{G} + \mathbf{\Gamma}$, with $\mathbf{G} = \mathbf{X}$ and $\mathbf{\Gamma} = \sqrt{w_\gamma}\mathbf{\Gamma}_0$, where $\mathbf{X}$ and $\mathbf{\Gamma}$ are generated as before. The factor loadings are generated independently from the standard normal distribution. To ensure that $c_{uu,l} > 0$, we set $\bar{c} = 6$ as a known upper bound for $\max_{l \leq p} |\boldsymbol{\beta}_l \mathbf{c}_{FF}|$.

We then proceed the two-step GMM method based on the moment conditions described in Example 4.2, where the quadratic variations are replaced by their sample analogues. It is easy to verify that in this case $\boldsymbol{\xi}^0_{mt,s}$ in Assumption 4.6 is given by

$$\boldsymbol{\xi}^0_{mt,s} = -\mathbf{c}^{-1}_{FF,t}\left(\frac{1}{s}(U_{m,t+s} - U_{mt})^2 - c_{uu,mt}\right) h_{t,m1}, \quad m \leq p.$$

where $c_{uu,mt}$ is the quadratic variation of U_{mt}, and U_{mt}'s are cross-sectionally independent. This leads to the estimated asymptotic variance components (using the "plug-in" method):

$$\begin{aligned}
\widehat{\mathbf{V}}_u &= \widehat{\mathbf{c}}^{-2}_{FF}\frac{1}{p}\sum_{m=1}^{p} h^2_{t,m1}\widehat{\text{Var}}(u^2_{mi}), \quad \widehat{\text{Var}}(u^2_{mi}) = \frac{1}{k_n}\sum_{i=1}^{k_n}\widehat{u}^4_{mi} - (\frac{1}{k_n}\sum_{i=1}^{k_n}\widehat{u}^2_{mi})^2 \\
\widehat{\mathbf{V}}_\gamma &= \frac{1}{p}\sum_{m=1}^{p} h^2_{ml}\widehat{\boldsymbol{\gamma}}_m\widehat{\boldsymbol{\gamma}}'_m.
\end{aligned}$$

We are interested in the inference about the idiosyncratic variance beta $\boldsymbol{\beta}_1$ for the first individual, in particular, testing whether it can be explained by its characteristics, which corresponds to the null hypothesis that $\mathbf{g}_1$, the first component of $\mathbf{G}$, is zero almost surely. So under the null hypothesis we set the true value $\mathbf{g}_1 = 0$. We still compare three methods: (i) the bootstrap confidence interval, (ii) the "over-coverage" confidence interval using the "plug-in" asymptotic variance $\frac{1}{k_n p}\widehat{\mathbf{V}}_u + \frac{1}{p}\widehat{\mathbf{V}}_\gamma$, and (iii) the "under-coverage" confidence interval using the "plug-in" asymptotic variance $\frac{1}{k_n p}\widehat{\mathbf{V}}_u$. For each method, we calculate the confidence intervals and reject $H_0 : \mathbf{g}_1 = 0$ if the interval does not cover zero.

Table 2 reports the rejection probabilities under the null using 2000 replications with 5% significance level. The numerical findings continues to being consistent with what our theory predicts: (1) The bootstrap confidence interval has rejection probabilities close to 5% uniformly over w_γ. (2) The "over-coverage" method is conservative when w_γ is small, whose size is very small , and the size increases as w_γ increases. (3) The "under-coverage" method has a size close to 5% when w_γ is near zero, but has large size as w_γ increases.

B Additional empirical results

B.1 Estimated $\mathbf{g}_t(\cdot)$ functions

We report the scatter plots of estimated cross-sectional $\mathbf{G}$'s versus cross-sectional instruments for different Fama-French factors and on four selected days in Figures 1-4 show. The black solid line is the sieve fitted $\mathbf{g}_t(\cdot)$ function using B-splines with degree 3 (Eilers and Marx,

Table 2: Size in idiosyncratic variance model for testing $H_0 : \mathbf{g}_1 = 0$, with significance level 5%

k_n	p		w_γ 0.001	0.1	1	3
50	50	bootstrap	0.076	0.074	0.045	0.059
		over-coverage	0.001	0.003	0.013	0.031
		under-coverage	0.051	0.056	0.195	0.350
100	50	bootstrap	0.056	0.049	0.064	0.057
		over-coverage	0.001	0.002	0.031	0.040
		under-coverage	0.045	0.054	0.254	0.426
300	50	bootstrap	0.064	0.076	0.053	0.058
		over-coverage	0.010	0.013	0.039	0.052
		under-coverage	0.056	0.077	0.425	0.652
50	100	bootstrap	0.057	0.067	0.063	0.064
		over-coverage	0.002	0.005	0.019	0.032
		under-coverage	0.043	0.054	0.214	0.300
100	100	bootstrap	0.057	0.072	0.061	0.062
		over-coverage	0.006	0.005	0.036	0.039
		under-coverage	0.047	0.057	0.220	0.360
300	100	bootstrap	0.053	0.052	0.060	0.066
		over-coverage	0.004	0.007	0.043	0.057
		under-coverage	0.053	0.071	0.391	0.571

This table reports the size of three tests for testing $H_0 : \mathbf{g}_1 = 0$: (i) the proposed bootstrap confidence interval, (ii) the "over-coverage" confidence interval using the plug-in asymptotic variance $\frac{1}{k_n p}\widehat{\mathbf{V}}_u + \frac{1}{p}\widehat{\mathbf{V}}_\gamma$, and (iii) the "under-coverage" confidence interval using the plug-in asymptotic variance $\frac{1}{k_n p}\widehat{\mathbf{V}}_u$.

1996). The plots show several interesting features. First, by comparing the subplots in the same column, one can see that the estimated G function is time-varying. Secondly, the estimated $\mathbf{g}_t(\cdot)$ functions have noticeable nonlinear patterns. As for the specific functions, for the market factor, there is a small downward slope, which is consistent with our finding that small-size firm slightly tends to have a G value larger than 1. A more noticable downward slope can be found on the third column of Figure 1, indicating a significant decreasing pattern of the size's effect on the SMB factor. We then focus on the $\mathbf{g}_t(\cdot)$ function of the relation "SMB vs Size". Figure 5 plots the estimated size effect on the SMB factor and the corresponding $\mathbf{g}_t$ function from January 3, 2006 to October 16, 2013, proceeding for every 50 trading days. We see that the nonlinear and downward trends are persistent on most of the representative days during the testing period. This is also consistent with our finding from the confidence intervals that large-size firms generally have negative values for the SMB factor's G.

Figure 2 gives the result with size replaced by value. The second column indicates that small-value firms tend to have insignificant or negative G values for the HML factor. Figure 3 presents the result for momentum. One can see an obvious downward slope for the HML factors' in July 1, 2008. In fact, this pattern is very persistent during the 2008-2009 crisis. The result shown in Figure 4 is consistent with our finding that large-volatility stocks in general have market G values larger than 1. Lastly, all figures with non-flat G functions are consistent with the results in Table 3.

Next, we compare the standard deviations of the estimated two components in beta. The cross-sectional standard deviation is the sample standard deviation of the estimated $\mathbf{g}_{lt}$ and $\boldsymbol{\gamma}_{lt}$ among firms in each group for each fixed day. Then averaging these standard deviations over all days leads to the "averaged cross-sectional standard deviations". On the other hand, the time-series standard deviation is the sample standard deviation of the estimated $\mathbf{g}_{lt}$ and $\boldsymbol{\gamma}_{lt}$ over time for each fixed firm. Then averaging these time-series standard deviation across firms in each group leads to the "averaged time-series standard deviations". Here groups (small, medium, large) are determined by either the size or the volatility of the firms. So they respectively measure the cross-sectional and time-series variations of the two beta components. The results are given in Table 3 below, and show several interesting patterns: (1) The characteristic beta always possess significantly smaller standard deviations, in both cross-sectional and time-series, than the idiosyncratic beta. It demonstrates that there are relatively smaller cross-sectional variations in the characteristic betas among S&P500 firms. In addition, characteristic betas, as they capture long-run beta movements, are much less volatile in the time domain. (2) On average, firms with larger size and smaller volatilities tend to have smaller cross-sectional and time series variations in both components of beta. These firms have larger market values, whose betas are often more stable than the others. (3) Even for firms with larger size and smaller volatilities, the time series standard deviations of the characteristic betas are noticeably different from zero, showing a significant degree of time-variations in betas.

C Organization of the proofs

C.1 The main organization

We prove the main results separately for the known and latent factor cases. As for the known factor case, it is based on the moment condition $\Psi(\boldsymbol{\beta}_{mt}, \mathbf{c}_{z,mt}) = 0$ with

$$\Psi(\boldsymbol{\beta}_{mt}, \mathbf{c}_{z,mt}) := \mathbf{c}_{FF,t}\boldsymbol{\beta}_{mt} - \mathbf{c}_{YF,mt},$$

so it is a special case of the general result Theorems 4.1 and 4.2. Therefore, we shall first prove Theorems 4.1 and 4.2. Next, we verify that the high-level conditions of these two proved theorems are satisfied by the linear factor model, which then proves Theorems 3.1 and 3.4 for the known factor case.

We then separately prove the unknown factor case, the bias correction, and the estimated integrated effect $\int \mathbf{g}_t dt$, which are Theorems 3.2, 3.3 and 3.5.

C.2 The organization for the unknown factor case

The case of the unknown factors is more sophisticated, as it also involves analyzing the high-dimensional eigenvectors with the new factor estimators (running PCA on the "projected data" in the continuous-time models). This involves an effect of bias as well as the issue of bias corrections. This section provides a roadmap of the proofs for the unknown factor case.

Step 1. Asymptotic expansion for the estimated factors.

We show that there is a rotation matrix $\mathbf{H}_{nt}$ and $\boldsymbol{\Xi}_t$ so that for all $i \in I_t^n$,

$$\widehat{\Delta_i^n \mathbf{F}} - \mathbf{H}_{nt}\Delta_i^n \mathbf{F} = \boldsymbol{\Xi}_t' \frac{1}{\sqrt{p}} \mathbf{P}_{i-1}\Delta_i^n \mathbf{U} + \text{higher order remainder terms.}$$

Step 2. Asymptotic expansion in the estimated factor case.

We show that the effect of estimating latent factors, $\boldsymbol{\Xi}_t' \frac{1}{\sqrt{p}} \mathbf{P}_{i-1}\Delta_i^n \mathbf{U}$, gives rise to a bias term: there is a rotation matrix $\boldsymbol{\Upsilon}_t = \mathbf{H}_{nt}^{'-1} + o_P(1)$ and a BIAS_g so that

$$\widehat{\mathbf{g}}_{lt}^{\text{latent}} - \boldsymbol{\Upsilon}_t \mathbf{g}_{lt} - \text{BIAS}_g = \boldsymbol{\Upsilon}_t(\mathbf{a}_1 + \mathbf{a}_2) + o_P((k_n p)^{-1/2}). \tag{C.1}$$

where

$$\begin{aligned}\mathbf{a}_1 &= \mathbf{c}_{FF,t}^{-1}\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{U}'\mathbf{P}_{t,l}\\ \mathbf{a}_2 &= \mathbf{\Gamma}_t'\mathbf{P}_{t,l}.\end{aligned}$$

Here $\mathbf{a}_1, \mathbf{a}_2$ jointly determine the limiting distribution. Here we pursue a direct proof for the unknown factor case, instead of applying Theorem 4.1.

Step 3. Bias correction.

Whether the bias is negligible depends on whether the cross-sectional variation of $\boldsymbol{\gamma}_{lt}$ is "strong". We then prove that the bias correction is first-order valid uniformly over the strength of $\boldsymbol{\gamma}_{lt}$. These results lead to the asymptotic distribution of the estimated $\mathbf{g}_{lt}$ in the unknown factor case (for the bias corrected estimator). Note that the validity of bias correction is under a general condition on the quadratic variation of $\Delta_t^n\mathbf{U}$. In Section E.2.5 we verify this condition under the more primitive sparse condition on the quadratic variation.

Step 4. Bootstrap validity.

We then study the bootstrapped estimators for the unknown factor case, and obtain bootstrap expansion:

$$\widehat{\mathbf{g}}_{lt}^{*\text{latent}} - \widehat{\mathbf{g}}_{lt}^{\text{latent}} = \mathbf{\Upsilon}_t(\mathbf{a}_1^* - \mathbf{a}_1) + \mathbf{\Upsilon}_t(\mathbf{a}_2^* - \mathbf{a}_2) + o_{P^*}((k_n p)^{-1/2}).$$

where $\mathbf{a}_1^*$ and $\mathbf{a}_2^*$ respectively denote the "bootstrap versions" of $\mathbf{a}_1$ and $\mathbf{a}_2$. Note that we use the cross-sectional bootstrap with $\mathbb{E}^*(\mathbf{a}_1^*, \mathbf{a}_2^*) = (\mathbf{a}_1, \mathbf{a}_2)$. The bootstrap distribution of $(\mathbf{a}_1^* - \mathbf{a}_1, \mathbf{a}_2^* - \mathbf{a}_2)$ conditioning on the original data admits a cross-sectional CLT, which is asymptotically the same as the sampling distribution of $(\mathbf{a}_1, \mathbf{a}_2)$. Thus the bootstrap distribution of $\widehat{\mathbf{g}}_{lt}^{*\text{latent}} - \widehat{\mathbf{g}}_{lt}^{\text{latent}}$ is asymptotically identical to the sampling distribution of $\widehat{\mathbf{g}}_{lt}^{\text{latent}} - \mathbf{\Upsilon}_t\mathbf{g}_{lt} - \text{BIAS}_g$. These results hold uniformly over the class of DGP under consideration and thus lead to a uniformly valid coverage property.

D Proofs for Section 4

D.1 Proof of Theorem 4.1

We first obtain an expansion of $\widehat{\mathbf{g}}_{lt} - \mathbf{g}_{lt}$. From $\widehat{\mathbf{G}}_t = \mathbf{P}_t \widehat{\boldsymbol{\beta}}_t$ and $\boldsymbol{\beta}_t = \mathbf{G}_t + \boldsymbol{\Gamma}_t$, we take the l'th row of $\widehat{\mathbf{G}}_t$ to obtain,

$$\widehat{\mathbf{g}}_{lt} - \mathbf{g}_{lt} = \boldsymbol{\Gamma}_t' \mathbf{P}_{t,l} + \frac{1}{p} \sum_{m=1}^{p} h_{t,ml} (\widehat{\boldsymbol{\beta}}_{mt} - \boldsymbol{\beta}_{mt}) + (\mathbf{G}_t' \mathbf{P}_{t,l} - \mathbf{g}_{tl}). \tag{D.1}$$

where $\mathbf{P}_{t,l}$ denotes the l th column of $\mathbf{P}_t$ and $h_{t,ml} = \boldsymbol{\phi}_{lt}' (\frac{1}{p} \boldsymbol{\Phi}_t' \boldsymbol{\Phi}_t)^{-1} \boldsymbol{\phi}_{mt}$. By the sieve approximation assumption, $\|\mathbf{G}_t' \mathbf{P}_{t,l} - \mathbf{g}_{lt}\| = O_P(J^{-\eta}) = o_P((k_n p)^{-1/2})$. The first term $\boldsymbol{\Gamma}_t' \mathbf{P}_l = \frac{1}{p} \sum_{m=1}^{p} \boldsymbol{\gamma}_{mt} h_{t,ml}$ is one of the leading terms in the expansion. The key task is to expand $\frac{1}{p} \sum_{m=1}^{p} h_{t,ml} (\widehat{\boldsymbol{\beta}}_{mt} - \boldsymbol{\beta}_{mt})$.

The first-order condition for

$$\widehat{\boldsymbol{\beta}}_{mt} := \arg\min_{\boldsymbol{\beta}} \Psi_m(\boldsymbol{\beta}, \widehat{\mathbf{c}}_{z,mt})' \boldsymbol{\Omega}_{mt} \Psi_m(\boldsymbol{\beta}, \widehat{\mathbf{c}}_{z,mt}), \quad m \leq p$$

combined with Taylor expansion and $\Psi_m(\boldsymbol{\beta}_{mt}, \mathbf{c}_{z,mt}) = 0$ yields

$$\begin{aligned} \widehat{\boldsymbol{\beta}}_{mt} - \boldsymbol{\beta}_{mt} &= -\mathbf{D}_{mt}(\widehat{\mathbf{c}}_{z,mt})^{-1} \nabla_\beta \Psi_m(\widehat{\mathbf{c}}_{z,mt})' \boldsymbol{\Omega}_{mt} \Psi_m(\boldsymbol{\beta}_{mt}, \widehat{\mathbf{c}}_{z,mt}) \\ &= -\mathbf{A}_{mt}(\mathbf{c}_{z,mt}) \nabla_c \Psi_m(\boldsymbol{\beta}_{mt}, \mathbf{c}_{z,mt}) \operatorname{vec}(\widehat{\mathbf{c}}_{z,mt} - \mathbf{c}_{z,mt}) \\ &\quad -[\mathbf{A}_{mt}(\widehat{\mathbf{c}}_{z,mt}) - \mathbf{A}_{mt}(\mathbf{c}_{z,mt})] \nabla_c \Psi_m(\boldsymbol{\beta}_{mt}, \mathbf{c}_{z,mt}) \operatorname{vec}(\widehat{\mathbf{c}}_{z,mt} - \mathbf{c}_{z,mt}) \\ &\quad +\mathbf{A}_{mt}(\widehat{\mathbf{c}}_{z,mt}) \mathbf{m}_{mt} \end{aligned} \tag{D.2}$$

where

$$\begin{aligned} \mathbf{D}_{mt}(\mathbf{c}) &= \nabla_\beta \Psi_m(\mathbf{c})' \boldsymbol{\Omega}_{mt} \nabla_\beta \Psi_m(\mathbf{c}) \\ \mathbf{A}_{mt}(\mathbf{c}) &= \mathbf{D}_{mt}(\mathbf{c})^{-1} \nabla_\beta \Psi_m(\mathbf{c})' \boldsymbol{\Omega}_{mt} \\ \mathbf{m}_{mt} &= \Psi_m(\boldsymbol{\beta}_{mt}, \widehat{\mathbf{c}}_{z,mt}) - \Psi_m(\boldsymbol{\beta}_{mt}, \mathbf{c}_{z,mt}) - \nabla_c \Psi_m(\boldsymbol{\beta}_{mt}, \mathbf{c}_{z,mt}) \operatorname{vec}(\widehat{\mathbf{c}}_{z,mt} - \mathbf{c}_{z,mt}) \end{aligned}$$

With slight abuse of notation, we write

$$\mathbf{A}_{m,i-1} := \mathbf{A}_{m,(i-1)\Delta_n}(\mathbf{c}_{z,m,(i-1)\Delta_n}), \quad \mathbf{c}_{z,m,i-1} := \mathbf{c}_{z,m,(i-1)\Delta_n}$$

Hence $\frac{1}{p}\sum_{m=1}^{p} h_{t,ml}(\widehat{\boldsymbol{\beta}}_{mt}-\boldsymbol{\beta}_{mt}) = \mathbf{d}_1+...+\mathbf{d}_5$, where

$$
\begin{aligned}
\mathbf{d}_1 &= -\frac{1}{p}\sum_{m=1}^{p} h_{t,ml}\frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{A}_{m,i-1}\nabla_c\Psi_m(\mathbf{c}_{z,m,i-1})\,\mathrm{vec}(\frac{1}{\Delta_n}\Delta_i^n\mathbf{Z}_m\Delta_i^n\mathbf{Z}_m'-\mathbf{c}_{z,m,i-1})\\
\mathbf{d}_2 &= \frac{1}{p}\sum_{m=1}^{p} h_{t,ml}\mathbf{A}_{mt}(\widehat{\mathbf{c}}_{z,mt})\mathbf{m}_{mt}\\
\mathbf{d}_3 &= -\frac{1}{p}\sum_{m=1}^{p} h_{t,ml}[\mathbf{A}_{mt}(\widehat{\mathbf{c}}_{z,mt})-\mathbf{A}_{mt}(\mathbf{c}_{z,mt})]\nabla_c\Psi_m(\mathbf{c}_{z,mt})\,\mathrm{vec}(\widehat{\mathbf{c}}_{z,mt}-\mathbf{c}_{z,mt})\\
\mathbf{d}_4 &= -\frac{1}{p}\sum_{m=1}^{p} h_{t,ml}\frac{1}{k_n}\sum_{i\in I_t^n}[\mathbf{A}_{mt}(\mathbf{c}_{z,mt})\nabla_c\Psi_m(\mathbf{c}_{z,mt})-\mathbf{A}_{m,i-1}\nabla_c\Psi_m(\mathbf{c}_{z,m,i-1})]\,\mathrm{vec}(\frac{\Delta_i^n\mathbf{Z}_m\Delta_i^n\mathbf{Z}_m'}{\Delta_n}-\mathbf{c}_{z,m,i-1})\\
\mathbf{d}_5 &= -\frac{1}{p}\sum_{m=1}^{p} h_{t,ml}\mathbf{A}_{mt}(\mathbf{c}_{z,mt})\nabla_c\Psi_m(\mathbf{c}_{z,mt})\,\mathrm{vec}(\frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{c}_{z,m,(i-1)\Delta_n}-\mathbf{c}_{z,mt}).
\end{aligned}
\tag{D.3}
$$

By Assumption 4.3, $\mathbf{d}_2 = o_P((k_np)^{-1/2})$ and $\mathbf{d}_3 = o_P((k_np)^{-1/2})$.

Next we show $\sqrt{pk_n}\mathbf{d}_4 = o_P(1)$. By results on spot volatility estimation (see, e.g., Jacod and Todorov (2010)), we have an absolute constant L, for any $m \le p$,

$$
\begin{aligned}
\|\mathbb{E}\big(\frac{1}{\Delta_n}\Delta_i^n\mathbf{Z}_m\Delta_i^n\mathbf{Z}_m'-\mathbf{c}_{z,m,i-1}\mid\mathcal{F}_{i-1}^n\big)\| &\le L\Delta_n,\\
\mathbb{E}\Big(\big\|\frac{1}{\Delta_n}\Delta_i^n\mathbf{Z}_m\Delta_i^n\mathbf{Z}_m'-\mathbf{c}_{z,m,i-1}\big\|^2\mid\mathcal{F}_{i-1}^n\Big) &\le L.
\end{aligned}
$$

Also, $\|\mathbf{A}(\mathbf{c}_{z,mt})\|\le L, \|h_{t,ml}\|\le L$. Provided $\nabla_c\Psi_m$ is continuously differentiable, the right continuous property of $\mathbf{c}_z$ and the uniform boundedness on its drift and diffusion coefficients together imply that

$$
\mathbb{E}\big((\mathbf{A}_{mt}(\mathbf{c}_{z,mt})\nabla_c\Psi_m(\mathbf{c}_{z,mt})-\mathbf{A}_{m,i-1}\nabla_c\Psi_m(\mathbf{c}_{z,m,i-1}))^2\mathcal{F}_t\big)\le Lk_n\Delta_n.
$$

The above results imply that almost surely,

$$
\begin{aligned}
\sqrt{pk_n}\|\mathbb{E}(\mathbf{d}_4|\mathcal{F}_t)\| &\le \sqrt{pk_n}\mathbb{E}\Big(\frac{1}{p}\sum_{m=1}^{p}\|h_{t,ml}\|\frac{1}{k_n}\sum_{i\in I_t^n}\|[\mathbf{A}_{mt}(\mathbf{c}_{z,mt})\nabla_c\Psi_m(\mathbf{c}_{z,mt})\\
&\quad -\mathbf{A}_{m,i-1}\nabla_c\Psi_m(\mathbf{c}_{z,m,i-1})]\|\|\mathbb{E}\big(\,\mathrm{vec}(\frac{\Delta_i^n\mathbf{Z}_m\Delta_i^n\mathbf{Z}_m'}{\Delta_n}-\mathbf{c}_{z,m,i-1})\mid\mathcal{F}_{i-1}^n\big)\|\Big|\mathcal{F}_t\Big)\\
&\le L\sqrt{pk_n\Delta_n^2}=o(1).
\end{aligned}
$$

Moreover, the cross-sectional asymptotic un-correlation of $\mathrm{vec}(\frac{\Delta_i^n \mathbf{Z}_m \Delta_i^n \mathbf{Z}_m'}{\Delta_n} - \mathbf{c}_{z,m,i-1})$ implies that

$$
\begin{aligned}
pk_n \mathbb{E}(\mathbf{d}_4^2 | \mathcal{F}_t) = & \frac{1}{pk_n} \mathbb{E}\Big(\sum_{m,m'=1}^{p} \sum_{i,i' \in I_t^n} h_{t,ml} h_{t,m'l} [\mathbf{A}_{mt}(\mathbf{c}_{z,mt}) \nabla_c \Psi_m(\mathbf{c}_{z,mt}) - \mathbf{A}_{m,i-1} \nabla_c \Psi_m(\mathbf{c}_{z,m,i-1})] \\
& \times [\mathbf{A}_{m't}(\mathbf{c}_{z,m't}) \nabla_c \Psi_{m'}(\mathbf{c}_{z,m't}) - \mathbf{A}_{m',i'-1} \nabla_c \Psi_{m'}(\mathbf{c}_{z,m',i'-1})] \\
& \times \mathbb{E}\Big((\frac{1}{\Delta_n} \Delta_i^n Z_m \Delta_i^n Z_m^\intercal - \mathbf{c}_{z,m,(i-1)\Delta_n})(\frac{1}{\Delta_n} \Delta_{i'}^n Z_{m'} \Delta_{i'}^n Z_{m'}^\intercal - \mathbf{c}_{z,m',(i'-1)\Delta_n}) | \mathcal{F}_{i-1}^n \Big) \Big| \mathcal{F}_t \Big) \\
\leq & \frac{L}{pk_n} \sum_{m=1}^{p} \sum_{i \in I_t^n} \mathbb{E}\Big(h_{t,ml}^2 [\mathbf{A}_{mt}(\mathbf{c}_{z,mt}) \nabla_c \Psi_m(\mathbf{c}_{z,mt}) - \mathbf{A}_{m,i-1} \nabla_c \Psi_m(\mathbf{c}_{z,m,i-1})]^2 \Big| \mathcal{F}_t^n \Big) \\
\leq & \frac{L}{pk_n} \sum_{m=1}^{p} \sum_{i \in I_t^n} k_n \Delta_n \leq L k_n \Delta_n = o(1).
\end{aligned}
$$

Hence, $\sqrt{pk_n}\mathbf{d}_4 = o_P(1)$.

As for $\mathbf{d}_5$, write

$$
W_i = \frac{1}{p} \sum_{m=1}^{p} \mathbf{A}(\mathbf{c}_{z,mt}) \nabla_c \Psi_m(\mathbf{c}_{z,mt}) \, \mathrm{vec}\big(\mathbf{c}_{z,m,(i-1)\Delta_n} - \mathbf{c}_{z,mt}\big) h_{t,ml}.
$$

We shall prove $\sqrt{pk_n}\mathbf{d}_5 = \sqrt{pk_n}\frac{1}{k_n}\sum_{i \in I_t} W_i = o_P(1)$. In fact, we have almost surely,

$$
\begin{aligned}
& \Big\| \mathbb{E}\Big(\sqrt{pk_n} \frac{1}{k_n} \sum_{i \in I_t} W_i \mid \mathcal{F}_t \Big) \Big\| \leq \sqrt{pk_n} \frac{1}{k_n} \sum_{i \in I_t} \big\| \mathbb{E}\big(W_i \mid \mathcal{F}_t \big) \big\| \\
\leq & \sqrt{pk_n} \frac{1}{k_n} \sum_{i \in I_t} \frac{1}{p} \sum_{m=1}^{p} \big\| \mathbf{A}(\mathbf{c}_{z,mt}) \nabla_c \Psi_m(\mathbf{c}_{z,mt}) \, \mathbb{E}\big(\mathrm{vec}(\mathbf{c}_{z,m,(i-1)\Delta_n} - \mathbf{c}_{z,mt}) \mid \mathcal{F}_t \big) h_{t,ml} \big\| \\
\leq & \sqrt{pk_n} \frac{1}{k_n} \sum_{i \in I_t} \frac{1}{p} \sum_{m=1}^{p} \big\| \mathbf{A}(\mathbf{c}_{z,mt}) \nabla_c \Psi_m(\mathbf{c}_{z,mt}) \, L_m h_{t,ml} \big\| \, k_n \Delta_n \\
= & \sqrt{pk_n^3 \Delta_n^2} \frac{1}{p} \sum_{m=1}^{p} \big\| \mathbf{A}(\mathbf{c}_{z,mt}) \nabla_c \Psi_m(\mathbf{c}_{z,mt}) \, L_m h_{t,ml} \big\| \leq L \sqrt{pk_n^3 \Delta_n^2} = o(1)
\end{aligned}
$$

where L is a finite number that does not dependent on m. Also, almost surely,

$$
\Big\| \mathbb{E}\Big(pk_n \frac{1}{k_n^2} \sum_{i,j \in I_t} W_i W_j' \mid \mathcal{F}_t \Big) \Big\| \leq \sqrt{pk_n} \frac{1}{k_n} \sum_{i,j \in I_t} \big\| \mathbb{E}\big(W_i W_j \mid \mathcal{F}_t \big) \big\|
$$

$$\le pk_n \frac{1}{k_n^2} \sum_{i,j\in I_t} \frac{1}{p^2} \sum_{m,q=1}^{p} \Big\| \mathbf{A}(\mathbf{c}_{z,mt}) \nabla_c \Psi_m(\mathbf{c}_{z,mt}) \, \mathbb{E}\Big(\operatorname{vec}(\mathbf{c}_{z,m,(i-1)\Delta_n} - \mathbf{c}_{z,mt}) h_{t,ml}$$

$$h_{t,ql} \operatorname{vec}(\mathbf{c}_{z,q,(j-1)\Delta_n} - \mathbf{c}_{z,qt})' \mid \mathcal{F}_t \Big) \nabla_c \Psi_q(\mathbf{c}_{z,qt})' \mathbf{A}(\mathbf{c}_{z,mt})' \Big\|$$

$$\le pk_n \frac{1}{k_n^2} \sum_{i,j\in I_t} \frac{1}{p^2} \sum_{m,q=1}^{p} L_{m,q} k_n \Delta_n = pk_n^2 \Delta_n \frac{1}{p^2} \sum_{m,q=1}^{p} L_{m,q} \le Lpk_n^2 \Delta_n = o(1).$$

This proves $\mathbf{d}_5 = o_P((k_n p)^{-1/2})$.

Hence substituting to (D.4) yields

$$\widehat{\mathbf{g}}_{lt} - \mathbf{g}_{lt} = \underbrace{-\frac{1}{p} \sum_{m=1}^{p} h_{t,ml} \frac{1}{k_n} \sum_{i\in I_t^n} \mathbf{A}_{m,i-1} \nabla_c \Psi_m(\mathbf{c}_{z,m,i-1}) \operatorname{vec}\Big(\frac{1}{\Delta_n} \Delta_i^n \mathbf{Z}_m \Delta_i^n \mathbf{Z}_m' - \mathbf{c}_{z,m,i-1}\Big)}_{\mathbf{a}_1} + \underbrace{\boldsymbol{\Gamma}_t' \mathbf{P}_l}_{\mathbf{a}_2} + o_P((k_n p)^{-1/2}) \tag{D.4}$$

Therefore, the limiting distribution is jointly determined by $\mathbf{a}_1, \mathbf{a}_2$. Lemma D.1 presents a central limit theorem (CLT) of the sum of multiple (possibly) dominating terms, which also allows the presence of the remainder term. We apply this lemma here by verifying its conditions.

For Condition (i), let $\mathcal{Z}$ be the multivariate standard normal of dimension $\mathbf{a}_2$. From Assumption 3.4 that $\sqrt{p}\mathbf{V}_{\gamma,t}^{-1/2}\mathbf{a}_2 | (\mathbf{X}_t, \mathbf{V}_{\gamma,t}) \stackrel{\mathcal{L}\text{-}s}{\to} N(0, \mathbf{I})$. It implies, for any bounded random variable Y and bounded continuous function f, we have, as $p \to \infty$,

$$F_p(\mathbf{X}_t, \mathbf{V}_{\gamma,t}) := \mathbb{E}[Y f(\sqrt{p}\mathbf{V}_{\gamma,t}^{-1/2}\mathbf{a}_2) | \mathbf{X}_t, \mathbf{V}_{\gamma,t}] \to \widetilde{\mathbb{E}}[Y f(\mathcal{Z})]$$

almost surely in $\mathbf{X}_t, \mathbf{V}_{\gamma,t}$, where $\widetilde{\mathbb{E}}$ denotes the expectation with respect to the extension of the probability measure (see Jacod and Protter (2011) for the definition of $\widetilde{\mathbb{E}}$). In addition, since Y and f are both bounded, $F_p(\mathbf{X}_t, \mathbf{V}_{\gamma,t})$ is almost surely dominated by a constant, and thus we can apply the dominated convergence theorem to have

$$\mathbb{E}[Y f(\sqrt{p}\mathbf{V}_{\gamma,t}^{-1/2}\mathbf{a}_2)] = \mathbb{E} F_p(\mathbf{X}_t, \mathbf{V}_{\gamma,t}) \to \widetilde{\mathbb{E}}[Y f(\mathcal{Z})].$$

This shows that $\sqrt{p}\mathbf{V}_{\gamma,t}^{-1/2}\mathbf{a}_2 \stackrel{\mathcal{L}\text{-}s}{\to} \mathcal{Z}$ unconditionally.

In addition, by Lemma D.3, we have $\sqrt{k_n p}\mathbf{V}_{u,t}^{-1/2}\mathbf{a}_1 \stackrel{\mathcal{L}\text{-}s}{\to} N(0, \mathbf{I}_K)$. Hence condition (i) of

Lemma D.1 is satisfied with $\mathbf{A}_n = \sqrt{k_n p}\mathbf{V}_{u,t}^{-1/2}$ and $\mathbf{B}_n = \sqrt{p}\mathbf{V}_{\gamma,t}^{-1/2}$, where $\mathbf{V}_{u,t}$ and $\mathbf{V}_{\gamma,t}$ are respectively defined in the main paper.

Condition (ii) follows since $\mathbf{a}_1$ and $\mathbf{a}_2$ are conditionally uncorrelated. For Condition (iii), note that $\min\{\|\mathbf{A}_n\|, \|\mathbf{B}_n\|\} \geq \|\sqrt{k_n p}\mathbf{V}_{u,t}^{-1/2}\|$, while

$$\|\sqrt{k_n p}\mathbf{V}_{u,t}^{-1/2}\| o_P((k_n p)^{-1/2}) = o_P(1)\|\mathbf{V}_{u,t}^{-1/2}\| = o_P(1)$$

given the assumption that $\lambda_{\min}(\mathbf{V}_{u,t}) \geq c$. Hence all conditions of Lemma D.1 are satisfied. We have

$$\left(\frac{1}{k_n p}\mathbf{V}_{\gamma,t} + \frac{1}{p}\mathbf{V}_{\gamma,t}\right)^{-1/2} (\widehat{\mathbf{g}}_{lt} - \mathbf{g}_{lt}) \stackrel{\mathcal{L}\text{-}s}{\to} N(0, \mathbf{I}).$$

D.2 Proof of Theorem 4.2

Define

$$\mathbf{\Sigma}_n = \frac{1}{k_n p}\mathbf{V}_{u,t} + \frac{1}{p}\mathbf{V}_{\gamma,t}.$$

By definition, $\mathbb{P}^*\left(|\mathbf{v}'\widehat{\mathbf{g}}_{lt}^* - \mathbf{v}'\widehat{\mathbf{g}}_{lt}| \leq q_\tau\right) = 1 - \tau$. Theorem 4.1 implies $(\mathbf{v}'\mathbf{\Sigma}_n\mathbf{v})^{-1/2}\mathbf{v}'\left(\widehat{\mathbf{g}}_{lt} - \mathbf{g}_{lt}\right) \stackrel{\mathcal{L}\text{-}s}{\to} N(0,1)$. In addition, Proposition D.1 below implies $(\mathbf{v}'\mathbf{\Sigma}_n\mathbf{v})^{-1/2}\mathbf{v}'\left(\widehat{\mathbf{g}}_{lt}^* - \widehat{\mathbf{g}}_{lt}\right) \stackrel{d^*}{\longrightarrow} N(0,1)$. We have

$$\begin{aligned} 1 - \tau &= \mathbb{P}^*\left((\mathbf{v}'\mathbf{\Sigma}_n\mathbf{v})^{-1/2}|\mathbf{v}'\widehat{\mathbf{g}}_{lt}^* - \mathbf{v}'\widehat{\mathbf{g}}_{lt}| \leq (\mathbf{v}'\mathbf{\Sigma}_n\mathbf{v})^{-1/2} q_\tau\right) \\ &= \mathbb{P}\left((\mathbf{v}'\mathbf{\Sigma}_n\mathbf{v})^{-1/2}|\mathbf{v}'\widehat{\mathbf{g}}_{lt} - \mathbf{v}'\mathbf{g}_{lt}| \leq (\mathbf{v}'\mathbf{\Sigma}_n\mathbf{v})^{-1/2} q_\tau\right) + o(1) \\ &= \mathbb{P}\left(\mathbf{v}'\mathbf{g}_{lt} \in CI_{nt,\tau}\right) + o(1). \end{aligned}$$

Now consider a DGP sequence $\{\mathbb{P}_n : n \geq 1\} \subset \mathcal{P}$, so that

$$\limsup_n \sup_{\mathbb{P}\in\mathcal{P}} |\mathbb{P}(\mathbf{v}'\mathbf{g}_{lt} \in CI_{nt,\tau}) - (1-\tau)| = \limsup_n |\mathbb{P}_n(\mathbf{v}'\mathbf{g}_{lt} \in CI_{nt,\tau}) - (1-\tau)|.$$

Such a sequence always exists. Now let $\{\mathbb{P}_{n_k} : k \geq 1\}$ be a subsequence of $\{\mathbb{P}_n : n \geq 1\}$ so that $\lim_k |\mathbb{P}_{n_k}(\mathbf{v}'\mathbf{g}_{lt} \in CI_{nt,\tau}) - (1-\tau)| = \limsup_n |\mathbb{P}_n(\mathbf{v}'\mathbf{g}_{lt} \in CI_{nt,\tau}) - (1-\tau)|$. Note that $\{\mathbb{P}_{n_k} : k \geq 1\} \subset \mathcal{P}$, and hence $\mathbb{P}_{n_k}$ satisfies the conditions of Theorem 3.4. It implies that

$$\mathbb{P}_{n_k}\left(\mathbf{v}'\mathbf{g}_{lt} \in CI_{nt,\tau}\right) \to 1 - \tau.$$

Hence $\limsup_n \sup_{\mathbb{P}\in\mathcal{P}} |\mathbb{P}(\mathbf{v}'\mathbf{g}_{lt} \in CI_{nt,\tau}) - (1-\tau)| = \lim_k |\mathbb{P}_{n_k}(\mathbf{v}'\mathbf{g}_{lt} \in CI_{nt,\tau}) - (1-\tau)| = 0$. Q.E.D.

Proposition D.1. *Under assumptions of Theorem 4.2,*

$$(\mathbf{v}'\boldsymbol{\Sigma}_n\mathbf{v})^{-1/2}\,\mathbf{v}'\big(\widehat{\mathbf{g}}_{lt}^* - \widehat{\mathbf{g}}_{lt}\big) \xrightarrow{d^*} N(0, \mathbf{I}_K),$$

where $\xrightarrow{d^*}$ *means "converge in the bootstrap distribution, generated from the simple random sample of the cross-sectional units with replacement, conditional on the original data.*

Proof. We aim to make inference about $\mathbf{g}_{lt}$ for a fixed $l \leq p$. In the designed cross-sectional bootstrap, we keep the index of the l th resampled cross-sectional unit being l. Hence $(\mathbf{G}_t^*)_l = \mathbf{g}_{lt}$.

step 1: expansion of $\widehat{\mathbf{g}}_{lt} - \mathbf{g}_{lt}$.

We first obtain an expansion of $\widehat{\mathbf{g}}_{lt}^* - \mathbf{g}_{lt}$. From $\widehat{\mathbf{G}}_t^* = \mathbf{P}_t^*\widehat{\boldsymbol{\beta}}_t^*$ and $\boldsymbol{\beta}_t^* = \mathbf{G}_t^* + \boldsymbol{\Gamma}_t^*$, we take the l'th row of $\widehat{\mathbf{G}}_t^*$ to obtain,

$$\widehat{\mathbf{g}}_{lt}^* - \mathbf{g}_{lt} \;=\; \boldsymbol{\Gamma}_t^{*'}\mathbf{P}_{t,l}^* + \frac{1}{p}\sum_{m=1}^{p} h_{t,ml}^*(\widehat{\boldsymbol{\beta}}_{mt}^* - \boldsymbol{\beta}_{mt}^*) + (\mathbf{G}_t^{'*}\mathbf{P}_{t,l}^* - \mathbf{g}_{tl}). \tag{D.5}$$

where $\mathbf{P}_{t,l}^*$ denotes th l th column of $\mathbf{P}_t^*$. By the sieve approximation assumption, $\|\mathbf{G}_t^{*'}\mathbf{P}_{t,l}^* - \mathbf{g}_{tl}^*\| = O_{P^*}(J^{-\eta}) = o_{P^*}((k_np)^{-1/2})$. On the other hand, resampling $\{\widehat{\boldsymbol{\beta}}_{mt} : m \leq p\}$ yields the same expansion as (D.2): for each resampled cross-sectional unit $\widehat{\boldsymbol{\beta}}_{mt}^*$, we have:

$$\begin{aligned}
\widehat{\boldsymbol{\beta}}_{mt}^* - \boldsymbol{\beta}_{mt}^* \;=\;& -\mathbf{A}_{mt}^*(\mathbf{c}_{z,mt}^*)\nabla_c\Psi_m^*(\boldsymbol{\beta}_{mt}^*, \mathbf{c}_{z,mt}^*)\,\mathrm{vec}(\widehat{\mathbf{c}}_{z,mt}^* - \mathbf{c}_{z,mt}^*)\\
&-[\mathbf{A}_{mt}^*(\widehat{\mathbf{c}}_{z,mt}^*) - \mathbf{A}_{mt}^*(\mathbf{c}_{z,mt}^*)]\nabla_c\Psi_m^*(\boldsymbol{\beta}_{mt}^*, \mathbf{c}_{z,mt}^*)\,\mathrm{vec}(\widehat{\mathbf{c}}_{z,mt}^* - \mathbf{c}_{z,mt}^*)\\
&+\mathbf{A}_{mt}^*(\widehat{\mathbf{c}}_{z,mt}^*)\mathbf{m}_{mt}^*,
\end{aligned}$$

where $\mathbf{c}_{z,mt}^* = \mathbf{c}_{z,m_i,t}$, $\mathbf{A}_{mt}^* = \mathbf{A}_{m_i,t}, \boldsymbol{\beta}_{mt}^* = \boldsymbol{\beta}_{m_i,t}$ with m_i being the index of the sampled cross-sectional unit so that $\widehat{\boldsymbol{\beta}}_{mt}^* = \widehat{\boldsymbol{\beta}}_{m_i,t}$. Hence $\frac{1}{p}\sum_{m=1}^{p} h_{t,ml}^*(\widehat{\boldsymbol{\beta}}_{mt}^* - \boldsymbol{\beta}_{mt}^*) = \mathbf{d}_1^* + \mathbf{d}_2^* + \mathbf{d}_3^*$, where

$$\begin{aligned}
\mathbf{d}_1^* \;=\;& -\frac{1}{p}\sum_{m=1}^{p} h_{t,ml}^*\mathbf{A}_{mt}^*(\mathbf{c}_{z,mt}^*)\nabla_c\Psi_m^*(\boldsymbol{\beta}_{mt}^*, \mathbf{c}_{z,mt}^*)\,\mathrm{vec}(\widehat{\mathbf{c}}_{z,mt}^* - \mathbf{c}_{z,mt}^*)\\
\mathbf{d}_2^* \;=\;& \frac{1}{p}\sum_{m=1}^{p} h_{t,ml}^*\mathbf{A}_{mt}^*(\widehat{\mathbf{c}}_{z,mt}^*)\mathbf{m}_{mt}^*\\
\mathbf{d}_3^* \;=\;& -\frac{1}{p}\sum_{m=1}^{p} h_{t,ml}^*[\mathbf{A}_{mt}(\widehat{\mathbf{c}}_{z,mt}^*) - \mathbf{A}_{mt}^*(\mathbf{c}_{z,mt}^*)]\nabla_c\Psi_m^*(\boldsymbol{\beta}_{mt}^*, \mathbf{c}_{z,mt}^*)\,\mathrm{vec}(\widehat{\mathbf{c}}_{z,mt}^* - \mathbf{c}_{z,mt}^*).
\end{aligned}$$

Assumption 4.5 shows that $\mathbf{d}_2^*, \mathbf{d}_3^* = o_{P^*}((k_np)^{-1/2})$. Lemma D.2 shows that $\mathbf{d}_1^* = \mathbf{a}_1^* +$

$o_{P^*}((k_np)^{-1/2})$, and $\boldsymbol{\Gamma}_t^{*'}\mathbf{P}_{t,l}^* = \mathbf{a}_2^* + o_{P^*}(p^{-1/2})\lambda_{\min}^{1/2}(\mathbf{V}_{\gamma,t})$ where

$$\begin{aligned}
\mathbf{a}_1^* &= \frac{1}{p}\sum_{m=1}^p \mathbf{a}_{1,m}^*, \quad \mathbf{a}_2^* = \frac{1}{p}\sum_{m=1}^p \mathbf{a}_{2,m}^*, \\
\mathbf{a}_{1,m}^* &:= -\mathbf{A}_{mt}^*(\mathbf{c}_{z,mt}^*)\nabla_c\Psi_m^*(\boldsymbol{\beta}_{mt}^*, \mathbf{c}_{z,mt}^*)\operatorname{vec}(\widehat{\mathbf{c}}_{z,mt}^* - \mathbf{c}_{z,mt}^*)\boldsymbol{\phi}_{mt}^{*'}(\frac{1}{p}\boldsymbol{\Phi}_t'\boldsymbol{\Phi}_t)^{-1}\boldsymbol{\phi}_{lt} \\
\mathbf{a}_{2,m}^* &:= \boldsymbol{\gamma}_{t,m}^*\boldsymbol{\phi}_{tm}^{*'}(\frac{1}{p}\boldsymbol{\Phi}_t'\boldsymbol{\Phi}_t)^{-1}\boldsymbol{\phi}_{tl}.
\end{aligned}$$

Therefore, (D.5) implies $\widehat{\mathbf{g}}_{lt}^* - \mathbf{g}_{lt} = \mathbf{a}_1^* + \mathbf{a}_2^* + o_{P^*}((k_np)^{-1/2}) + o_{P^*}(p^{-1/2})\lambda_{\min}^{1/2}(\mathbf{V}_{\gamma,t})$. Now recall that (D.3) and (D.4) also imply $\widehat{\mathbf{g}}_{lt} - \mathbf{g}_{lt} = \mathbf{a}_0 + \mathbf{a}_2 + o_P((k_np)^{-1/2})$ where $\mathbf{a}_2 = \boldsymbol{\Gamma}_t'\mathbf{P}_l$, and

$$\mathbf{a}_0 = -\frac{1}{p}\sum_{m=1}^p h_{t,ml}\mathbf{A}_{mt}(\mathbf{c}_{z,mt})\nabla_c\Psi_m(\mathbf{c}_{z,mt})\operatorname{vec}(\widehat{\mathbf{c}}_{z,mt} - \mathbf{c}_{z,mt}).$$

So we have

$$\widehat{\mathbf{g}}_{lt}^* - \widehat{\mathbf{g}}_{lt} = (\mathbf{a}_1^* - \mathbf{a}_0) + (\mathbf{a}_2^* - \mathbf{a}_2) + o_{P^*}((k_np)^{-1/2}) + o_{P^*}(p^{-1/2})\lambda_{\min}^{1/2}(\mathbf{V}_{\gamma,t}). \tag{D.6}$$

We note that $\mathbb{E}^*(\mathbf{a}_1^* - \mathbf{a}_0) = \mathbb{E}^*(\mathbf{a}_2^* - \mathbf{a}_2) = 0$. We now calculate their bootstrap covariances.

step 2: calculate the bootstrap variances $\operatorname{var}^*(\mathbf{a}_{2,m}^*)$ and $\operatorname{var}^*(\mathbf{a}_{2,m}^*)$.

By Assumption 4.6, there is $\boldsymbol{\xi}_{mt,s}^0$, so that conditionally on $\mathbf{X}_t$, $\{(\boldsymbol{\gamma}_{mt}, \boldsymbol{\xi}_{mt,s}^0) : m \le p\}$ are cross-sectionally uncorrelated, and $\max_{s\in[0,\Delta_n]}\frac{1}{p}\sum_{m\le p}\mathbb{E}(\|\boldsymbol{\xi}_{mt,s} - \boldsymbol{\xi}_{mt,s}^0\|^4|\mathcal{F}_t) = O_P((pk_n)^{-2})$. Now let $(\mathbf{e}_{m,i}, \mathbf{e}_{m,i}^0)$ be $(\boldsymbol{\xi}_{mt,s}, \boldsymbol{\xi}_{mt,s}^0)$ by setting $s = \Delta_n$ and $t = (i-1)\Delta_n$. In particular,

$$\mathbf{e}_{m,i} := \mathbf{A}_{m,i-1}\nabla_c\Psi_m(\mathbf{c}_{z,m,i-1})\operatorname{vec}\left(\frac{1}{\Delta_n}\Delta_i^n\mathbf{Z}_m\Delta_i^n\mathbf{Z}_m' - \mathbf{c}_{z,m,(i-1)\Delta_n}\right).$$

Then conditionally on $\mathbf{X}_t$, $\{(\boldsymbol{\gamma}_{mt}, \mathbf{e}_{mt}^0) : m \le p\}$ are cross-sectionally uncorrelated. Note that $\mathbf{e}_{m,i}$ is equivalent to $\boldsymbol{\xi}_{mt,s}$ with $t = (i-1)\Delta_n$ and $s = \Delta_n$, so is $\mathbf{e}_{m,i}^0$ to $\boldsymbol{\xi}_{mt,s}^0$.

Write $\mathbf{M}_{mt} := \mathbf{A}_{mt}(\mathbf{c}_{z,mt})\nabla_c\Psi_m(\mathbf{c}_{z,mt})$ and $\mathbf{M}_{m,i-1} := \mathbf{M}_{mt}$,

$$\begin{aligned}
\boldsymbol{\delta}_{em,i} :=& \mathbf{e}_{m,i} - \mathbf{e}_{m,i}^0 + \mathbf{M}_{m,t}(\mathbf{c}_{z,m,(i-1)\Delta_n} - \mathbf{c}_{z,mt}) + (\mathbf{M}_{m,i-1} - \mathbf{M}_{m,t})(\mathbf{c}_{z,m,(i-1)\Delta_n} - \mathbf{c}_{z,mt}) \\
&+ (\mathbf{M}_{mt} - \mathbf{M}_{m,i-1})\operatorname{vec}(\frac{1}{\Delta_n}\Delta_i^n\mathbf{Z}_m\Delta_i^n\mathbf{Z}_m' - \mathbf{c}_{z,mt}).
\end{aligned}$$

We now prove $k_n^2\frac{1}{p}\sum_{m=1}^p\frac{1}{k_n}\sum_{i\in I_t^n}\|\boldsymbol{\delta}_{em,i}\|^4 = o_P(1)$, which will be used in step 3 below.

First,

$$
\begin{aligned}
&\frac{1}{pk_n}\sum_{i\in I_t^n}\sum_{m\leq p}\mathbb{E}(\|\mathbf{e}_{m,i}-\mathbf{e}_{m,i}^0\|^4|\mathcal{F}_{i-1})\\
=&\frac{1}{pk_n}\sum_{i\in I_t^n}\sum_{m\leq p}\mathbb{E}(\|\boldsymbol{\xi}_{m,(i-1)\Delta_n,\Delta_n}-\boldsymbol{\xi}_{m,(i-1)\Delta_n,\Delta_n}^0\|^4|\mathcal{F}_{i-1})\\
\leq&\max_{t\in[0,T],s\in[0,\Delta_n]}\frac{1}{p}\sum_{m\leq p}\mathbb{E}(\|\boldsymbol{\xi}_{mt,s}-\boldsymbol{\xi}_{mt,s}^0\|^4|\mathcal{F}_t)=O_P((pk_n)^{-2}).
\end{aligned}
$$

Note that, for some finite number K, we have

$$
\begin{aligned}
\|\boldsymbol{\delta}_{em,i}\|^4\leq K\big(&\|\mathbf{e}_{m,i}-\mathbf{e}_{m,i}^0\|^4+\|\mathbf{M}_{m,t}(\mathbf{c}_{z,m,(i-1)\Delta_n}-\mathbf{c}_{z,mt})\|^4\\
&+\|(\mathbf{M}_{m,i-1}-\mathbf{M}_{m,t})(\mathbf{c}_{z,m,(i-1)\Delta_n}-\mathbf{c}_{z,mt})\|^4\\
&+\|(\mathbf{M}_{mt}-\mathbf{M}_{m,i-1})\operatorname{vec}(\frac{1}{\Delta_n}\Delta_i^n\mathbf{Z}_m\Delta_i^n\mathbf{Z}_m'-\mathbf{c}_{z,mt})\|^4\big).
\end{aligned}
$$

The desired result about the first term readily follows from the above result. For the second one, the assumption on $\mathbf{c}_z$ implies that (uniformly in m)

$$
\mathbb{E}(\|\mathbf{M}_{m,t}(\mathbf{c}_{z,m,(i-1)\Delta_n}-\mathbf{c}_{z,mt})\|^4\big|\mathcal{F}_t)\leq Lk_n^2\Delta_n^2.
$$

It then follows that

$$
k_n^2\frac{1}{p}\sum_{m=1}^{p}\frac{1}{k_n}\sum_{i\in I_t^n}\mathbb{E}\big(\|\mathbf{M}_{m,t}(\mathbf{c}_{z,m,(i-1)\Delta_n}-\mathbf{c}_{z,mt})\|^4\big|\mathcal{F}_t\big)\leq Lk_n^3\Delta_n^2\to 0.
$$

Similarly, we have

$$
\mathbb{E}\big(\|(\mathbf{M}_{m,i-1}-\mathbf{M}_{m,t})(\mathbf{c}_{z,m,(i-1)\Delta_n}-\mathbf{c}_{z,mt})\|^4\big|\mathcal{F}_t)\leq Lk_n^2\Delta_n^2,
$$

and

$$
\begin{aligned}
&\mathbb{E}\big(\|(\mathbf{M}_{mt}-\mathbf{M}_{m,i-1})\operatorname{vec}(\frac{1}{\Delta_n}\Delta_i^n\mathbf{Z}_m\Delta_i^n\mathbf{Z}_m'-\mathbf{c}_{z,mt})\|^4\mid\mathcal{F}_t\big)\\
=&\mathbb{E}\big(\|(\mathbf{M}_{mt}-\mathbf{M}_{m,i-1})\|^4\,\mathbb{E}(\|\operatorname{vec}(\frac{1}{\Delta_n}\Delta_i^n\mathbf{Z}_m\Delta_i^n\mathbf{Z}_m'-\mathbf{c}_{z,mt})\|^4\mid\mathcal{F}_{i-1}^n)\mid\mathcal{F}_t\big)\\
\leq&L\,\mathbb{E}\big(\|(\mathbf{M}_{mt}-\mathbf{M}_{m,i-1})\|^4\mid\mathcal{F}_t\big)\leq Lk_n^2\Delta_n^2.
\end{aligned}
$$

The corresponding results follow a similar argument. Hence, the desired result about $\boldsymbol{\delta}_{em,i}$ readily follows.

Then

$$\mathbf{M}_{mt}\,\mathrm{vec}(\widehat{\mathbf{c}}_{z,mt}-\mathbf{c}_{z,mt}) = \frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{e}_{m,i}^0 + \frac{1}{k_n}\sum_{i\in I_t^n}\boldsymbol{\delta}_{em,i},$$

So the bootstrap variances of $\mathbf{a}_2^*$ and $\mathbf{a}_1^*$ are respectively given by

$$\begin{aligned}
\mathbf{V}_{\gamma,t}^* &= \mathrm{var}^*(\mathbf{a}_{2,m}^*) = \frac{1}{p}\sum_{m=1}^p \boldsymbol{\gamma}_{t,m}\boldsymbol{\gamma}_{t,m}' h_{t,ml}^2 \\
\mathbf{V}_{u,t}^* &= k_n\mathrm{var}^*(\mathbf{a}_{1,m}^*) = k_n\frac{1}{p}\sum_{m=1}^p \mathbf{M}_{mt}\,\mathrm{vec}(\widehat{\mathbf{c}}_{z,mt}-\mathbf{c}_{z,mt})\,\mathrm{vec}(\widehat{\mathbf{c}}_{z,mt}-\mathbf{c}_{z,mt})'\mathbf{M}_{mt}h_{t,ml}^2 \\
&= \frac{1}{p}\sum_{m=1}^p\frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{e}_{m,i}^0\mathbf{e}_{m,i}^{0\prime}h_{t,ml}^2 + \frac{1}{p}\sum_{m=1}^p\frac{1}{k_n}\sum_{i\neq j\in I_t^n}\mathbf{e}_{m,i}^0\mathbf{e}_{m,j}^{0\prime}h_{t,ml}^2 \\
&\quad + \frac{1}{p}\sum_{m=1}^p\frac{1}{k_n}\sum_{i\neq j\in I_t^n}(\boldsymbol{\delta}_{em,i}\mathbf{e}_{m,j}^{0\prime} + \mathbf{e}_{m,i}^0\boldsymbol{\delta}_{em,j}')h_{t,ml}^2 \\
&:= \mathbf{V}_{u,t,1}^* + \mathbf{V}_{u,t,2}^* + \mathbf{V}_{u,t,3}^*.
\end{aligned}$$

step 3: $\mathbf{V}_{u,t}^* = \mathbf{V}_{u,t}(1+o_P(1))$ and $\mathbf{V}_{\gamma,t}^* = \mathbf{V}_{\gamma,t}(1+o_P(1))$.

First, on the other hand, under the assumption that conditionally on $\mathbf{X}_t$, $\{(\boldsymbol{\gamma}_{mt},\mathbf{e}_{mt}^0) : m\leq p\}$ are cross-sectionally independent at each fixed $t\in[0,T]$,

$$\begin{aligned}
\mathbf{V}_{\gamma,t} &= \frac{1}{p}\sum_{m=1}^p h_{t,ml}^2\,\mathrm{Var}(\boldsymbol{\gamma}_{mt}|\mathbf{X}_t) \\
\mathbf{V}_{u,t} &= \frac{1}{k_n}\sum_{i\in I_t^n}\mathrm{Var}\left(\frac{1}{\sqrt{p}}\sum_{m=1}^p h_{t,ml}(\mathbf{e}_{m,i}^0+\boldsymbol{\delta}_{em,i})\middle|\mathcal{F}_t\right) \\
&= \frac{1}{k_n}\sum_{i\in I_t^n}\frac{1}{p}\sum_{m=1}^p h_{t,ml}^2\,\mathrm{Var}(\mathbf{e}_{m,i}^0|\mathcal{F}_t) + o_P(1)
\end{aligned}$$

where the last equality follows from $\frac{1}{pk_n}\sum_{i\in I_t^n}\sum_{m\leq p}\|\boldsymbol{\delta}_{em,i}\|^2 = o_P(p^{-1})$.

Then by Assumption 3.7,

$$\mathbb{E}\|\mathbf{V}_{\gamma,t}^*-\mathbf{V}_{\gamma,t}\|^2 = \sum_{k_1,k_2\leq K}\frac{1}{p^2}\sum_{m=1}^p\mathbb{E}\mathrm{var}[\gamma_{mt,k_1}\gamma_{mt,k_2}h_{t,ml}^2|\mathbf{X}_t] \leq C\frac{1}{p^2}\sum_{m=1}^p\mathbb{E}\|\boldsymbol{\gamma}_{mt}\|^4h_{t,ml}^4$$

$$\leq \quad \frac{C}{p}\lambda_{\min}^2(\frac{1}{p}\sum_{m=1}^p h_{t,ml}^2 \mathrm{var}(\boldsymbol{\gamma}_{mt}|\mathbf{X}_t)) \leq \frac{C}{p}\lambda_{\min}^2(\mathbf{V}_{\gamma,t}).$$

This implies $\|\mathbf{V}_{\gamma,t}^* - \mathbf{V}_{\gamma,t}\| = o_P(1)\lambda_{\min}(\mathbf{V}_{\gamma,t})$.

As for $\mathbf{V}_{u,t}^*$, by Lemma D.4 $\mathbf{V}_{u,t,2}^* = o_P(1)$. Note that $\lambda_{\min}(\mathbf{V}_{u,t})$ is bounded away from zero. Hence $\mathbf{V}_{u,t,2}^* = o_P(1)\lambda_{\min}(\mathbf{V}_{u,t})$. Also, by Assumption 4.6,

$$\begin{aligned}
\|\mathbf{V}_{u,t,3}^*\|^2 &= \|\frac{1}{p}\sum_{m=1}^p \frac{1}{k_n}\sum_{i\neq j\in I_t^n}(\boldsymbol{\delta}_{em,i}\mathbf{e}_{m,j}^{0\prime} + \mathbf{e}_{m,i}^0\boldsymbol{\delta}_{em,j}')h_{t,ml}^2\|^2 \\
&\leq k_n^2\frac{1}{p}\sum_{m=1}^p\frac{1}{k_n}\sum_{i\in I_t^n}\|\boldsymbol{\delta}_{em,i}\|^4\frac{1}{p}\sum_{m=1}^p\frac{1}{k_n}\sum_{j\in I_t^n}\|\mathbf{e}_{m,j}^0\|^4\frac{1}{p}\sum_{m=1}^p h_{t,ml}^4 \\
&= O_P(k_n^2)\frac{1}{p}\sum_{m=1}^p\frac{1}{k_n}\sum_{i\in I_t^n}\|\boldsymbol{\delta}_{em,i}\|^4 \\
&= o_P(1).
\end{aligned}$$

So $\mathbf{V}_{u,t,3}^* = o_P(1)\lambda_{\min}(\mathbf{V}_{u,t})$. Next,

$$\begin{aligned}
\mathbb{E}(\mathbf{V}_{u,t,1}^*|\mathcal{F}_t) &= \frac{1}{p}\sum_{m=1}^p\frac{1}{k_n}\sum_{i\in I_t^n}\mathrm{Var}(\mathbf{e}_{m,i}^0|\mathcal{F}_t)h_{t,ml}^2 = \mathbf{V}_{ut} + o_P(1) \\
\mathrm{Var}(\mathbf{V}_{u,t,1}^*|\mathcal{F}_t) &\leq \frac{C}{p}\sum_{m=1}^p h_{t,ml}^4\frac{1}{k_n^2}\sum_{i\in I_t^n}\frac{1}{p}\sum_{m=1}^p\mathbb{E}\left(\|\mathrm{vec}(\mathbf{Z}_{mi}\mathbf{Z}_{mi}' - \mathbf{c}_{z,mt})\|^4|\mathcal{F}_t\right) = O_P(1).
\end{aligned}$$

Thus $\mathbf{V}_{u,t,1}^* - \mathbf{V}_{u,t} = o_P(1)\lambda_{\min}(\mathbf{V}_{u,t})$. Together $\mathbf{V}_{u,t}^* - \mathbf{V}_{u,t} = o_P(1)\lambda_{\min}(\mathbf{V}_{u,t})$.

Next, let

$$\boldsymbol{\xi}_n := p\mathbb{E}^*(\mathbf{a}_1^* - \mathbf{a}_0)(\mathbf{a}_2^* - \mathbf{a}_2)'.$$

$$\boldsymbol{\Sigma}_n^* := \frac{1}{k_n p}\mathbf{V}_{u,t}^* + \frac{1}{p}\mathbf{V}_{\gamma,t}^* + \frac{2}{p}\boldsymbol{\xi}_n.$$

step 4: $p^{-1}\lambda_{\min}^{-1}(\boldsymbol{\Sigma}_n)\|\boldsymbol{\xi}_n\| = o_P(1)$**.** To see this,

$$\begin{aligned}
\frac{1}{p}\boldsymbol{\xi}_n &= \mathbb{E}^*(\mathbf{a}_1^* - \mathbf{a}_0)(\mathbf{a}_2^* - \mathbf{a}_2)' = \frac{1}{p^2}\sum_{m=1}^p\sum_{q=1}^p\mathbb{E}^*(\mathbf{a}_{1,m}^* - \mathbf{a}_{1,m})(\mathbf{a}_{2,q}^* - \mathbf{a}_{2,q})' \\
&= \frac{1}{p^2}\sum_m\mathbb{E}^*(\mathbf{a}_{1,m}^* - \mathbf{a}_{1,m})(\mathbf{a}_{2,m}^* - \mathbf{a}_{2,m})' + \frac{1}{p^2}\sum_m\sum_{q\neq m}\mathbb{E}^*(\mathbf{a}_{1m}^* - \mathbf{a}_{1m})\mathbb{E}^*(\mathbf{a}_{2q}^* - \mathbf{a}_{2q})' \\
&= \frac{1}{p^2}\sum_{m=1}^p\mathbb{E}^*(\mathbf{a}_{1,m}^* - \mathbf{a}_{1,m})(\mathbf{a}_{2,m}^* - \mathbf{a}_{2,m})' = \frac{2}{p^2}\sum_{m=1}^p\mathbf{a}_{1,m}\mathbf{a}_{2,m}' - \frac{2}{p^2}\sum_{m=1}^p\mathbf{a}_{1,m}\frac{1}{p}\sum_{q=1}^p\mathbf{a}_{2,q}'.
\end{aligned}$$

(D.7)

where

$$\begin{aligned}\mathbf{a}_{1,m} &:= -\mathbf{M}_{mt}\,\mathrm{vec}(\widehat{\mathbf{c}}_{z,mt}-\mathbf{c}_{z,mt})h_{t,ml}\\ \mathbf{a}_{2,m} &:= \boldsymbol{\gamma}_{t,m}h_{t,ml}.\end{aligned}$$

Then let $\mathbb{E}_{X_t}$ and $\mathbb{E}_{\mathcal{F}_t}$ respectively denote the conditional mean given $\mathbf{X}_t$ and $\mathcal{F}_t$. Let Var_{X_t} and $\mathrm{Var}_{\mathcal{F}_t}$ denote the corresponding variances. Let $[\mathbf{e}]_j$ denote the j th element of a vector $\mathbf{e}$. Because conditionally on $\mathbf{X}_t$, $(\mathbf{e}^0_{mt},\boldsymbol{\gamma}_{mt})$ are cross-sectionally independent,

$$\begin{aligned}
&\|\frac{1}{p}\sum_{m=1}^p \mathbf{a}_{1,m}\mathbf{a}_{2,m}'\|^2 = O_P(1)\mathbb{E}_{X_t}\mathbb{E}_{\mathcal{F}_t}\|\frac{1}{p}\sum_{m=1}^p \mathbf{M}_{mt}\,\mathrm{vec}(\widehat{\mathbf{c}}_{z,mt}-\mathbf{c}_{z,mt})\boldsymbol{\gamma}_{mt}'h_{t,ml}^2\|^2\\
=\;& O_P(1)\mathbb{E}_{X_t}\mathbb{E}_{\mathcal{F}_t}\|\frac{1}{p}\sum_{m=1}^p\frac{1}{k_n}\sum_{i\in I_t^n}(\mathbf{e}^0_{m,i}+\boldsymbol{\delta}_{em,i})\boldsymbol{\gamma}_{mt}'h_{t,ml}^2\|^2\\
=\;& O_P(1)\sum_{j,r}\frac{1}{p^2}\sum_{m=1}^p h_{t,ml}^4\mathbb{E}_{X_t}\gamma_{mt,r}^2\mathbb{E}_{\mathcal{F}_t}(\frac{1}{k_n}\sum_{i\in I_t^n}[\mathbf{e}^0_{m,i}]_j)^2 + O_P(1)\mathbb{E}_{X_t}\mathbb{E}_{\mathcal{F}_t}\|\frac{1}{p}\sum_{m=1}^p\frac{1}{k_n}\sum_{i\in I_t^n}\boldsymbol{\delta}_{em,i}\boldsymbol{\gamma}_{mt}'h_{t,ml}^2\|^2\\
=\;& O_P(1)\sum_{j,r}\frac{1}{p^2}\sum_{m=1}^p h_{t,ml}^4\mathbb{E}_{X_t}\gamma_{mt,r}^2\frac{1}{k_n^2}\sum_{i\in I_t^n}\mathrm{Var}_{\mathcal{F}_t}([\mathbf{e}^0_{m,i}]_j)\\
&+O_P(1)\frac{1}{p}\sum_{m=1}^p\frac{1}{k_n}\sum_{i\in I_t^n}\|\boldsymbol{\delta}_{em,i}\|^2\frac{1}{p}\sum_{m=1}^p h_{t,ml}^4\mathbb{E}_{X_t}\|\boldsymbol{\gamma}_{mt}\|^2\\
\leq\;& O_P(\frac{1}{pk_n})\frac{1}{p}\sum_{m=1}^p h_{t,ml}^4\mathbb{E}_{X_t}\|\boldsymbol{\gamma}_{mt}\|^2 \quad (\text{ because } \frac{1}{p}\sum_{m=1}^p\frac{1}{k_n}\sum_{i\in I_t^n}\|\boldsymbol{\delta}_{em,i}\|^2 = O_P((pk_n)^{-1}))\\
\leq\;& O_P(\frac{1}{pk_n})(\frac{1}{p}\sum_{m=1}^p h_{t,ml}^2)^{1/2}(\frac{1}{p}\sum_{m=1}^p h_{t,ml}^4\mathbb{E}_{X_t}\|\boldsymbol{\gamma}_{mt}\|^4)^{1/2}\leq O_P(1)\frac{1}{pk_n}\lambda_{\min}(\mathbf{V}_\gamma).
\end{aligned}$$

This implies

$$p^{-1}\|\frac{1}{p}\sum_{m=1}^p\mathbf{a}_{1,m}\mathbf{a}_{2,m}'\|\leq O_P(1)\frac{1}{p}\frac{1}{\sqrt{pk_n}}\lambda_{\min}^{1/2}(\mathbf{V}_\gamma)\leq O_P(1)\frac{1}{\sqrt{p}}[\frac{1}{k_np}+\frac{1}{p}\lambda_{\min}(\mathbf{V}_\gamma)]\leq O_P(1)\frac{1}{\sqrt{p}}\lambda_{\min}(\boldsymbol{\Sigma}_n).$$

Similarly, $p^{-1}\|\frac{1}{p}\sum_{m=1}^p\mathbf{a}_{1,m}\frac{1}{p}\sum_{q=1}^p\mathbf{a}_{2,q}'\|\leq C\frac{1}{\sqrt{p}}\lambda_{\min}(\boldsymbol{\Sigma}_n)$. Thus $p^{-1}\|\boldsymbol{\xi}_n\|\leq C\lambda_{\min}(\boldsymbol{\Sigma}_n)p^{-1/2}$.

step 5: complete the proof. Note that

$$(\mathbf{v}'\boldsymbol{\Sigma}_n\mathbf{v})^{-1/2}\,\mathbf{v}'\big(\widehat{\mathbf{g}}^*_{lt}-\widehat{\mathbf{g}}_{lt}\big) = (\mathbf{v}'\boldsymbol{\Sigma}_n\mathbf{v})^{-1/2}\,\mathbf{v}'[(\mathbf{a}_1^*-\mathbf{a}_1)+(\mathbf{a}_2^*-\mathbf{a}_2)]+R$$

$$= \left(\mathbf{v}'\mathbf{\Sigma}_n\mathbf{v}\right)^{-1/2}\left(\mathbf{v}'\mathbf{\Sigma}_n^*\mathbf{v}\right)^{1/2} Z_n^* + R$$

where $Z_n^* := \left(\mathbf{v}'\mathbf{\Sigma}_n^*\mathbf{v}\right)^{-1/2}\mathbf{v}'[(\mathbf{a}_1^*-\mathbf{a}_1)+(\mathbf{a}_2^*-\mathbf{a}_2)] \xrightarrow{d^*} N(0,1)$ and

$$\begin{aligned} R &= \left(\mathbf{v}'\mathbf{\Sigma}_n\mathbf{v}\right)^{-1/2}[o_{P^*}((k_np)^{-1/2})+o_{P^*}(p^{-1/2})\lambda_{\min}^{1/2}(\mathbf{V}_{\gamma,t})] \\ &\leq \min\{\|(\frac{1}{k_np}\mathbf{V}_{u,t})^{-1/2}\|,\|(\frac{1}{p}\mathbf{V}_{\gamma,t})^{-1/2}\|\}[o_{P^*}((k_np)^{-1/2})+o_{P^*}(p^{-1/2})\lambda_{\min}^{1/2}(\mathbf{V}_{\gamma,t})] \\ &\leq o_{P^*}(1)+\|\mathbf{V}_{\gamma,t}^{-1/2}\|o_{P^*}(1)\lambda_{\min}^{1/2}(\mathbf{V}_{\gamma,t}) = o_{P^*}(1). \end{aligned}$$

It remains to show that $\left(\mathbf{v}'\mathbf{\Sigma}_n\mathbf{v}\right)^{-1/2}\left(\mathbf{v}'\mathbf{\Sigma}_n^*\mathbf{v}\right)^{1/2}-1 = o_P(1)$, which then implies $\left(\mathbf{v}'\mathbf{\Sigma}_n\mathbf{v}\right)^{-1/2}\mathbf{v}'\left(\widehat{\mathbf{g}}_{lt}^*-\widehat{\mathbf{g}}_{lt}\right) \xrightarrow{d^*} N(0,\mathbf{I}_K)$. In fact,

$$\begin{aligned} &\left(\mathbf{v}'\mathbf{\Sigma}_n\mathbf{v}\right)^{-1/2}\left(\mathbf{v}'\mathbf{\Sigma}_n^*\mathbf{v}\right)^{1/2}-1 = \left(\mathbf{v}'\mathbf{\Sigma}_n\mathbf{v}\right)^{-1/2}[\left(\mathbf{v}'\mathbf{\Sigma}_n^*\mathbf{v}\right)^{1/2}-\left(\mathbf{v}'\mathbf{\Sigma}_n\mathbf{v}\right)^{1/2}] \\ =\ & \left(\mathbf{v}'\mathbf{\Sigma}_n\mathbf{v}\right)^{-1/2}[\left(\mathbf{v}'\mathbf{\Sigma}_n^*\mathbf{v}\right)^{1/2}+\left(\mathbf{v}'\mathbf{\Sigma}_n\mathbf{v}\right)^{1/2}]^{-1}\mathbf{v}'(\mathbf{\Sigma}_n^*-\mathbf{\Sigma}_n)\mathbf{v} \\ =\ & \left(\mathbf{v}'\mathbf{\Sigma}_n\mathbf{v}\right)^{-1/2}[\left(\mathbf{v}'\mathbf{\Sigma}_n^*\mathbf{v}\right)^{1/2}+\left(\mathbf{v}'\mathbf{\Sigma}_n\mathbf{v}\right)^{1/2}]^{-1}\mathbf{v}'(o_P(1)\mathbf{\Sigma}_n+\frac{2}{p}\boldsymbol{\xi}_n)\mathbf{v} \\ \leq\ & o_P(1)\left(\mathbf{v}'\mathbf{\Sigma}_n\mathbf{v}\right)^{-1}\lambda_{\min}(\mathbf{\Sigma}_n)+\frac{2}{p}\left(\mathbf{v}'\mathbf{\Sigma}_n\mathbf{v}\right)^{-1}\mathbf{v}'\boldsymbol{\xi}_n\mathbf{v} \leq o_P(1)+Cp^{-1}\lambda_{\min}^{-1}(\mathbf{\Sigma}_n)\|\boldsymbol{\xi}_n\|. \end{aligned}$$

By step 3, the last term on the right hand side is $o_P(1)$.

□

Lemma D.1. *Consider random (vector) sequences $\mathbf{X}_n,\mathbf{Y}_n,\mathbf{Z}_n$. Also, let $\mathbf{A}_n,\mathbf{B}_n$ be nonrandom (matrix) sequences. Suppose:*
(i) $\mathbf{A}_n\mathbf{X}_n \overset{\mathcal{L}\text{-}s}{\to} N(0,\mathbf{I})$; $\mathbf{B}_n\mathbf{Y}_n \overset{\mathcal{L}\text{-}s}{\to} N(0,\mathbf{I})$.
(ii) $\mathbf{X}_n$ *and* $\mathbf{Y}_n$ *are uncorrelated.*
(iii) $\min\{\|\mathbf{A}_n\|,\|\mathbf{B}_n\|\}\|\mathbf{Z}_n\| \xrightarrow{P} 0$. *Then*

$$[(\mathbf{A}_n'\mathbf{A}_n)^{-1}+(\mathbf{B}_n'\mathbf{B}_n)^{-1}]^{-1/2}(\mathbf{X}_n+\mathbf{Y}_n+\mathbf{Z}_n) \overset{\mathcal{L}\text{-}s}{\to} N(0,1).$$

Proof. **Step 1:** prove $\|[(\mathbf{A}_n'\mathbf{A}_n)^{-1}+(\mathbf{B}_n'\mathbf{B}_n)^{-1}]^{-1/2}\| \leq \min\{\|\mathbf{A}_n\|,\|\mathbf{B}_n\|\}$. In fact,

$$\begin{aligned} &\|[(\mathbf{A}_n'\mathbf{A}_n)^{-1}+(\mathbf{B}_n'\mathbf{B}_n)^{-1}]^{-1/2}\|^2 = \|[(\mathbf{A}_n'\mathbf{A}_n)^{-1}+(\mathbf{B}_n'\mathbf{B}_n)^{-1}]^{-1}\| = \lambda_{\min}^{-1}((\mathbf{A}_n'\mathbf{A}_n)^{-1}+(\mathbf{B}_n'\mathbf{B}_n)^{-1}) \\ \leq\ & (\lambda_{\min}((\mathbf{A}_n'\mathbf{A}_n)^{-1})+\lambda_{\min}((\mathbf{B}_n'\mathbf{B}_n)^{-1}))^{-1} = (\|\mathbf{A}_n'\mathbf{A}_n\|^{-1}+\|\mathbf{B}_n'\mathbf{B}_n\|^{-1})^{-1} \\ \leq\ & (\max(\|\mathbf{A}_n'\mathbf{A}_n\|^{-1},\|\mathbf{B}_n'\mathbf{B}_n\|^{-1}))^{-1} = \min(\|\mathbf{A}_n'\mathbf{A}_n\|,\|\mathbf{B}_n'\mathbf{B}_n\|) = \min(\|\mathbf{A}_n\|,\|\mathbf{B}_n\|)^2. \end{aligned}$$

Hence $\|[(\mathbf{A}_n'\mathbf{A}_n)^{-1}+(\mathbf{B}_n'\mathbf{B}_n)^{-1}]^{-1/2}\| \leq \min\{\|\mathbf{A}_n\|,\|\mathbf{B}_n\|\}$.

Step 2: prove $[(\mathbf{A}_n'\mathbf{A}_n)^{-1}+(\mathbf{B}_n'\mathbf{B}_n)^{-1}]^{-1/2}\mathbf{Z}_n = o_P(1)$.

By step 1 and (iii), $\|[(\mathbf{A}_n'\mathbf{A}_n)^{-1}+(\mathbf{B}_n'\mathbf{B}_n)^{-1}]^{-1/2}\mathbf{Z}_n\| \leq \min\{\|\mathbf{A}_n\|,\|\mathbf{B}_n\|\}\|\mathbf{Z}_n\| = o_P(1)$.

Step 3: prove $[(\mathbf{A}_n'\mathbf{A}_n)^{-1}+(\mathbf{B}_n'\mathbf{B}_n)^{-1}]^{-1/2}(\mathbf{X}_n+\mathbf{Y}_n) \overset{\mathcal{L}\text{-}s}{\to} N(0,\mathbf{I})$. Let

$$\mathbf{D}_n = [(\mathbf{A}_n'\mathbf{A}_n)^{-1}+(\mathbf{B}_n'\mathbf{B}_n)^{-1}]^{-1/2}\left((\mathbf{A}_n'\mathbf{A}_n)^{-1}\mathbf{A}_n',(\mathbf{B}_n'\mathbf{B}_n)^{-1}\mathbf{B}_n'\right), \quad \mathcal{Z}_n = \begin{pmatrix}\mathbf{A}_n\mathbf{X}_n\\ \mathbf{B}_n\mathbf{Y}_n\end{pmatrix}.$$

Then (i)(ii) imply $\mathcal{Z}_n \overset{\mathcal{L}\text{-}s}{\to} N(0,\mathbf{I})$. Also $[(\mathbf{A}_n'\mathbf{A}_n)^{-1}+(\mathbf{B}_n'\mathbf{B}_n)^{-1}]^{-1/2}(\mathbf{X}_n+\mathbf{Y}_n) = \mathbf{D}_n\mathcal{Z}_n$. Let $\mathcal{Z}$ be a standard normal random vector. Since $\mathbf{D}_n\mathbf{D}_n' = \mathbf{I}$, $\mathbf{D}_n\mathcal{Z} =^d (\mathbf{D}_n\mathbf{D}_n')^{1/2}\mathcal{Z} =^d \mathcal{Z}$. Hence $\mathbf{D}_n\mathcal{Z}_n \overset{\mathcal{L}\text{-}s}{\to} \mathcal{Z}$, which is

$$[(\mathbf{A}_n'\mathbf{A}_n)^{-1}+(\mathbf{B}_n'\mathbf{B}_n)^{-1}]^{-1/2}(\mathbf{X}_n+\mathbf{Y}_n) \overset{\mathcal{L}\text{-}s}{\to} N(0,\mathbf{I}).$$

The result then follows from step 2,3.

□

Lemma D.2. *Suppose $J^2 = o(p)$. (i) $\|(\frac{1}{p}\mathbf{\Phi}_t^*\mathbf{\Phi}_t^{*'})^{-1}\| = O_{P^*}(1)$.*
(ii) $\|\mathbf{d}_1^* - \mathbf{a}_1^*\| = o_{P^*}((k_np)^{-1/2})$, where

$$\begin{aligned}
\mathbf{d}_1^* &= -\frac{1}{p}\sum_{m=1}^p h_{t,ml}^*\mathbf{A}_{mt}^*(\mathbf{c}_{z,mt}^*)\nabla_c\Psi_m^*(\boldsymbol{\beta}_{mt}^*,\mathbf{c}_{z,mt}^*)\,\mathrm{vec}(\widehat{\mathbf{c}}_{z,mt}^*-\mathbf{c}_{z,mt}^*)\\
\mathbf{a}_1^* &= -\frac{1}{p}\sum_{m=1}^p \mathbf{A}_{mt}^*(\mathbf{c}_{z,mt}^*)\nabla_c\Psi_m^*(\boldsymbol{\beta}_{mt}^*,\mathbf{c}_{z,mt}^*)\,\mathrm{vec}(\widehat{\mathbf{c}}_{z,mt}^*-\mathbf{c}_{z,mt}^*)\boldsymbol{\phi}_{mt}^{*'}(\frac{1}{p}\mathbf{\Phi}_t'\mathbf{\Phi}_t)^{-1}\boldsymbol{\phi}_{lt}
\end{aligned}$$

(iii) $\|\mathbf{\Gamma}_t^{*'}\mathbf{P}_{t,l}^* - \mathbf{a}_2^*\| = o_{P^*}(p^{-1/2})\lambda_{\min}^{1/2}(\mathbf{V}_{\gamma,t})$, where $\mathbf{a}_2^* = \frac{1}{p}\sum_{m=1}^p\boldsymbol{\gamma}_{t,m}^*\boldsymbol{\phi}_{tm}^{*'}(\frac{1}{p}\mathbf{\Phi}_t'\mathbf{\Phi}_t)^{-1}\boldsymbol{\phi}_{tl}$.

Proof. (i) First, we have $\|\frac{1}{p}\mathbf{\Phi}_t^{*'}\mathbf{\Phi}_t^* - \frac{1}{p}\mathbf{\Phi}_t'\mathbf{\Phi}_t\| = O_{P^*}(\sqrt{\frac{J}{p}})$, which implies $\lambda_{\min}(\frac{1}{p}\mathbf{\Phi}_t^{*'}\mathbf{\Phi}_t^*) \geq \lambda_{\min}(\frac{1}{p}\mathbf{\Phi}_t'\mathbf{\Phi}_t) - o_P(1)$. Hence

$$\begin{aligned}
&\|(\frac{1}{p}\mathbf{\Phi}_t^*\mathbf{\Phi}_t^{*'})^{-1}\| = O_{P^*}(1)\\
&\|(\frac{1}{p}\mathbf{\Phi}_t^*\mathbf{\Phi}_t^{*'})^{-1} - (\frac{1}{p}\mathbf{\Phi}_t\mathbf{\Phi}_t')^{-1}\| \leq \|(\frac{1}{p}\mathbf{\Phi}_t^{*'}\mathbf{\Phi}_t^*)^{-1}\|\|(\frac{1}{p}\mathbf{\Phi}_t'\mathbf{\Phi}_t)^{-1}\|\|\frac{1}{p}\mathbf{\Phi}_t^{*'}\mathbf{\Phi}_t^* - \frac{1}{p}\mathbf{\Phi}_t'\mathbf{\Phi}_t\| = O_{P^*}(\sqrt{\frac{J}{p}}).
\end{aligned} \tag{D.8}$$

(ii) Define:

$$\mathbf{B}_t^* := \frac{1}{p}\sum_{m=1}^p \mathbf{A}_{mt}^*(\mathbf{c}_{z,mt}^*)\nabla_c\Psi_m^*(\boldsymbol{\beta}_{mt}^*,\mathbf{c}_{z,mt}^*)\,\mathrm{vec}(\widehat{\mathbf{c}}_{z,mt}^*-\mathbf{c}_{z,mt}^*)\boldsymbol{\phi}_{mt}^{*'}$$

$$\mathbf{B}_t \;:=\; \frac{1}{p}\sum_{m=1}^{p}\mathbf{A}_{mt}(\mathbf{c}_{z,mt})\nabla_c\Psi_m(\boldsymbol{\beta}_{mt},\mathbf{c}_{z,mt})\,\mathrm{vec}(\widehat{\mathbf{c}}_{z,mt}-\mathbf{c}_{z,mt})\boldsymbol{\phi}_{mt}',$$

Next, we use the inequality: $\mathrm{Var}^*(\frac{1}{p}\sum_{m=1}^{p}x_m^*)\le\frac{1}{p}\sum_{m=1}^{p}(x_m)^2$ for a random variable in the bootstrap world x_m^* that is cross-sectionally resampled from x_m. Note $\mathbb{E}^*\mathbf{B}_t^*=\mathbf{B}_t$.

$$\begin{aligned}
\|\mathrm{Var}^*(\mathbf{B}_t^*)\| &= \|\frac{1}{p^2}\sum_{m=1}^{p}\mathrm{Var}(\boldsymbol{\phi}_{mt}^*\mathbf{A}_{mt}^*(\mathbf{c}_{z,mt}^*)\nabla_c\Psi_m^*(\boldsymbol{\beta}_{mt}^*,\mathbf{c}_{z,mt}^*)\,\mathrm{vec}(\widehat{\mathbf{c}}_{z,mt}^*-\mathbf{c}_{z,mt}^*))\|\\
&\le \frac{1}{p^2}\sum_{m=1}^{p}\|\boldsymbol{\phi}_{mt}\|^2\|\mathbf{A}_{mt}(\mathbf{c}_{z,mt})\|^2\|\nabla_c\Psi_m(\boldsymbol{\beta}_{mt},\mathbf{c}_{z,mt})\|^2\|\widehat{\mathbf{c}}_{z,mt}-\mathbf{c}_{z,mt}\|^2\\
&\le O_P(\frac{1}{p})(\frac{1}{p}\sum_{m=1}^{p}\mathbb{E}\|\boldsymbol{\phi}_{mt}\|^4\|\nabla_\beta\Psi_m(\mathbf{c}_{z,mt})\|^4\|\nabla_c\Psi_m(\boldsymbol{\beta}_{mt},\mathbf{c}_{z,mt})\|^4)^{1/2}(\frac{1}{p}\sum_{m=1}^{p}\|\widehat{\mathbf{c}}_{z,mt}-\mathbf{c}_{z,mt}\|^4)^{1/2}\\
&\le O_P(\frac{J}{p})(\frac{1}{p}\sum_{m=1}^{p}\|\widehat{\mathbf{c}}_{z,mt}-\mathbf{c}_{z,mt}\|^4)^{1/2}= O_P(\frac{J}{pk_n}).
\end{aligned}$$

where we used $\max_m\mathbb{E}\|\widehat{\mathbf{c}}_{z,mt}-\mathbf{c}_{z,mt}\|^4=O(k_n^{-2})$ from Jacod and Protter (2011). This implies $\|\mathbf{B}_t^*-\mathbf{B}_t\|=O_P(J^{1/2}(pk_n)^{-1/2})$. So

$$\begin{aligned}
\mathbf{d}_1^*-\mathbf{a}_1^* &= \mathbf{B}_t^*[(\frac{1}{p}\boldsymbol{\Phi}_t'\boldsymbol{\Phi}_t)^{-1}-(\frac{1}{p}\boldsymbol{\Phi}_t^{*\prime}\boldsymbol{\Phi}_t^*)^{-1}]\boldsymbol{\phi}_{lt}=O_{P^*}(\sqrt{\frac{J}{p}})[\|\mathbf{B}_t^*-\mathbf{B}_t\|+\|\mathbf{B}_t\|]\\
&\le o_{P^*}(\frac{1}{\sqrt{pk_n}})+O_{P^*}(\sqrt{\frac{J}{p}})\|\frac{1}{p}\sum_{m=1}^{p}\mathbf{A}_{mt}(\mathbf{c}_{z,mt})\nabla_c\Psi_m(\boldsymbol{\beta}_{mt},\mathbf{c}_{z,mt})\,\mathrm{vec}(\widehat{\mathbf{c}}_{z,mt}-\mathbf{c}_{z,mt})\boldsymbol{\phi}_{mt}'\|\\
&\le o_{P^*}(\frac{1}{\sqrt{pk_n}}).
\end{aligned}$$

where we bounded the last term:

$$\begin{aligned}
&\mathbb{E}_{X_t}\mathbb{E}_{\mathcal{F}_t}\|\frac{1}{p}\sum_{m=1}^{p}\mathbf{A}_{mt}(\mathbf{c}_{z,mt})\nabla_c\Psi_m(\boldsymbol{\beta}_{mt},\mathbf{c}_{z,mt})\,\mathrm{vec}(\widehat{\mathbf{c}}_{z,mt}-\mathbf{c}_{z,mt})\boldsymbol{\phi}_{mt}'\|^2\\
=&\sum_r\frac{1}{p^2}\sum_{m=1}^{p}\mathbb{E}_{X_t}\frac{1}{k_n^2}\sum_{i\in I_t^n}\mathrm{Var}_{\mathcal{F}_t}([\mathbf{e}_{m,i}^0]_r)\|\boldsymbol{\phi}_{mt}\|^2+O_P(1)\frac{1}{pk_n}\sum_{i\in I_t^n}\sum_{m\le p}\|\mathbf{e}_{m,i}-\mathbf{e}_{m,i}^0\|^2\\
=&O_P(\frac{1}{k_np})\frac{1}{p}\sum_{m=1}^{p}h_{t,mm}^2+O_P(\frac{1}{k_np})=O_P(\frac{1}{k_np}).
\end{aligned}$$

(iii) Recall that $\mathbb{E}_{X_t}$ denotes the conditional expectation given $\mathbf{X}_t$.

$$\begin{aligned}
&\mathbb{E}_{X_t}\mathbb{E}^*\sum_{d\leq K}\|\boldsymbol{\phi}_{tl}\frac{1}{p}\sum_{m=1}^p\gamma_{t,md}^*\boldsymbol{\phi}_{tm}^{*'}\|^2 \leq \mathbb{E}_{X_t}\|\boldsymbol{\phi}_{tl}\|^2\sum_{d\leq K}\frac{1}{p^2}\sum_{m=1}^p\mathbb{E}^*\gamma_{t,md}^{*2}\boldsymbol{\phi}_{tm}^{*'}\boldsymbol{\phi}_{tm}^*\\
&+\mathbb{E}_{X_t}\|\boldsymbol{\phi}_{tl}\|^2\mathbb{E}^*\sum_{d\leq K}\frac{1}{p}\sum_{m=1}^p\gamma_{t,md}^*\boldsymbol{\phi}_{tm}^{*'}\mathbb{E}^*\frac{1}{p}\sum_{r\neq m}\gamma_{t,rd}^*\boldsymbol{\phi}_{tr}^*\\
\leq\ & \mathbb{E}_{X_t}\sum_{d\leq K}\frac{1}{p^2}\sum_{m=1}^p\gamma_{t,md}^2\|\boldsymbol{\phi}_{tm}\|^2\|\boldsymbol{\phi}_{tl}\|^2+\mathbb{E}_{X_t}\sum_{d\leq K}\|\frac{1}{p}\sum_{m=1}^p\gamma_{t,md}\boldsymbol{\phi}_{tm}\|^2\|\boldsymbol{\phi}_{tl}\|^2\\
\leq\ & \mathbb{E}_{X_t}\sum_{d\leq K}\frac{1}{p^2}\sum_{m=1}^p\gamma_{t,md}^2h_{t,mm}h_{t,ll}\lambda_{\max}^2(\frac{1}{p}\boldsymbol{\Phi}_t'\boldsymbol{\Phi}_t)+\mathbb{E}_{X_t}\sum_{d\leq K}\|\frac{1}{p}\sum_{m=1}^p\gamma_{t,md}\boldsymbol{\phi}_{tm}\|^2h_{t,ll}\lambda_{\max}(\frac{1}{p}\boldsymbol{\Phi}_t'\boldsymbol{\Phi}_t)\\
\leq\ & \frac{C}{p}\|\frac{1}{p}\sum_{m=1}^p\mathrm{Var}(\boldsymbol{\gamma}_{mt}|\mathbf{X}_t)h_{t,mm}h_{t,ll}\|\leq\frac{C}{p}\lambda_{\min}(\mathbf{V}_{\gamma,t})\quad\text{ by Assumption 3.7.}
\end{aligned}$$

Hence

$$\begin{aligned}
\|\boldsymbol{\Gamma}_t^{*'}\mathbf{P}_{t,l}^*-\mathbf{a}_2^*\|^2 &= \sum_{d\leq K}(\frac{1}{p}\sum_{m=1}^p\gamma_{t,md}^*\boldsymbol{\phi}_{tm}^{*'}[(\frac{1}{p}\boldsymbol{\Phi}_t^{*'}\boldsymbol{\Phi}_t^*)^{-1}-(\frac{1}{p}\boldsymbol{\Phi}_t'\boldsymbol{\Phi}_t)^{-1}]\boldsymbol{\phi}_{tl})^2\\
&\leq \sum_{d\leq K}\|\boldsymbol{\phi}_{tl}\frac{1}{p}\sum_{m=1}^p\gamma_{t,md}^*\boldsymbol{\phi}_{tm}^{*'}\|^2O_{P^*}(\frac{J}{p})\leq o_{P^*}(1)\frac{1}{p}\lambda_{\min}(\mathbf{V}_{\gamma,t}).
\end{aligned}$$

□

Lemma D.3. *Recall that $\mathbf{A}_{m,i-1}:=\mathbf{A}_{mt}(\mathbf{c}_{z,mt})$ and $\nabla_c\Psi_m(\mathbf{c}_{z,m,i-1})=\nabla_c\Psi_m(\mathbf{c}_{z,mt})$ when $t=(i-1)\Delta_n$.*

$$\sqrt{k_np}\mathbf{V}_{u,t}^{-1/2}\frac{1}{p}\sum_{m=1}^p h_{t,ml}\frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{A}_{m,i-1}\nabla_c\Psi_m(\boldsymbol{\beta}_{mt},\mathbf{c}_{z,m,i-1})\,\mathrm{vec}(\frac{1}{\Delta_n}\Delta_i^n\mathbf{Z}_m\Delta_i^n\mathbf{Z}_m'-\mathbf{c}_{z,m,(i-1)\Delta_n})\stackrel{\mathcal{L}\text{-}s}{\to}N(0,\mathbf{I}_K).$$

Proof. Write $W(1)_i=\frac{1}{p}\sum_{m=1}^p h_{t,ml}\mathbf{e}_{m,i}$, where

$$\mathbf{e}_{m,i}\ :=\ \mathbf{A}_{m,i-1}\nabla_c\Psi_m(\mathbf{c}_{z,m,i-1})\,\mathrm{vec}\left(\frac{1}{\Delta_n}\Delta_i^n\mathbf{Z}_m\Delta_i^n\mathbf{Z}_m'-\mathbf{c}_{z,m,i-1}\right).$$

Write

$$\mathbf{V}_{u,t}\ =\ \lim_{s\to0}\mathrm{Var}\left(\frac{1}{\sqrt{p}}\sum_{m=1}^p h_{t,ml}\boldsymbol{\xi}_{mt,s}\middle|\mathcal{F}_t\right)$$

$$\mathbf{V}_{e,t} = \frac{1}{k_n}\sum_{i\in I_t^n}\mathrm{Var}\left(\frac{1}{\sqrt{p}}\sum_{m=1}^{p}h_{t,ml}\mathbf{e}_{m,i}\middle|\mathcal{F}_{i-1}\right).$$

By Assumption 4.4, $\|\mathbf{V}_{u,t}^{-1/2}\| < C$. Also, as $\Delta_n \to 0$, due to the right continuity of the quadratic variation process $\mathbf{c}_{c,mt}$ with respect to t, $\|\mathbf{V}_{u,t} - \mathbf{V}_{e,t}\| = o_P(1)$. Then it suffices to prove $\sqrt{k_n p}\mathbf{V}_{e,t}^{-1/2}\frac{1}{k_n}\sum_{i\in I_t^n} W(1)_i \stackrel{\mathcal{L}\text{-}s}{\to} N(0,\mathbf{I}_K)$. We apply Theorems 2.2.13 and 2.2.15 in Jacod and Protter (2011) by checking all its conditions.

First we have $\sqrt{k_n p}\mathbb{E}(\mathbf{V}_{e,t}^{-1/2}\frac{1}{k_n}\sum_{i\in I_t^n} W(1)_i|\mathcal{F}_{i-1}) = O_P(\Delta_n\sqrt{pk_n}) = o_P(1)$, and

$$\frac{1}{k_n}\sum_{i\in I_t^n}\mathrm{Var}(\mathbf{V}_{e,t}^{-1/2}\frac{1}{\sqrt{p}}\sum_{m=1}^{p}h_{t,ml}\mathbf{e}_{m,i}|\mathcal{F}_{i-1}) \stackrel{P}{\longrightarrow} \mathbf{I}_K.$$

Also by Assumption 4.4, $\max_{t\in[0,T],s\in[0,\Delta_n]}\mathbb{E}(\|\frac{1}{\sqrt{p}}\sum_{m=1}^{p}h_{t,ml}\boldsymbol{\xi}_{mt,s}\|^4|\mathcal{F}_t) = O_P(1)$, where

$$\boldsymbol{\xi}_{mt,s} := \mathbf{A}_{mt}(\mathbf{c}_{z,mt})\nabla_c\Psi_m(\mathbf{c}_{z,mt})\,\mathrm{vec}\left(\frac{1}{s}(\mathbf{Z}_{m,t+s}-\mathbf{Z}_{mt})(\mathbf{Z}_{m,t+s}-\mathbf{Z}_{mt})' - \mathbf{c}_{z,mt}\right).$$

Recall that $\mathbf{e}_{m,i}$ is equivalent to $\boldsymbol{\xi}_{mt,s}$ with $t = (i-1)\Delta_n$ and $s = \Delta_n$. It then implies $\frac{1}{k_n}\sum_{i\in I_t^n}\mathbb{E}(\|\frac{1}{\sqrt{p}}\sum_{m=1}^{p}h_{t,ml}\mathbf{e}_{m,i}\|^4|\mathcal{F}_{i-1}) = O_P(1)$.

Next, we need to show, for an arbitrary continuous martingale M on the same filtered probability space,

$$\sqrt{pk_n}\sum_{i\in I_t}\mathbb{E}(W(1)_i\,\Delta_i^n M \mid \mathcal{F}_{i-1}^n) = o_P(1).$$

In fact for $a_{mi} = h_{t,ml}\mathbf{A}_{m,i-1}\nabla_c\Psi_m(\mathbf{c}_{z,m,i-1})$, the left hand side equals, by the Itô's formula, regardless of whether the brownian motion components of M and $\mathbf{Z}_m$ are correlated,

$$\begin{aligned}&\sqrt{pk_n}\,\frac{1}{p}\sum_{m=1}^{p}\sum_{i\in I_t}a_{mi}\mathbb{E}\left(\mathrm{vec}(\frac{1}{\Delta_n}\Delta_i^n\mathbf{Z}_m\Delta_i^n\mathbf{Z}_m' - \mathbf{c}_{z,m,(i-1)\Delta_n})\Delta_i^n M \mid \mathcal{F}_{i-1}^n\right)\\ \leq\;& O_P(\sqrt{pk_n\Delta_n}) = o_P(1).\end{aligned}$$

So all conditions of Theorem 2.2.13 and 2.2.15 in Jacod and Protter (2011) are satisfied. Thus we have

$$\sqrt{k_n p}\mathbf{V}_{e,t}^{-1/2}\frac{1}{k_n}\sum_{i\in I_t^n}W(1)_i \stackrel{\mathcal{L}\text{-}s}{\to} N(0,\mathbf{I}_K).$$

$\square$

The following lemma is needed to prove that the second term for $\mathbf{V}^*_{u,t}$ is negligible, so that $\mathbf{V}^*_{u,t}$ "mimics" $\mathbf{V}_{u,t}$.

Lemma D.4.

$$\frac{1}{p}\sum_{m=1}^{p}\frac{1}{k_n}\sum_{i\neq j\in I_t^n}\mathbf{e}^0_{m,i}\mathbf{e}^{0\prime}_{m,j}h^2_{t,ml}=o_P(1)$$

Proof. We assume $\dim(\mathbf{e}^0_{m,i})=1$ for simplicity.

$$\begin{aligned}&\mathbb{E}\Big\|\frac{1}{p}\sum_{m=1}^{p}\frac{1}{k_n}\sum_{i\neq j\in I_t^n}\mathbf{e}^0_{m,i}\mathbf{e}^0_{m,j}h^2_{t,ml}\Big\|^2\\ =&\frac{1}{p^2k_n^2}\sum_{m,m'=1}^{p}\sum_{\substack{i,j\in I_t^n\\ i\neq j}}\sum_{\substack{i',j'\in I_t^n\\ i'\neq j'}}\mathbb{E}\big(\mathbf{e}^0_{m,i}\mathbf{e}^0_{m,j}\mathbf{e}^0_{m',i'}\mathbf{e}^0_{m',j'}h^2_{t,ml}h^2_{t,m'l}\big).\end{aligned}$$

The main argument we shall use is that for $i\neq j$, $\mathbf{e}^0_{m,i}$ and $\mathbf{e}^0_{m,j}$ are mutually independent not matter $m=m'$ or not. Moreover, all such standardized increments of $\mathbf{e}^0_{m,i}$ are independent of the filtration $\mathcal{F}^n_{i-1}$. Hence, their $\mathcal{F}^n_{i-1}$-conditional expectation is zero. Lastly, according to the localization argument, their $\mathcal{F}^n_{i-1}$-conditional variance is bounded over the time interval $[0,T]$. Now let us proceed this argument in different cases. Note that i,j,i',j' may take 2, 3, or 4 different values. In view the above properties of $\mathbf{e}^0_{m,i}$'s, when they take 4 different values, then it is relatively easy to see that the above expectation is zero, regardless of $m=m'$ or not. When they take 3 different values, for example $j=i'$. Then in the summand, we have $\mathbf{e}^0_{m,i}\mathbf{e}^0_{m,j}\mathbf{e}^0_{m',j}\mathbf{e}^0_{m',j'}$. Then no matter m and m' are the same or not, one can show the above expectation is zero too. Same conclusion holds for other possibilities. When they take 2 different values, for example $i=i'$ and $j=j'$, the term inside the expectation symbol becomes (note that $i\neq j$)

$$\mathbf{e}^0_{m,i}\mathbf{e}^0_{m',i}\mathbf{e}^0_{m,j}\mathbf{e}^0_{m',j}h^2_{t,ml}h^2_{t,m'l}$$

The expectation of the above term is only non-zero when $m=m'$. In this case, we have

$$\mathbb{E}(\mathbf{e}^{02}_{m,i}\mathbf{e}^{02}_{m,j}h^2_{j-1,ml}h^2_{i-1,ml})\leq L,$$

where L is some finite number that may change value from line to line.

The above discussion readily yields that

$$\frac{1}{p^2k_n^2}\sum_{m,m'=1}^{p}\sum_{\substack{i,j\in I_t^n\\ i\neq j}}\sum_{\substack{i',j'\in I_t^n\\ i'\neq j'}}\mathbb{E}\big(\mathbf{e}_{m,i}^0\mathbf{e}_{m,j}^0 h_{j-1,ml}h_{i-1,ml}\mathbf{e}_{m',i'}^0\mathbf{e}_{m',j'}^0 h_{j'-1,m'l}h_{i'-1,m'l}\big)$$

$$=\frac{2}{p^2k_n^2}\sum_{m=1}^{p}\sum_{\substack{i,j\in I_t^n\\ i\neq j}}\mathbb{E}(\mathbf{e}_{m,i}^{02}\mathbf{e}_{m,j}^{02}h_{j-1,ml}^2h_{i-1,ml}^2)\leq L\,\frac{1}{p^2k_n^4}\,p\,k_n^2=\frac{L}{p}=o(1).$$

Since both the mean and the variance of the above sum is zero, it converges to zero in probability. This completes the proof.

□

E Proofs for Section 3

Recall that

$$\Psi_m(\boldsymbol{\beta}_{mt},\mathbf{c}_{z,mt})=\mathbf{c}_{FF,t}\boldsymbol{\beta}_{mt}-\mathbf{c}_{YF,mt}.$$

We aim to verify Assumptions 4.2 - 4.4, 4.5, 4.6. Once this is done, then Theorem 3.1 immediately follows from Theorems 4.1; the known factor case of Theorem 3.4 then follows from Theorem 4.2. Because $\Psi_l(\boldsymbol{\beta}_{mt},\mathbf{c}_{z,mt})$ does not depend on $\mathbf{c}_{YY,lt}$, the quadratic variation of $\mathbf{Y}_l$, thus we simply write $\mathbf{c}_{z,mt}=(\mathbf{c}_{FF,t},\mathbf{c}_{YF,mt})$, with abuse of notation. In addition, for a generic $\mathbf{c}$, write $\mathbf{c}=(\mathbf{c}_{ff},\mathbf{c}_{yf})$, corresponding to $\mathbf{c}_{z,mt}=(\mathbf{c}_{FF,t},\mathbf{c}_{YF,mt})$ when it is evaluate at $\mathbf{c}_{z,mt}$. Furthermore, because $\boldsymbol{\beta}_{mt}$ is exactly identified by $\Psi_m(\boldsymbol{\beta}_{mt},\mathbf{c}_{z,mt})=0$, the first-order condition does not depend on $\boldsymbol{\Omega}_{mt}$ and we can take $\boldsymbol{\Omega}_{mt}=\mathbf{I}_K$. In fact (D.4) in this case can be written as

$$\widehat{\mathbf{g}}_{lt}-\mathbf{g}_{lt}=\boldsymbol{\Gamma}_t'\mathbf{P}_{t,l}+\mathbf{c}_{FF,t}^{-1}\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{U}'\mathbf{P}_{t,l}+o_P((k_np)^{-1/2}). \tag{E.1}$$

E.1 Proofs of Theorem 3.1 and Theorem 3.4(known factors)

Proof. Verify Assumption 4.2. Also, $\nabla_\beta\Psi_m(\boldsymbol{\beta}_{mt},\mathbf{c})=\mathbf{c}_{ff}$, and $\nabla_c\Psi_m(\boldsymbol{\beta}_{mt},\mathbf{c})=(\boldsymbol{\beta}_{mt},-\mathbf{I})$. Then for (i) $\Psi_m(.,.)$ is indeed twice continuously differentiable. For (ii), we have

$$\max_{m\leq p}\|(\nabla_\beta\Psi_m(\mathbf{c}_{z,mt})'\boldsymbol{\Omega}_{mt}\nabla_\beta\Psi_m(\mathbf{c}_{z,mt}))^{-1}\|=\|\mathbf{c}_{FF,t}^{-2}\|<C$$

with the assumption that $\lambda_{\min}(\mathbf{c}_{FF,t}) > c_0$.

Verify Assumption 4.3. We have $\mathbf{A}_{mt}(\mathbf{c}) = \mathbf{c}_{ff}^{-1}$. In addition, write $x_{li} = \Delta_i^n x_l/\sqrt{\Delta_n}$. Then $y_{li} = \boldsymbol{\beta}_{l,i-1}\mathbf{f}_i + u_{li} + \psi_{li}$, where $\psi_{li} = \frac{1}{\sqrt{\Delta_n}}\int_{(i-1)\Delta_n}^{i\Delta_n} \alpha_{ls} ds + \frac{1}{\sqrt{\Delta_n}}\int_{(i-1)\Delta_n}^{i\Delta_n} (\boldsymbol{\beta}_{ls} - \boldsymbol{\beta}_{l,(i-1)\Delta_n})' d\mathbf{F}_s$.

We have

$$\widehat{\mathbf{c}}_{YF,mt} = \frac{1}{k_n}\sum_{i\in I_t^n} \mathbf{f}_i y_{li} = \frac{1}{k_n}\sum_{i\in I_t^n} \mathbf{f}_i\mathbf{f}_i'\boldsymbol{\beta}_{m,i-1} + \frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{f}_i u_{mi} + \frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{f}_i\psi_{mi}.$$

Therefore,

$$\begin{aligned}
& \nabla_c\Psi_m(\mathbf{c}_{z,mt})\,\mathrm{vec}(\widehat{\mathbf{c}}_{z,mt} - \mathbf{c}_{z,mt}) = (\widehat{\mathbf{c}}_{FF,t} - \mathbf{c}_{FF,t})\boldsymbol{\beta}_{mt} - (\widehat{\mathbf{c}}_{YF,mt} - \mathbf{c}_{YF,mt}) \\
= \; & \widehat{\mathbf{c}}_{FF,t}\boldsymbol{\beta}_{mt} - \widehat{\mathbf{c}}_{YF,mt} = -\frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{f}_i u_{mi} + \frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{f}_i\mathbf{f}_i'(\boldsymbol{\beta}_{mt} - \boldsymbol{\beta}_{m,i-1}) - \frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{f}_i\psi_{mi}.
\end{aligned}$$

Condition (i) holds since $\Psi_m(\widehat{\mathbf{c}}_{z,mt}) - \Psi_m(\mathbf{c}_{z,mt}) = (\widehat{\mathbf{c}}_{FF,t} - \mathbf{c}_{FF,t})\boldsymbol{\beta}_{mt} - (\widehat{\mathbf{c}}_{YF,mt} - \mathbf{c}_{YF,mt})$ as well. As for condition (ii), we have

$$\begin{aligned}
& \frac{1}{p}\sum_{m=1}^p h_{t,ml}[\mathbf{A}_{mt}(\widehat{\mathbf{c}}_{z,mt}) - \mathbf{A}_{mt}(\mathbf{c}_{z,mt})]\nabla_c\Psi_m(\mathbf{c}_{z,mt})\,\mathrm{vec}(\widehat{\mathbf{c}}_{z,mt} - \mathbf{c}_{z,mt}) := -(b_1 + b_2 + b_3), \\
b_1 \; := \; & \frac{1}{p}\sum_{m=1}^p h_{t,ml}[\widehat{\mathbf{c}}_{FF,t}^{-1} - \mathbf{c}_{FF,t}^{-1}]\frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{f}_i u_{mi} \\
b_2 \; := \; & \frac{1}{p}\sum_{m=1}^p h_{t,ml}[\widehat{\mathbf{c}}_{FF,t}^{-1} - \mathbf{c}_{FF,t}^{-1}]\frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{f}_i\mathbf{f}_i'(\boldsymbol{\beta}_{mt} - \boldsymbol{\beta}_{m,i-1}) \\
b_3 \; := \; & \frac{1}{p}\sum_{m=1}^p h_{t,ml}[\widehat{\mathbf{c}}_{FF,t}^{-1} - \mathbf{c}_{FF,t}^{-1}]\frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{f}_i\psi_{mi}.
\end{aligned} \tag{E.2}$$

We now prove each term is $o_P((k_n p)^{-1/2})$. First note that $\|\widehat{\mathbf{c}}_{FF,t}^{-1} - \mathbf{c}_{FF,t}^{-1}\| = O_P(k_n^{-1/2})$. For each $r \le K$, let f_{ir} be the r th component of $\mathbf{f}_i$; let $\mathbf{u}_i = (u_{1i}, ..., u_{pi})'$ and $\mathbf{h}_{t,l} = (h_{t,1l}, ..., h_{t,pl})'$.

$$\mathbb{E}_{X_t}\mathbb{E}_{\mathcal{F}_t}(\frac{1}{p}\sum_{m=1}^p\frac{1}{k_n}\sum_{i\in I_t^n} f_{ir}u_{mi}h_{t,ml})^2 = \frac{1}{p^2}\mathbb{E}_{X_t}\mathbf{h}_{t,l}'\frac{1}{k_n^2}\sum_{i\in I_t^n}\mathbb{E}f_{ir}^2\mathbb{E}(\mathbf{u}_i\mathbf{u}_i')\mathbf{h}_{t,l} \le O_P(\frac{1}{pk_n})\frac{1}{p}\sum_m h_{t,ml}^2 \tag{E.3}$$

So $b_1 = o_P((k_n p)^{-1/2})$.

On the other hand, for all $j, s \in I_t^n$, write $\boldsymbol{\beta}_{m,j-1} - \boldsymbol{\beta}_{m,s-1} = \mathbf{M}_{m,j-1,s-1}^{BM} + \mathbf{M}_{m,j-1,s-1}^{df}$, which are the Brownian motion and drift decomposition; see Chapter 2 of Jacod and Protter

(2011) for detailed definitions of each component. Then by the Burkholder-Davis-Grundy inequality (Chapter 2 of Jacod and Protter (2011)), $\max_{m\leq p}\mathbb{E}\max_{s,j\in I_t^n}\|\mathbf{M}^{BM}_{m,j-1,s-1}\|^4 = O(k_n^2\Delta_n^2)$, and $\max_{m\leq p}\mathbb{E}\max_{s,j\in I_t^n}\|\mathbf{M}^{df}_{m,j-1,s-1}\|^4 = O(k_n^4\Delta_n^4)$, so

$$\max_{m\leq p}\mathbb{E}\max_{s,j\in I_t^n}\|\boldsymbol{\beta}_{m,j-1}-\boldsymbol{\beta}_{m,s-1}\|^4 = O(k_n^2\Delta_n^2).$$

Then

$$\begin{aligned}\|b_2\| &\leq O_P(\frac{1}{\sqrt{k_n}})(\frac{1}{p}\sum_{m=1}^p\frac{1}{k_n}\sum_{i\in I_t^n}\mathbb{E}\|\boldsymbol{\beta}_{mt}-\boldsymbol{\beta}_{m,i-1}\|^2)^{1/2} = O_P(\sqrt{\Delta_n}) = o_P((k_np)^{-1/2})\\ \|b_3\| &\leq O_P(\frac{\sqrt{\Delta_n}}{\sqrt{k_n}}) + O_P(k_n^{-1/2})(\frac{1}{p}\sum_m\mathbb{E}\max_{s,j\in I_t^n}\|\boldsymbol{\beta}_{ms}-\boldsymbol{\beta}_{mj}\|^2)^{-1/2} = o_P((k_np)^{-1/2})\end{aligned}$$

with the assumption $k_np\Delta_n = o(1)$.

Verify Assumption 4.4. We only need to verify condition (ii). It is straightforward to verify:

$$\begin{aligned}\boldsymbol{\xi}_{mt,s} &= \boldsymbol{\xi}^0_{mt,s}+\widetilde{\boldsymbol{\delta}}_{mt,s},\\ \boldsymbol{\xi}^0_{mt,s} &:= -\mathbf{c}^{-1}_{FF,t}\frac{1}{s}(\mathbf{F}_{t+s}-\mathbf{F}_t)(U_{m,t+s}-U_{mt})\\ \widetilde{\boldsymbol{\delta}}_{mt,s} &:= -\mathbf{c}^{-1}_{FF,t}\frac{1}{s}(\mathbf{F}_{t+s}-\mathbf{F}_t)\int_t^{t+s}\alpha_{mr}dr + \mathbf{c}^{-1}_{FF,t}(\mathbf{F}_{t+s}-\mathbf{F}_s)\int_t^{t+s}(\boldsymbol{\beta}_{mr}-\boldsymbol{\beta}_{mt})'d\mathbf{F}_r\end{aligned}\tag{E.4}$$

Then

$$\begin{aligned}&\max_{s\in[0,\Delta_n]}\mathbb{E}(\|\frac{1}{\sqrt{p}}\sum_{m=1}^p h_{t,ml}\boldsymbol{\xi}_{mt,s}\|^4|\mathcal{F}_t)\\ \leq\ & O(1)\max_{s\in[0,\Delta_n]}\mathbb{E}(\|\frac{1}{\sqrt{p}}\sum_{m=1}^p h_{t,ml}\boldsymbol{\xi}^0_{mt,s}\|^4|\mathcal{F}_t) + O(p^2)\max_{s\in[0,\Delta_n]}\mathbb{E}(\|\frac{1}{p}\sum_{m=1}^p h_{t,ml}\widetilde{\boldsymbol{\delta}}_{mt,s}\|^4|\mathcal{F}_t).\end{aligned}$$

The first term is $O_P(1)$ due to Assumption 3.4; the second term is $O_P(1)$ due to

$$\max_{s\in[0,\Delta_n]}\frac{1}{p}\sum_{m\leq p}\mathbb{E}(\|\widetilde{\boldsymbol{\delta}}_{mt,s}\|^4|\mathcal{F}_t) = O_P((pk_n)^{-2}),$$

to be proved in (E.5) below.

Verify Assumption 4.5. Let $\mathbb{E}^*$ denote the conditional mean with respect to the bootstrap resampling scheme, given the original data. Condition (i) follows immediately because

$\Psi_m^*(\widehat{\mathbf{c}}_{z,mt}^*) - \Psi_m^*(\mathbf{c}_{z,mt}^*) - \nabla_c \Psi_m^*(\mathbf{c}_{z,mt}^*) \operatorname{vec}(\widehat{\mathbf{c}}_{z,mt}^* - \mathbf{c}_{z,mt}^*) = 0$. As for condition (ii), similar to (E.2)

$$
\begin{aligned}
&\frac{1}{p}\sum_{m=1}^{p} h_{t,ml}^*[\mathbf{A}_{mt}^*(\widehat{\mathbf{c}}_{z,mt}^*) - \mathbf{A}_{mt}^*(\mathbf{c}_{z,mt}^*)]\nabla_c \Psi_m^*(\mathbf{c}_{z,mt}^*) \operatorname{vec}(\widehat{\mathbf{c}}_{z,mt}^* - \mathbf{c}_{z,mt}^*) := -(b_1^* + b_2^* + b_3^*),\\
b_1^* \;&:= \; \frac{1}{p}\sum_{m=1}^{p} h_{t,ml}^*[\widehat{\mathbf{c}}_{FF,t}^{-1} - \mathbf{c}_{FF,t}^{-1}]\frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{f}_i u_{mi}^*\\
b_2^* \;&:= \; \frac{1}{p}\sum_{m=1}^{p} h_{t,ml}^*[\widehat{\mathbf{c}}_{FF,t}^{-1} - \mathbf{c}_{FF,t}^{-1}]\frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{f}_i \mathbf{f}_i'(\boldsymbol{\beta}_{mt}^* - \boldsymbol{\beta}_{m,i-1}^*)\\
b_3^* \;&:= \; \frac{1}{p}\sum_{m=1}^{p} h_{t,ml}^*[\widehat{\mathbf{c}}_{FF,t}^{-1} - \mathbf{c}_{FF,t}^{-1}]\frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{f}_i \psi_{mi}^*.
\end{aligned}
$$

where $\psi_{mi}^* = \frac{1}{\sqrt{\Delta_n}}\int_{(i-1)\Delta_n}^{i\Delta_n}\alpha_{ms}^* ds + \frac{1}{\sqrt{\Delta_n}}\int_{(i-1)\Delta_n}^{i\Delta_n}(\boldsymbol{\beta}_{ms}^* - \boldsymbol{\beta}_{m,(i-1)\Delta_n}^*)' d\mathbf{F}_s$. First of all,

$$
\begin{aligned}
\frac{1}{p}\sum_m h_{t,ml}^{*2} \;&=\; \frac{1}{p}\sum_m \|\boldsymbol{\phi}_{mt}^*\|^2 \|(\frac{1}{p}\boldsymbol{\Phi}_t^{*'}\boldsymbol{\Phi}_t^*)^{-1}\boldsymbol{\phi}_{lt}\|^2 \le O_{P^*}(1) h_{t,ll}\frac{1}{p}\sum_m \mathbb{E}^*\|\boldsymbol{\phi}_{mt}^*\|^2\\
&\le\; O_{P^*}(1) h_{t,ll}\frac{1}{p}\sum_m h_{t,mm} = O_{P^*}(1).
\end{aligned}
$$

Next, as for b_1^*, first note that for each fixed $r \le K$,

$$
\frac{1}{p}\sum_{m=1}^{p}\frac{1}{k_n}\sum_{i\in I_t^n} f_{ir}\boldsymbol{\phi}_{mt}^* u_{mi}^* = \frac{1}{p}\sum_{m=1}^{p}\frac{1}{k_n}\sum_{i\in I_t^n} f_{ir}\boldsymbol{\phi}_{mt} u_{mi} + \frac{1}{p}\sum_{m=1}^{p}\frac{1}{k_n}\sum_{i\in I_t^n} f_{ir}(\boldsymbol{\phi}_{mt}^* u_{mi}^* - \boldsymbol{\phi}_{mt} u_{mi}).
$$

On one hand, we have $\mathbb{E}_{X_t}\mathbb{E}_{\mathcal{F}_t}\|\frac{1}{p}\sum_{m=1}^{p}\frac{1}{k_n}\sum_{i\in I_t^n} f_{ik}\boldsymbol{\phi}_{mt} u_{mi}\|^2 = \sum_{j=1}^{J}\frac{1}{p^2 k_n}\|\boldsymbol{\Phi}_t\|_F^2 \le O_P(\frac{J}{pk_n})$. On the other hand, we have $\mathbb{E}^*\frac{1}{p}\sum_{m=1}^{p}\frac{1}{k_n}\sum_{i\in I_t^n} f_{ir}(\boldsymbol{\phi}_{mt}^* u_{mi}^* - \boldsymbol{\phi}_{mt} u_{mi}) = 0$, and

$$
\begin{aligned}
&\|\operatorname{Var}^*(\frac{1}{p}\sum_{m=1}^{p}\frac{1}{k_n}\sum_{i\in I_t^n} f_{ir}\boldsymbol{\phi}_{mt}^* u_{mi}^*)\| = O_{P^*}(\frac{1}{p})\frac{1}{p}\sum_{m=1}^{p}\|\frac{1}{k_n}\sum_{i\in I_t^n} f_{ir}\boldsymbol{\phi}_{mt} u_{mi}\|^2\\
\le\;& O_{P^*}(\frac{1}{pk_n})\frac{1}{p}\sum_{m=1}^{p}\frac{1}{k_n}\sum_{i\in I_t^n}\operatorname{Var}(f_{ir}u_{mi})\|\boldsymbol{\phi}_{mt}\|^2 = O_{P^*}(\frac{1}{pk_n})\frac{1}{p}\sum_{m=1}^{p} h_{t,mm} = O_{P^*}(\frac{1}{pk_n}).
\end{aligned}
$$

Hence $\frac{1}{p}\sum_{m=1}^{p}\frac{1}{k_n}\sum_{i\in I_t^n}\boldsymbol{\phi}_{mt}^*\mathbf{f}_i u_{mi}^* = O_{P^*}(\sqrt{\frac{J}{pk_n}})$. Hence together with (E.3),

$$\begin{aligned}
\frac{1}{p}\sum_{m=1}^{p}\frac{1}{k_n}\sum_{i\in I_t^n}\boldsymbol{\phi}_{mt}^*\mathbf{f}_i u_{mi}^* &= O_{P^*}(\sqrt{\frac{J}{pk_n}}) \\
\frac{1}{p}\sum_{m=1}^{p}\frac{1}{k_n}\sum_{i\in I_t^n} f_{ir}(\boldsymbol{\phi}_{mt}^* u_{mi}^* - \boldsymbol{\phi}_{mt}u_{mi}) &= O_{P^*}(\frac{1}{\sqrt{pk_n}}) \\
\frac{1}{p}\sum_{m=1}^{p} h_{t,ml}\frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{f}_i u_{mi} &= O_{P^*}(\frac{1}{\sqrt{pk_n}}).
\end{aligned}$$

This implies, when $J = O(p)$,

$$\begin{aligned}
&\frac{1}{p}\sum_{m=1}^{p} h_{t,ml}^*\frac{1}{k_n}\sum_{i\in I_t^n} f_{ir}u_{mi}^* = \boldsymbol{\phi}_{lt}'[(\frac{1}{p}\boldsymbol{\Phi}_t^{*'}\boldsymbol{\Phi}_t^*)^{-1} - (\frac{1}{p}\boldsymbol{\Phi}_t'\boldsymbol{\Phi}_t)^{-1}]\frac{1}{p}\sum_{m=1}^{p}\boldsymbol{\phi}_{mt}^*\frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{f}_i u_{mi}^* \\
&+\boldsymbol{\phi}_{lt}'(\frac{1}{p}\boldsymbol{\Phi}_t'\boldsymbol{\Phi}_t)^{-1}\frac{1}{p}\sum_{m=1}^{p}\frac{1}{k_n}\sum_{i\in I_t^n}[\boldsymbol{\phi}_{mt}^*\mathbf{f}_i u_{mi}^* - \boldsymbol{\phi}_{mt}\mathbf{f}_i u_{mi}] + \frac{1}{p}\sum_{m=1}^{p} h_{t,ml}\frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{f}_i u_{mi} \\
= \;& O_{P^*}(\frac{1}{\sqrt{pk_n}}).
\end{aligned}$$

This implies $b_1^* = o_{P^*}((k_n p)^{-1/2})$. As for b_2^*, b_3^*, similar for the bound for b_2, b_3 of (E.2), we have

$$\begin{aligned}
\|b_2^*\| &\leq O_{P^*}(\frac{1}{\sqrt{k_n}})(\frac{1}{p}\sum_{m=1}^{p}\frac{1}{k_n}\sum_{i\in I_t^n}\mathbb{E}^*\|\boldsymbol{\beta}_{mt}^* - \boldsymbol{\beta}_{m,i-1}^*\|^2)^{1/2} = O_{P^*}(\sqrt{\Delta_n}) = o_{P^*}((k_n p)^{-1/2}) \\
\|b_3^*\| &\leq O_{P^*}(\frac{\sqrt{\Delta_n}}{\sqrt{k_n}}) = o_{P^*}((k_n p)^{-1/2}).
\end{aligned}$$

Verify Assumption 4.6.

First, we have $\boldsymbol{\xi}_{mt,s} = \boldsymbol{\xi}_{mt,s}^0 + \widetilde{\boldsymbol{\delta}}_{mt,s}$, where $\boldsymbol{\xi}_{mt,s}^0, \widetilde{\boldsymbol{\delta}}_{mt,s}$ are defined in (E.4):

$$\begin{aligned}
\boldsymbol{\xi}_{mt,s}^0 &:= -\mathbf{c}_{FF,t}^{-1}\frac{1}{s}(\mathbf{F}_{t+s} - \mathbf{F}_t)(U_{m,t+s} - U_{mt}) \\
\widetilde{\boldsymbol{\delta}}_{mt,s} &:= -\mathbf{c}_{FF,t}^{-1}\frac{1}{s}(\mathbf{F}_{t+s} - \mathbf{F}_t)\int_t^{t+s}\alpha_{mr}dr + \mathbf{c}_{FF,t}^{-1}\frac{1}{s}(\mathbf{F}_{t+s} - \mathbf{F}_t)\int_t^{t+s}(\boldsymbol{\beta}_{mr} - \boldsymbol{\beta}_{mt})'d\mathbf{F}_r.
\end{aligned}$$

Then condition (i) follows immediately from Assumption 3.6. As for condition (ii), we now prove

$$\max_{s\in[0,\Delta_n]} \frac{1}{p}\sum_{m\le p} \mathbb{E}(\|\widetilde{\boldsymbol{\delta}}_{mt,s}\|^4|\mathcal{F}_t) = O_P((pk_n)^{-2}). \tag{E.5}$$

To do so, first note that the boundedness assumption of $\mathbf{c}_{FF}$ and α_{mt}, together with the Burkholder-Davis-Grundy inequality, imply that

$$\begin{aligned}
&\max_{s\in[0,\Delta_n]} \frac{1}{p}\sum_{m\le p} \mathbb{E}\big(\|\mathbf{c}_{FF,t}^{-1}\frac{1}{s}(\mathbf{F}_{t+s}-\mathbf{F}_t)\int_t^{t+s}\alpha_{mr}dr\|^4|\mathcal{F}_t\big)\\
\le &\max_{s\in[0,\Delta_n]} \frac{1}{p}\sum_{m\le p} L\,\mathbb{E}\big(\|(\mathbf{F}_{t+s}-\mathbf{F}_t)\|^4|\mathcal{F}_t\big) \le \max_{s\in[0,\Delta_n]} \frac{1}{p}\sum_{m\le p} Ls^2 \le L\Delta_n^2.
\end{aligned}$$

Since we assume $pk_n\Delta_n \to 0$, the above term is $o_P((pk_n)^{-2})$.

Itô's formula yields that

$$\begin{aligned}
&(\mathbf{F}_{t+s}-\mathbf{F}_t)\int_t^{t+s}(\boldsymbol{\beta}_{mr}-\boldsymbol{\beta}_{mt})'d\mathbf{F}_r = \int_t^{t+s} d\mathbf{F}_u\int_t^{t+s}(\boldsymbol{\beta}_{mr}-\boldsymbol{\beta}_{mt})'d\mathbf{F}_r\\
=&\int_t^{t+s}(\boldsymbol{\beta}_{mr}-\boldsymbol{\beta}_{mt})'\mathbf{c}_{FF,r}dr + \int_t^{t+s}(\mathbf{F}_r-\mathbf{F}_t)(\boldsymbol{\beta}_{mr}-\boldsymbol{\beta}_{mt})'d\mathbf{F}_r + \int_t^{t+s}\int_t^{t+u}(\boldsymbol{\beta}_{mr}-\boldsymbol{\beta}_{mt})'d\mathbf{F}_r d\mathbf{F}_u.
\end{aligned}$$

One can check that (by using the Burkholder-Davis-Grundy inequality)

$$\begin{aligned}
&\max_{s\in[0,\Delta_n]} \frac{1}{p}\sum_{m\le p} \mathbb{E}\big(\|\mathbf{c}_{FF,t}^{-1}\frac{1}{s}\int_t^{t+s}(\boldsymbol{\beta}_{mr}-\boldsymbol{\beta}_{mt})'\mathbf{c}_{FF,r}dr\|^4\big|\mathcal{F}_t\big)\\
\le &L\max_{s\in[0,\Delta_n]} \frac{1}{p}\sum_{m\le p} \mathbb{E}\big(\|\sup_{u\in[0,s]}|(\boldsymbol{\beta}_{mr}-\boldsymbol{\beta}_{mt})|\|^4\big|\mathcal{F}_t\big) \le \max_{s\in[0,\Delta_n]} \frac{1}{p}\sum_{m\le p} Ls^2 \le L\Delta_n^2.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\max_{s\in[0,\Delta_n]} \frac{1}{p}\sum_{m\le p} \mathbb{E}\big(\|\mathbf{c}_{FF,t}^{-1}\frac{1}{s}\int_t^{t+s}(\mathbf{F}_{t+r}-\mathbf{F}_t)(\boldsymbol{\beta}_{mr}-\boldsymbol{\beta}_{mt})'d\mathbf{F}_r\|^4\big|\mathcal{F}_t\big)\\
\le &L\max_{s\in[0,\Delta_n]} \frac{1}{p}\sum_{m\le p} \mathbb{E}\big(\|\frac{1}{s^2}\int_t^{t+s}\|\mathbf{F}_r-\mathbf{F}_t\|^2\|\boldsymbol{\beta}_{mr}-\boldsymbol{\beta}_{mt}\|^2dr\|^2\big|\mathcal{F}_t\big)\\
\le &L\max_{s\in[0,\Delta_n]} \frac{1}{p}\sum_{m\le p} \mathbb{E}\big(\|\frac{1}{s^2}\int_t^{t+s}\|\mathbf{F}_r-\mathbf{F}_t\|^2\|\boldsymbol{\beta}_{mr}-\boldsymbol{\beta}_{mt}\|^2dr\|^2\big|\mathcal{F}_t\big)
\end{aligned}$$

$$
\begin{aligned}
&\le L \max_{s\in[0,\Delta_n]} \frac{1}{p}\sum_{m\le p}\mathbb{E}\big(\frac{1}{s^2}\sup_{u\in[0,s]}\|\mathbf{F}_{t+u}-\mathbf{F}_t\|^4\|\boldsymbol{\beta}_{m,t+u}-\boldsymbol{\beta}_{mt}\|^4\big|\mathcal{F}_t\big)\\
&\le L \max_{s\in[0,\Delta_n]} \frac{1}{p}\sum_{m\le p} s^2 \le L\Delta_n^2,
\end{aligned}
$$

and

$$
\begin{aligned}
&\max_{s\in[0,\Delta_n]} \frac{1}{p}\sum_{m\le p}\mathbb{E}\big(\|\mathbf{c}_{FF,t}^{-1}\frac{1}{s}\int_t^{t+s}\int_t^{t+u}(\boldsymbol{\beta}_{mr}-\boldsymbol{\beta}_{mt})'d\mathbf{F}_r d\mathbf{F}_u\|^4\big|\mathcal{F}_t\big)\\
\le{}& L \max_{s\in[0,\Delta_n]} \frac{1}{p}\sum_{m\le p}\mathbb{E}\big(\Big\|\frac{1}{s^2}\int_t^{t+s}\|\int_t^{t+u}(\boldsymbol{\beta}_{mr}-\boldsymbol{\beta}_{mt})'d\mathbf{F}_r\Big\|^2 du\Big\|^2|\mathcal{F}_t\big)\\
\le{}& L \max_{s\in[0,\Delta_n]} \frac{1}{p}\sum_{m\le p}\mathbb{E}\big(\frac{1}{s^2}\sup_{u\in[0,s]}\|\int_t^{t+u}(\boldsymbol{\beta}_{mr}-\boldsymbol{\beta}_{mt})'d\mathbf{F}_r\Big\|^4|\mathcal{F}_t\big)\\
\le{}& L \max_{s\in[0,\Delta_n]} \frac{1}{p}\sum_{m\le p}\mathbb{E}\big(\frac{1}{s^2}\|\int_t^{t+s}\big\|\boldsymbol{\beta}_{mr}-\boldsymbol{\beta}_{mt}\|^2 dr\big\|^2|\mathcal{F}_t\big) \le L \max_{s\in[0,\Delta_n]} \frac{1}{p}\sum_{m\le p} s^2 \le L\Delta_n^2.
\end{aligned}
$$

Then the desired results readily follows.

Finally, we show

$$
\lim_{s\to 0}\left\|\mathrm{Var}\left(\frac{1}{\sqrt{p}}\sum_{m=1}^{p} h_{t,ml}\boldsymbol{\xi}_{mt,s}\middle|\mathcal{F}_t\right)-\mathrm{Var}\left(\frac{1}{\sqrt{p}}\sum_{m=1}^{p} h_{t,ml}\boldsymbol{\xi}^0_{mt,s}\middle|\mathcal{F}_t\right)\right\| = o_P(1).
$$

In fact this is implied by (E.5) that $p\lim_{s\to 0}\mathbb{E}(\frac{1}{p}\sum_{m=1}^{p}\|\widetilde{\boldsymbol{\delta}}_{mt,s}\|^2|\mathcal{F}_t) = o_P(1)$.

Since Assumptions 4.2 - 4.4 are verified, Theorem 3.1 immediately follows from Theorems 4.1. In addition, since Assumptions 4.5, 4.6 are verified, the known factor case of Theorem 3.4 then follows from Theorem 4.2.

□

E.2 Proofs of Theorems 3.2-3.4 (unknown factors)

The proofs below cite several technical lemmas as intermediate steps, which are presented at Section E.4.

E.2.1 Estimating the latent factors

Here we derive the expansion of the estimated latent factors. Let $\widehat{\mathbf{V}}_t$ be the $K \times K$ diagonal matrix, whose diagonal elements are the first K eigenvalues of $\frac{1}{pk_n\Delta_n}(\mathbf{P}\Delta^n\mathbf{Y})_t'(\mathbf{P}\Delta\mathbf{Y})_t$. Then by definition of eigenvectors/eigenvalues,

$$\frac{1}{pk_n\Delta_n}(\mathbf{P}\Delta^n\mathbf{Y})_t'(\mathbf{P}\Delta\mathbf{Y})_t\widehat{\Delta^n\mathbf{F}} = \widehat{\Delta^n\mathbf{F}}\widehat{\mathbf{V}}_t \tag{E.6}$$

Take a fixed row of the above equality, we have, for each fixed $i \in I_t^n$,

$$\frac{1}{pk_n\Delta_n}\Delta_i^n\mathbf{Y}'\mathbf{P}_{i-1}\sum_{j\in I_t^n}\mathbf{P}_{j-1}\Delta_j^n\mathbf{Y}\widehat{\Delta_j^n\mathbf{F}}' = \widehat{\Delta_i^n\mathbf{F}}'\widehat{\mathbf{V}}_t.$$

Substitute in $\Delta_j^n\mathbf{Y} = \boldsymbol{\beta}_{j-1}\Delta_j^n\mathbf{F} + \Delta_j^n\mathbf{U} + \Delta_n\boldsymbol{\psi}_j$, where

$$\boldsymbol{\psi}_j = \frac{1}{\Delta_n}\int_{(j-1)\Delta_n}^{j\Delta_n}\boldsymbol{\alpha}_s ds + \frac{1}{\Delta_n}\int_{(j-1)\Delta_n}^{j\Delta_n}(\boldsymbol{\beta}_s - \boldsymbol{\beta}_{(j-1)\Delta_n})d\mathbf{F}_s,$$

then

$$\begin{aligned}
\widehat{\Delta_i^n\mathbf{F}}'\widehat{\mathbf{V}}_t &= \frac{1}{pk_n}\boldsymbol{\psi}_i'\mathbf{P}_{i-1}\sum_{j\in I_t^n}\mathbf{P}_{j-1}\boldsymbol{\psi}_j\Delta_n\widehat{\Delta_j^n\mathbf{F}}' + \frac{1}{pk_n}\boldsymbol{\psi}_i'\mathbf{P}_{i-1}\sum_{j\in I_t^n}\mathbf{P}_{j-1}\boldsymbol{\beta}_{j-1}\Delta_j^n\mathbf{F}\widehat{\Delta_j^n\mathbf{F}}' \\
&+\frac{1}{pk_n}\boldsymbol{\psi}_i'\mathbf{P}_{i-1}\sum_{j\in I_t^n}\mathbf{P}_{j-1}\Delta_j^n\mathbf{U}\widehat{\Delta_j^n\mathbf{F}}' + \frac{1}{pk_n\Delta_n}\Delta_i^n\mathbf{F}'\boldsymbol{\beta}_{i-1}'\mathbf{P}_{i-1}\sum_{j\in I_t^n}\mathbf{P}_{j-1}\boldsymbol{\psi}_j\Delta_n\widehat{\Delta_j^n\mathbf{F}}' \\
&+\frac{1}{pk_n\Delta_n}\Delta_i^n\mathbf{F}'\boldsymbol{\beta}_{i-1}'\mathbf{P}_{i-1}\sum_{j\in I_t^n}\mathbf{P}_{j-1}\boldsymbol{\beta}_{j-1}\Delta_j^n\mathbf{F}\widehat{\Delta_j^n\mathbf{F}}' \\
&+\frac{1}{pk_n\Delta_n}\Delta_i^n\mathbf{F}'\boldsymbol{\beta}_{i-1}'\mathbf{P}_{i-1}\sum_{j\in I_t^n}\mathbf{P}_{j-1}\Delta_j^n\mathbf{U}\widehat{\Delta_j^n\mathbf{F}}' + \frac{1}{pk_n\Delta_n}\Delta_i^n\mathbf{U}'\mathbf{P}_{i-1}\sum_{j\in I_t^n}\mathbf{P}_{j-1}\boldsymbol{\psi}_j\Delta_n\widehat{\Delta_j^n\mathbf{F}}' \\
&+\frac{1}{pk_n\Delta_n}\Delta_i^n\mathbf{U}'\mathbf{P}_{i-1}\sum_{j\in I_t^n}\mathbf{P}_{j-1}\boldsymbol{\beta}_{j-1}\Delta_j^n\mathbf{F}\widehat{\Delta_j^n\mathbf{F}}' + \frac{1}{pk_n\Delta_n}\Delta_i^n\mathbf{U}'\mathbf{P}_{i-1}\sum_{j\in I_t^n}\mathbf{P}_{j-1}\Delta_j^n\mathbf{U}\widehat{\Delta_j^n\mathbf{F}}'.
\end{aligned}$$

Right multiplying by $\widehat{\mathbf{V}}_t^{-1}$ and rearranging the terms on the right hand side yields:

$$\widehat{\Delta_i^n\mathbf{F}} - \mathbf{H}_{nt}\Delta_i^n\mathbf{F} = \boldsymbol{\Xi}_{3,t}'\left(\frac{1}{\sqrt{p}}\Delta_n\mathbf{P}_{i-1}\boldsymbol{\psi}_i + \frac{1}{\sqrt{p}}\mathbf{P}_{i-1}\Delta_i^n\mathbf{U}\right) + \boldsymbol{\Xi}_{3,t}'\frac{1}{\sqrt{p}}(\mathbf{P}_{i-1}\boldsymbol{\beta}_{i-1} - \mathbf{P}_t\boldsymbol{\beta}_t)\Delta_i^n\mathbf{F}, \tag{E.7}$$

where

$$
\begin{aligned}
\boldsymbol{\Xi}_{1,t} &= \frac{1}{\sqrt{p}k_n\Delta_n}\sum_{j\in I_t^n}\mathbf{P}_{j-1}\boldsymbol{\psi}_j\Delta_n\widehat{\Delta_j^n\mathbf{F}}'\widehat{\mathbf{V}}_t^{-1} + \frac{1}{\sqrt{p}k_n\Delta_n}\sum_{j\in I_t^n}\mathbf{P}_{j-1}\Delta_j^n\mathbf{U}\widehat{\Delta_j^n\mathbf{F}}'\widehat{\mathbf{V}}_t^{-1} \\
\boldsymbol{\Xi}_{2,t} &= \frac{1}{\sqrt{p}k_n\Delta_n}\sum_{j\in I_t^n}\mathbf{P}_{j-1}\boldsymbol{\beta}_{j-1}\Delta_j^n\mathbf{F}\widehat{\Delta_j^n\mathbf{F}}'\widehat{\mathbf{V}}_t^{-1} \\
\boldsymbol{\Xi}_{3,t} &= \boldsymbol{\Xi}_{1,t} + \boldsymbol{\Xi}_{2,t}, \qquad \mathbf{H}_{nt}' = \frac{1}{\sqrt{p}}\boldsymbol{\beta}_t'\mathbf{P}_t\boldsymbol{\Xi}_{3,t}.
\end{aligned}
$$

We shall apply the above equality when deriving the asymptotic distribution of $\widehat{\mathbf{g}}_{lt}$.

E.2.2 Proof of Theorem 3.2: limiting distribution

Proof. Define $\boldsymbol{\Upsilon}_{nt} = \frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\widehat{\Delta_i^n\mathbf{F}}\Delta_i^n\mathbf{F}'$. By definition, $\widehat{\mathbf{G}}_t^{\text{latent}} = \mathbf{P}_t\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\Delta_i^n\mathbf{Y}\widehat{\Delta_i^n\mathbf{F}}'$. Hence for each fixed $l \le p$, substitute in $\Delta_i^n\mathbf{Y}$,

$$
\begin{aligned}
\widehat{\mathbf{g}}_{lt}^{\text{latent}} - \boldsymbol{\Upsilon}_{nt}\mathbf{g}_{lt} &= \boldsymbol{\Upsilon}_{nt}\boldsymbol{\Gamma}_t'\mathbf{P}_{t,l} + \mathbf{H}_{nt}\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{U}'\mathbf{P}_{t,l} \\
&\quad + \frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\widehat{\Delta_i^n\mathbf{F}}\boldsymbol{\psi}_i'\mathbf{P}_{t,l}\Delta_n + \frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\widehat{\Delta_i^n\mathbf{F}}\Delta_i^n\mathbf{F}'[\mathbf{G}_{i-1}' - \mathbf{G}_t']\mathbf{P}_{t,l} \\
&\quad + \boldsymbol{\Upsilon}_{nt}(\mathbf{G}_{t,l}'\mathbf{P}_{t,l} - \mathbf{g}_{lt}) + \frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\widehat{\Delta_i^n\mathbf{F}}\Delta_i^n\mathbf{F}'[\boldsymbol{\Gamma}_{i-1}' - \boldsymbol{\Gamma}_t']\mathbf{P}_{t,l} \\
&\quad + \frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}(\widehat{\Delta_i^n\mathbf{F}} - \mathbf{H}_{nt}\Delta_i^n\mathbf{F})\Delta_i^n\mathbf{U}'\mathbf{P}_{t,l} \\
&= \boldsymbol{\Upsilon}_{nt}\boldsymbol{\Gamma}_t'\mathbf{P}_{t,l} + \mathbf{H}_{nt}\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{U}'\mathbf{P}_{t,l} \\
&\quad + \frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}(\widehat{\Delta_i^n\mathbf{F}} - \mathbf{H}_{nt}\Delta_i^n\mathbf{F})\Delta_i^n\mathbf{U}'\mathbf{P}_{t,l} + o_P((k_np)^{-1/2}). \qquad \text{(E.8)}
\end{aligned}
$$

Lemma E.2 below shows that the third through the sixth terms on the right hand side are $o_P((k_np)^{-1/2})$. We now work on the last term on the right hand side. By (E.7),

$$
\begin{aligned}
&\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}(\widehat{\Delta_i^n\mathbf{F}} - \mathbf{H}_{nt}\Delta_i^n\mathbf{F})\Delta_i^n\mathbf{U}'\mathbf{P}_{i-1,l} - \text{BIAS}_g \\
= \;& \boldsymbol{\Xi}_{1,t}'\frac{1}{k_n\sqrt{p}}\sum_{i\in I_t^n}\mathbf{P}_{i-1}\mathbf{c}_{uu,i}\mathbf{P}_{i-1,l} + \frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\boldsymbol{\Xi}_{3,t}'\frac{1}{\sqrt{p}}\Delta_n\mathbf{P}_{i-1}\boldsymbol{\psi}_i\Delta_i^n\mathbf{U}'\mathbf{P}_{i-1,l} \\
&+ \frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\boldsymbol{\Xi}_{3,t}'\frac{1}{\sqrt{p}}(\mathbf{P}_{i-1}\boldsymbol{\beta}_{i-1} - \mathbf{P}_t\boldsymbol{\beta}_t)\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{U}'\mathbf{P}_{i-1,l}
\end{aligned}
$$

$$+\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\mathbf{\Xi}'_{3,t}\frac{1}{\sqrt{p}}\mathbf{P}_{i-1}(\Delta_i^n\mathbf{U}\Delta_i^n\mathbf{U}'-\mathbb{E}\Delta_i^n\mathbf{U}\Delta_i^n\mathbf{U}')\mathbf{P}_{i-1,l},$$

where BIAS_g is defined in the main paper. By Lemma E.3, all terms on the right hand side are $o_P((k_np)^{-1/2})$. Therefore, (E.8) implies

$$\widehat{\mathbf{g}}_{lt}^{\text{latent}}-\mathbf{\Upsilon}_{nt}\mathbf{g}_{lt}-\text{BIAS}_g=\mathbf{\Upsilon}_{nt}\mathbf{\Gamma}'_t\mathbf{P}_{t,l}+\mathbf{H}_{nt}\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{U}'\mathbf{P}_{i-1,l}+o_P((k_np)^{-1/2}).$$

In addition, Lemma E.4 implies $\mathbf{H}_{nt}-\mathbf{\Upsilon}_{nt}\mathbf{c}_{FF,t}^{-1}=o_P(1)$ and $\mathbf{\Upsilon}_{nt}-\mathbf{\Upsilon}_t=o_P(1)$, where $\mathbf{\Upsilon}_t$ is non-random conditional on $\mathbf{X}_t$ whose definition is given in Lemma E.4. Hence

$$\widehat{\mathbf{g}}_{lt}^{\text{latent}}-\mathbf{\Upsilon}_{nt}\mathbf{g}_{lt}-\text{BIAS}_g = \mathbf{\Upsilon}_t\mathbf{\Gamma}'_t\mathbf{P}_{t,l}+\mathbf{\Upsilon}_t\mathbf{c}_{FF,t}^{-1}\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{U}'\mathbf{P}_{t,l}+o_P((k_np)^{-1/2}). \tag{E.9}$$

The right hand side is the same expansion (E.1) as in the known factor case, up to $\mathbf{\Upsilon}_t$, which is the same as the expansion (D.4) in the general case. Then the same argument as in the proof based on (D.4) in Theorem 3.1 implies

$$\mathbf{S}^{-1/2}\left(\widehat{\mathbf{g}}_{lt}^{\text{latent}}-\mathbf{\Upsilon}_{nt}\mathbf{g}_{lt}-\text{BIAS}_g\right)\overset{\mathcal{L}\text{-}s}{\to}N(0,\mathbf{I}).$$

where $\mathbf{S}:=\mathbf{\Upsilon}_t[\frac{1}{k_np}\mathbf{V}_{\gamma,t}+\frac{1}{p}\mathbf{V}_{\gamma,t}]\mathbf{\Upsilon}'_t$. Now let $\widehat{\mathbf{S}}=\mathbf{\Upsilon}_{nt}[\frac{1}{k_np}\mathbf{V}_{\gamma,t}+\frac{1}{p}\mathbf{V}_{\gamma,t}]\mathbf{\Upsilon}'_{nt}$. We aim to show $\widehat{\mathbf{S}}^{-1/2}\left(\widehat{\mathbf{g}}_{lt}^{\text{latent}}-\mathbf{\Upsilon}_{nt}\mathbf{g}_{lt}-\text{BIAS}_g\right)\overset{\mathcal{L}\text{-}s}{\to}N(0,\mathbf{I})$.

We now show that $\widehat{\mathbf{S}}=\mathbf{S}(1+o_P(1))$, meaning that $\|\widehat{\mathbf{S}}-\mathbf{S}\|=o_P(1)\lambda_{\min}(\mathbf{S})$. In fact, let $\mathbf{\Sigma}_n=\frac{1}{k_np}\mathbf{V}_{u,t}+\frac{1}{p}\mathbf{V}_{\gamma,t}$, we have

$$\begin{aligned}\|\widehat{\mathbf{S}}-\mathbf{S}\| &\leq [2\|\mathbf{\Upsilon}_t-\mathbf{\Upsilon}_{nt}\|\|\mathbf{\Upsilon}_t\|+\|\mathbf{\Upsilon}_t-\mathbf{\Upsilon}_{nt}\|^2]\|\mathbf{\Sigma}_n\|\\ &= o_P(1)\|\mathbf{\Sigma}_n\|\leq o_P(1)[\|\frac{1}{k_np}\mathbf{V}_{u,t}\|+\|\frac{1}{p}\mathbf{V}_{\gamma,t}\|]\leq o_P(1)[\lambda_{\min}(\frac{1}{k_np}\mathbf{V}_{u,t})+\lambda_{\min}(\frac{1}{p}\mathbf{V}_{\gamma,t})]\\ &\leq o_P(1)\lambda_{\min}(\mathbf{\Sigma}_n)\leq o_P(1)\lambda_{\min}(\mathbf{\Sigma}_n)\lambda_{\min}(\mathbf{\Upsilon}_t\mathbf{\Upsilon}'_t)\leq o_P(1)\lambda_{\min}(\mathbf{S}).\end{aligned}$$

where $\|\frac{1}{p}\mathbf{V}_{\gamma,t}\|\leq C_1\lambda_{\min}(\frac{1}{p}\mathbf{V}_{\gamma,t})$ follows from Assumption 3.4 that $\lambda_{\max}(\mathbf{V}_{\gamma,t})\leq C_1\lambda_{\min}(\mathbf{V}_{\gamma,t})$ for all t. Now let $\mathbf{C}:=\mathbf{S}-\widehat{\mathbf{S}}$ and $\mathbf{D}=\mathbf{C}\mathbf{S}^{-1}$, then $\|\mathbf{D}\|\leq\|\mathbf{C}\|\lambda_{\min}^{-1}(\mathbf{S})=o_P(1)$, and $\mathbf{C}=\mathbf{D}\mathbf{C}+\mathbf{D}\widehat{\mathbf{S}}$ implies $\mathbf{C}=(\mathbf{I}-\mathbf{D})^{-1}\mathbf{D}\widehat{\mathbf{S}}$. Note that $\|(\mathbf{I}-\mathbf{D})^{-1}\mathbf{D}\|=o_P(1)$, which implies $\|\mathbf{C}\|=o_P(1)\|\widehat{\mathbf{S}}\|=o_P(1)\lambda_{\min}(\widehat{\mathbf{S}})$. Therefore, $\mathbf{S}=\widehat{\mathbf{S}}(1+o_P(1))$. It also implies

$\mathbf{S}^{1/2} = \widehat{\mathbf{S}}^{1/2}(1 + o_P(1))$. Hence $\widehat{\mathbf{S}}^{-1/2}\mathbf{S}^{1/2} = \widehat{\mathbf{S}}^{-1/2}\widehat{\mathbf{S}}^{1/2}(1 + o_P(1)) = \mathbf{I}(1 + o_P(1))$. Finally,

$$\widehat{\mathbf{S}}^{-1/2}\left(\widehat{\mathbf{g}}_{lt}^{\text{latent}} - \mathbf{\Upsilon}_{nt}\mathbf{g}_{lt} - \text{BIAS}_g\right) = [\widehat{\mathbf{S}}^{-1/2}\mathbf{S}^{1/2} - \mathbf{I}]\mathbf{S}^{-1/2}\left(\widehat{\mathbf{g}}_{lt}^{\text{latent}} - \mathbf{\Upsilon}_{nt}\mathbf{g}_{lt} - \text{BIAS}_g\right)$$
$$+\mathbf{S}^{-1/2}\left(\widehat{\mathbf{g}}_{lt}^{\text{latent}} - \mathbf{\Upsilon}_{nt}\mathbf{g}_{lt} - \text{BIAS}_g\right) \stackrel{\mathcal{L}\text{-}s}{\to} N(0, \mathbf{I}).$$

□

E.2.3 Proof of Theorem 3.3: bias correction

Proof. It suffices to prove that $\widehat{\text{BIAS}}_g - \text{BIAS} = o_P((k_n p)^{-1/2})$ for $\widehat{\text{BIAS}}_g$ defined in CASE I and CASE II. Write $\widehat{\text{BIAS}}_g = \widehat{\mathbf{M}}_t \frac{1}{k_n\sqrt{p}} \sum_{i\in I_t^n} \mathbf{P}_{i-1}\mathbf{A}_i\mathbf{P}_{i-1,l}$, we have

$$\text{BIAS}_g - \widehat{\text{BIAS}}_g = (\mathbf{M}_t - \widehat{\mathbf{M}}_t)\frac{1}{k_n\sqrt{p}}\sum_{i+1\in I_t}\mathbf{P}_i\,\mathbf{c}_{uu,i}\,\mathbf{P}_{i,l} + \widehat{\mathbf{M}}_t\frac{1}{k_n\sqrt{p}}\sum_{i+1\in I_t}\mathbf{P}_{i-1}\left(\mathbf{c}_{uu,i} - \mathbf{A}_i\right)\mathbf{P}_{i-1,l}.$$

To bound the first term on the right hand side, first note that

$$\begin{aligned}\widehat{\mathbf{G}}_t^{\text{latent}} &= \frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\mathbf{P}_{i-1}\Delta_i^n\mathbf{Y}\widehat{\Delta_i^n\mathbf{F}}' = \frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\mathbf{P}_{i-1}\boldsymbol{\psi}_i\Delta_n\widehat{\Delta_i^n\mathbf{F}}' \\ &\quad+\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\mathbf{P}_{i-1}\boldsymbol{\beta}_{i-1}\Delta_i^n\mathbf{F}\widehat{\Delta_i^n\mathbf{F}}' + \frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\mathbf{P}_{i-1}\Delta_i^n\mathbf{U}\widehat{\Delta_i^n\mathbf{F}}'.\end{aligned}$$

Hence $\mathbf{M}_t - \widehat{\mathbf{M}}_t = \widehat{\mathbf{V}}_t^{-1}[\frac{1}{k_n\Delta_n\sqrt{p}}\sum_{i\in I_t}\widehat{\Delta_i^n\mathbf{F}}\,\Delta_i^n\mathbf{F}'\,\boldsymbol{\beta}_{i-1}'\mathbf{P}_{i-1} - \frac{1}{\sqrt{p}}\widehat{\mathbf{G}}_t'] = -\mathbf{\Xi}_{1,t}'$. The first term on the right hand side of $\text{BIAS}_g - \widehat{\text{BIAS}}_g$ is then bounded by (for $h_{i,ll} = \boldsymbol{\phi}_{il}'(\frac{1}{p}\mathbf{\Phi}_i'\mathbf{\Phi}_i)^{-1}\boldsymbol{\phi}_{il}$)

$$\frac{C}{p}\|\mathbf{M}_t - \widehat{\mathbf{M}}_t\|\frac{1}{k_n}\sum_{i+1\in I_t}h_{i,ll}^{1/2} = O_P(p^{-1})\|\mathbf{\Xi}_{1,t}\| = o_P((k_n p)^{-1/2})$$

by Lemma E.3(iii). As for the second term on the right hand side, note that $\widehat{\mathbf{M}}_t = O_P(1)$.

Hence it remains to prove $\|\frac{1}{\sqrt{k_n}}\sum_{i+1\in I_t}\mathbf{P}_{i-1}\left(\mathbf{c}_{uu,i} - \mathbf{A}_i\right)\mathbf{P}_{i-1,l}\| = o_P(1)$. We respectively look at two cases.

CASE I: $\mathbf{A}_i = \text{diag}\{\widehat{\Delta_i^n\mathbf{U}}\widehat{\Delta_i^n\mathbf{U}}'\}\Delta_n^{-1}$. Write $\mathbf{u}_i = \Delta_n^{-1/2}\Delta_i^n\mathbf{U}$ and $\widehat{\mathbf{u}}_i = \Delta_n^{-1/2}\widehat{\Delta_i^n\mathbf{U}}$

$$\begin{aligned}&\|\frac{1}{\sqrt{k_n}}\sum_{i+1\in I_t}\mathbf{P}_{i-1}\left(\mathbf{c}_{uu,i} - \mathbf{A}_i\right)\mathbf{P}_{i-1,l}\|^2 \le \frac{2}{k_n p^2}\sum_{m=1}^{p}[\sum_{i+1\in I_t}\frac{1}{p}\sum_{d=1}^{p}h_{i-1,ld}h_{i-1,md}(\mathbb{E}u_{di}^2 - u_{di}^2)]^2 \\ &+\frac{2}{k_n p^2}\sum_{m=1}^{p}[\sum_{i+1\in I_t}\frac{1}{p}\sum_{d=1}^{p}h_{i-1,ld}h_{i-1,md}(\widehat{u}_{di}^2 - u_{di}^2)]^2.\end{aligned}$$

As for the first term on the right hand side,

$$\begin{aligned}
&\mathbb{E}\frac{2}{k_n p^2}\sum_{m=1}^p [\sum_{i+1\in I_t}\frac{1}{p}\sum_{d=1}^p h_{i-1,ld}h_{i-1,md}(\mathbb{E}u_{di}^2 - u_{di}^2)]^2 \\
= &\frac{2}{p^2}\sum_{m=1}^p \frac{1}{k_n}\sum_{i+1\in I_t}\text{var}[\frac{1}{p}\sum_{d=1}^p h_{i-1,ld}h_{i-1,md}(\mathbb{E}u_{di}^2 - u_{di}^2)] = o(1).
\end{aligned}$$

The second term on the right is shown to be $o_P(1)$ in Lemma E.5. This finishes the proof for CASE I.

CASE II: $\mathbf{A}_i = \widehat{\mathbf{c}}_{u,t}$. By the assumption, $\max_{i\in I_t^n}\|\widehat{\mathbf{c}}_{uu,t} - \mathbf{c}_{u,i}\| = o_P(\sqrt{p/k_n})$. Hence

$$\begin{aligned}
&\|\frac{1}{\sqrt{k_n}}\sum_{i+1\in I_t}\mathbf{P}_{i-1}\left(\mathbf{c}_{uu,i} - \mathbf{A}_i\right)\mathbf{P}_{i-1,l}\| \leq \frac{1}{\sqrt{k_n p}}\|\mathbf{c}_{uu,t} - \mathbf{c}_{u,t}\|\sum_{i+1\in I_t} h_{i-1,ll}^{1/2} \\
\leq &\frac{\sqrt{k_n}}{\sqrt{p}}\|\mathbf{c}_{uu,t} - \mathbf{c}_{u,t}\| = o_P(1).
\end{aligned}$$

We shall provide primitive conditions for $\max_{i\in I_t^n}\|\widehat{\mathbf{c}}_{uu,t} - \mathbf{c}_{u,i}\| = o_P(\sqrt{p/k_n})$ in Section E.2.5. □

E.2.4 Proof of Theorem 3.4 (estimated factors)

Proof. By definition, $\mathbb{P}^*\left(|\mathbf{v}'\widehat{\mathbf{g}}_{lt}^{*\,\text{latent}} - \mathbf{v}'\widehat{\mathbf{g}}_{lt}^{\text{latent}}| \leq q_\tau^{\text{latent}}\right) = 1-\tau$. In addition, the proof of Theorem 3.2 also implies $\left(\mathbf{v}'\boldsymbol{\Upsilon}_{nt}\boldsymbol{\Sigma}_n\boldsymbol{\Upsilon}_{nt}'\mathbf{v}\right)^{-1/2}\mathbf{v}'\left(\widehat{\mathbf{g}}_{lt}^{\text{latent}} - \mathbf{g}_{lt} - \text{BIAS}_g\right) \stackrel{\mathcal{L}\text{-}s}{\to} N(0,1)$. The proof of Theorem 3.3 also implies $\text{BIAS}_g - \widehat{\text{BIAS}}_g = o_P((k_n p)^{-1/2})$. Hence

$$\left(\mathbf{v}'\boldsymbol{\Upsilon}_{nt}\boldsymbol{\Sigma}_n\boldsymbol{\Upsilon}_{nt}'\mathbf{v}\right)^{-1/2}\mathbf{v}'\left(\widehat{\mathbf{g}}_{lt}^{\text{latent}} - \mathbf{g}_{lt} - \widehat{\text{BIAS}}_g\right) \stackrel{\mathcal{L}\text{-}s}{\to} N(0,1).$$

Hence from Proposition E.1, we have

$$\begin{aligned}
1-\tau &= \mathbb{P}^*\left(\left(\mathbf{v}'\boldsymbol{\Upsilon}_{nt}\boldsymbol{\Sigma}_n\boldsymbol{\Upsilon}_{nt}'\mathbf{v}\right)^{-1/2}|\mathbf{v}'\widehat{\mathbf{g}}_{lt}^{*\,\text{latent}} - \mathbf{v}'\widehat{\mathbf{g}}_{lt}^{\text{latent}}| \leq \left(\mathbf{v}'\boldsymbol{\Upsilon}_{nt}\boldsymbol{\Sigma}_n\boldsymbol{\Upsilon}_{nt}'\mathbf{v}\right)^{-1/2} q_\tau^{\text{latent}}\right) \\
&= \mathbb{P}\left(\left(\mathbf{v}'\boldsymbol{\Upsilon}_{nt}\boldsymbol{\Sigma}_n\boldsymbol{\Upsilon}_{nt}'\mathbf{v}\right)^{-1/2}|\mathbf{v}'\widehat{\mathbf{g}}_{lt}^{\text{latent}} - \mathbf{v}'\mathbf{g}_{lt} - \mathbf{v}'\widehat{\text{BIAS}}_g| \leq \left(\mathbf{v}'\boldsymbol{\Upsilon}_{nt}\boldsymbol{\Sigma}_n\boldsymbol{\Upsilon}_{nt}'\mathbf{v}\right)^{-1/2} q_\tau^{\text{latent}}\right) + o(1) \\
&= \mathbb{P}\left(\mathbf{v}'\mathbf{g}_{lt} \in CI_{nt,\tau}^{\text{latent}}\right) + o(1).
\end{aligned}$$

The remaining proof about the uniformity is the same as that of Theorem 3.4 (known factors case), so is omitted for brevity. □

Proposition E.1. *With $\boldsymbol{\Upsilon}_{nt}, \boldsymbol{\Sigma}_n$ as defined in Theorem 3.2,*

$$(\mathbf{v}'\boldsymbol{\Upsilon}_{nt}\boldsymbol{\Sigma}_n\boldsymbol{\Upsilon}_{nt}'\mathbf{v})^{-1/2}\,\mathbf{v}'[\widehat{\mathbf{g}}_{lt}^{*\,\text{latent}} - \widehat{\mathbf{g}}_{lt}^{\text{latent}}] \xrightarrow{d^*} N(0,1).$$

Proof. We have $\Delta_i^n\mathbf{Y}^* = \boldsymbol{\psi}_i^*\Delta_n + \mathbf{G}_{i-1}^*\Delta_i^n\mathbf{F} + \boldsymbol{\Gamma}_{i-1}^*\Delta_i^n\mathbf{F} + \Delta_i^n\mathbf{U}^*$. Hence similar to (E.8),

$$\begin{aligned}
&\widehat{\mathbf{g}}_{lt}^{*\,\text{latent}} - \boldsymbol{\Upsilon}_{nt}\mathbf{g}_{lt} = \boldsymbol{\Upsilon}_{nt}\boldsymbol{\Gamma}_t^{*'}\mathbf{P}_{t,l}^* + \frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\mathbf{H}_{nt}\Delta_i^n\mathbf{F}\,\Delta_i^n\mathbf{U}^{*'}\mathbf{P}_{t,l}^*\\
&+\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\widehat{\Delta_i^n\mathbf{F}}\,\Delta_n\boldsymbol{\psi}_i^{*'}\mathbf{P}_{t,l}^* + \frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\widehat{\Delta_i^n\mathbf{F}}\Delta_i^n\mathbf{F}'[\mathbf{G}_{i-1}^{*'} - \mathbf{G}_t^{*'}]\mathbf{P}_{t,l}^*\\
&+\boldsymbol{\Upsilon}_{nt}(\mathbf{G}_t^{*'}\mathbf{P}_{t,l}^* - \mathbf{g}_{lt}) + \frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\widehat{\Delta_i^n\mathbf{F}}\Delta_i^n\mathbf{F}'(\boldsymbol{\Gamma}_{i-1}^{*'} - \boldsymbol{\Gamma}_t^{*'})\mathbf{P}_{t,l}^*\\
&+\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}(\widehat{\Delta_i^n\mathbf{F}} - \mathbf{H}_{nt}\Delta_i^n\mathbf{F})\Delta_i^n\mathbf{U}^{*'}\mathbf{P}_{t,l}^*\\
=^{(1)}\;\; &\boldsymbol{\Upsilon}_{nt}\boldsymbol{\Gamma}_t^{*'}\mathbf{P}_{t,l}^* + \frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\mathbf{H}_{nt}\Delta_i^n\mathbf{F}\,\Delta_i^n\mathbf{U}^{*'}\mathbf{P}_{i-1,l}^* + \frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}(\widehat{\Delta_i^n\mathbf{F}} - \mathbf{H}_{nt}\Delta_i^n\mathbf{F})\Delta_i^n\mathbf{U}^{*'}\mathbf{P}_{i-1,l}^*\\
&+o_{P^*}((k_np)^{-1/2}), \qquad\qquad (\text{E.10})
\end{aligned}$$

where (1) follows from Lemma E.7. Now we investigate the last term, the effect of estimating factors:

$$\begin{aligned}
&\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}(\widehat{\Delta_i^n\mathbf{F}} - \mathbf{H}_{nt}\Delta_i^n\mathbf{F})\Delta_i^n\mathbf{U}^{*'}\mathbf{P}_{i-1,l}^* - \text{BIAS}_g\\
=\;\; &\frac{1}{k_n}\sum_{i\in I_t^n}\boldsymbol{\Xi}_{1,t}'\frac{1}{\sqrt{p}}\mathbf{P}_{i-1}\mathbf{c}_{uu,t}\mathbf{P}_{i-1,l} + \frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\boldsymbol{\Xi}_{3,t}'\frac{1}{\sqrt{p}}\mathbf{P}_{i-1}[\Delta_i^n\mathbf{U}\Delta_i^n\mathbf{U}' - \mathbb{E}\Delta_i^n\mathbf{U}\Delta_i^n\mathbf{U}']\mathbf{P}_{i-1,l}\\
&+\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\boldsymbol{\Xi}_{3,t}'\frac{1}{\sqrt{p}}\Delta_n\mathbf{P}_{i-1}\boldsymbol{\psi}_i\Delta_i^n\mathbf{U}^{*'}\mathbf{P}_{i-1,l}^*\\
&+\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\boldsymbol{\Xi}_{3,t}'\frac{1}{\sqrt{p}}(\mathbf{P}_{i-1}\boldsymbol{\beta}_{i-1} - \mathbf{P}_t\boldsymbol{\beta}_t)\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{U}^{*'}\mathbf{P}_{i-1,l}^*\\
&+\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\boldsymbol{\Xi}_{3,t}'\frac{1}{\sqrt{p}}\mathbf{P}_{i-1}\Delta_i^n\mathbf{U}[\Delta_i^n\mathbf{U}^{*'}\mathbf{P}_{i-1,l}^* - \frac{1}{p}\sum_{m=1}^{p}\Delta_i^nU_m^*\boldsymbol{\phi}_{i-1,m}^{*'}(\frac{1}{p}\boldsymbol{\Phi}_{i-1}'\boldsymbol{\Phi}_{i-1})^{-1}\boldsymbol{\phi}_{i-1,l}]\\
&+\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\boldsymbol{\Xi}_{3,t}'\frac{1}{\sqrt{p}}\mathbf{P}_{i-1}\Delta_i^n\mathbf{U}\frac{1}{p}\sum_{m=1}^{p}[\Delta_i^nU_m^*\boldsymbol{\phi}_{i-1,m}^{*'}(\frac{1}{p}\boldsymbol{\Phi}_{i-1}'\boldsymbol{\Phi}_{i-1})^{-1}\boldsymbol{\phi}_{i-1,l} - \Delta_i^nU_mh_{i-1,ml}]
\end{aligned}$$

The first two terms are bounded in Lemma E.3 while all other terms are bounded in Lemma

E.8. By these two lemmas, all terms are $o_{P^*}((k_n p)^{-1/2})$. Hence (E.10) implies

$$\widehat{\mathbf{g}}_{lt}^{*\,\text{latent}} - \mathbf{\Upsilon}_{nt}\mathbf{g}_{lt} - \text{BIAS}_g = \frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\mathbf{H}_{nt}\Delta_i^n\mathbf{F}\,\Delta_i^n\mathbf{U}^{*'}\mathbf{P}_{t,l}^* + \mathbf{\Upsilon}_{nt}\mathbf{\Gamma}_t^{*'}\mathbf{P}_{t,l}^* + o_{P^*}((k_n p)^{-1/2}).$$

Next, define

$$\begin{aligned}
\mathbf{a}_1^* &= \mathbf{c}_{FF,t}^{-1}\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\Delta_i^n\mathbf{F}\frac{1}{p}\sum_{m=1}^p\Delta_i^n U_m^*\boldsymbol{\phi}_{mt}^{*'}(\frac{1}{p}\mathbf{\Phi}_t'\mathbf{\Phi}_t)^{-1}\boldsymbol{\phi}_{lt}\\
\mathbf{a}_2^* &= \frac{1}{p}\sum_{m=1}^p\boldsymbol{\gamma}_{t,m}^*\boldsymbol{\phi}_{tm}^{*'}(\frac{1}{p}\mathbf{\Phi}_t'\mathbf{\Phi}_t)^{-1}\boldsymbol{\phi}_{tl}\\
\mathbf{a}_1 &= \mathbf{c}_{FF,t}^{-1}\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{U}'\mathbf{P}_{t,l}\\
\mathbf{a}_2 &= \mathbf{\Gamma}_t'\mathbf{P}_{t,l}.
\end{aligned}$$

Lemma D.2 shows that $\|\mathbf{\Gamma}_t^{*'}\mathbf{P}_{t,l}^* - \mathbf{a}_2^*\| = o_{P^*}(p^{-1/2})\lambda_{\min}^{1/2}(\mathbf{V}_{\gamma,t})$. Also, the similar proof as Lemma D.2(i) gives

$$\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\Delta_i^n\mathbf{F}\,\Delta_i^n\mathbf{U}^{*'}\mathbf{P}_{t,l}^* = \frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\Delta_i^n\mathbf{F}\frac{1}{p}\sum_{m=1}^p\Delta_i^n U_m^*\boldsymbol{\phi}_{mt}^{*'}(\frac{1}{p}\mathbf{\Phi}_t'\mathbf{\Phi}_t)^{-1}\boldsymbol{\phi}_{lt} + o_{P^*}((k_n p)^{-1/2}).$$

So

$$\begin{aligned}
\widehat{\mathbf{g}}_{lt}^{*\,\text{latent}} - \mathbf{\Upsilon}_{nt}\mathbf{g}_{lt} - \text{BIAS}_g &= \frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\mathbf{H}_{nt}\Delta_i^n\mathbf{F}\,\Delta_i^n\mathbf{U}^{*'}\mathbf{P}_{t,l}^* + \mathbf{\Upsilon}_{nt}\mathbf{\Gamma}_t^{*'}\mathbf{P}_{t,l}^* + o_{P^*}((k_n p)^{-1/2})\\
&= \mathbf{H}_{nt}\mathbf{c}_{FF,t}\mathbf{a}_1^* + \mathbf{\Upsilon}_{nt}\mathbf{a}_2^* + o_{P^*}((k_n p)^{-1/2}) + o_{P^*}(p^{-1/2})\lambda_{\min}^{1/2}(\mathbf{V}_{\gamma,t})\\
&= \mathbf{\Upsilon}_t\mathbf{a}_1^* + \mathbf{\Upsilon}_{nt}\mathbf{a}_2^* + o_{P^*}((k_n p)^{-1/2}) + o_{P^*}(p^{-1/2})\lambda_{\min}^{1/2}(\mathbf{V}_{\gamma,t})\\
&= \mathbf{\Upsilon}_t\mathbf{a}_1^* + \mathbf{\Upsilon}_t\mathbf{a}_2^* + o_{P^*}((k_n p)^{-1/2}) + o_{P^*}(p^{-1/2})\lambda_{\min}^{1/2}(\mathbf{V}_{\gamma,t}).
\end{aligned}$$

The third equality is due to $\mathbf{H}_{nt}\mathbf{c}_{FF,t} - \mathbf{\Upsilon}_{nt} = o_P(1)$, and $\mathbf{\Upsilon}_{nt} - \mathbf{\Upsilon}_t = o_P(1)$ (Lemma E.4). The last equality is due to $\mathbf{a}_2^* = O_{P^*}(p^{-1/2})\lambda_{\min}^{1/2}(\mathbf{V}_{\gamma,t})$, which comes from $\mathbf{a}_2^* - \mathbf{a}_2 = O_{P^*}(p^{-1/2})\lambda_{\min}^{1/2}(\mathbf{V}_{\gamma,t})$ and $\mathbf{a}_2 = O_{P^*}(p^{-1/2})\lambda_{\min}^{1/2}(\mathbf{V}_{\gamma,t})$.

Now recall (E.9),

$$\widehat{\mathbf{g}}_{lt}^{\text{latent}} - \mathbf{\Upsilon}_{nt}\mathbf{g}_{lt} - \text{BIAS}_g = \mathbf{\Upsilon}_t\mathbf{a}_1 + \mathbf{\Upsilon}_t\mathbf{a}_2 + o_P((k_n p)^{-1/2}) + o_P(p^{-1/2})\lambda_{\min}^{1/2}(\mathbf{V}_{\gamma,t}).$$

Hence

$$\widehat{\mathbf{g}}_{lt}^{*\,\text{latent}} - \widehat{\mathbf{g}}_{lt}^{\text{latent}} = \boldsymbol{\Upsilon}_t(\mathbf{a}_1^* - \mathbf{a}_1) + \boldsymbol{\Upsilon}_t(\mathbf{a}_2^* - \mathbf{a}_2) + o_{P^*}((k_n p)^{-1/2}) + o_{P^*}(p^{-1/2})\lambda_{\min}^{1/2}(\mathbf{V}_{\gamma,t}).$$

In addition, the same proof of Proposition D.1 (specifically, starting from (D.6)) implies

$$\left(\mathbf{v}'\boldsymbol{\Upsilon}_t\boldsymbol{\Sigma}_n\boldsymbol{\Upsilon}_t'\mathbf{v}\right)^{-1/2}\mathbf{v}'\boldsymbol{\Upsilon}_t(\mathbf{a}_1^* - \mathbf{a}_1 + \mathbf{a}_2^* - \mathbf{a}_2) \xrightarrow{d^*} N(0,1),$$

and we also have $\left(\mathbf{v}'\boldsymbol{\Upsilon}_t\boldsymbol{\Sigma}_n\boldsymbol{\Upsilon}_t'\mathbf{v}\right)^{-1/2}[o_{P^*}((k_n p)^{-1/2}) + o_{P^*}(p^{-1/2})\lambda_{\min}^{1/2}(\mathbf{V}_{\gamma,t})] = o_{P^*}(1)$. Therefore,

$$\left(\mathbf{v}'\boldsymbol{\Upsilon}_t\boldsymbol{\Sigma}_n\boldsymbol{\Upsilon}_t'\mathbf{v}\right)^{-1/2}\mathbf{v}'[\widehat{\mathbf{g}}_{lt}^{*\,\text{latent}} - \widehat{\mathbf{g}}_{lt}^{\text{latent}}] \xrightarrow{d^*} N(0,1).$$

Since $\boldsymbol{\Upsilon}_{nt} \xrightarrow{P} \boldsymbol{\Upsilon}_t$, we can apply the same argument as at the end of the Proof of Theorem 3.2 to reach that $\left(\mathbf{v}'\boldsymbol{\Upsilon}_{nt}\boldsymbol{\Sigma}_n\boldsymbol{\Upsilon}_{nt}'\mathbf{v}\right)^{-1/2}\mathbf{v}'[\widehat{\mathbf{g}}_{lt}^{*\,\text{latent}} - \widehat{\mathbf{g}}_{lt}^{\text{latent}}] \xrightarrow{d^*} N(0,1)$.

□

E.2.5 Sparse covariance estimation of $\mathbf{c}_{uu,i}$.

In this subsection, we provide primitive conditions to verify $\max_{i\in I_t^n}\|\widehat{\mathbf{c}}_{uu,t}-\mathbf{c}_{uu,i}\| = o_P(\sqrt{\frac{p}{Jk_n}})$ in CASE II of Theorem 3.3. The key assumption is the "uniform sparsity" of the $p\times p$ instantaneous quadratic variation $\mathbf{c}_{uu,t}$. At any time t, let $c_{u,t,dl}$ be the (d,l) th entry of $\mathbf{c}_{uu,t}$. Define:

$$m_{p,t} = \max_{i\in I_t^n}\max_{l\le p}\sum_{d\le p} 1\{c_{u,i,dl} \neq 0\},\ \text{the maximum number of nonzeros in rows.} \tag{E.11}$$

We shall assume that $m_{p,t}$ is either bounded or slowly grows as $p\to\infty$. Thus $\mathbf{c}_{uu,i}$ is a sparse matrix uniformly in the window $i\in I_t^n$. This condition is relatively reasonable since $\mathbf{c}_{uu,i}$ represents the quadratic variance of $p\times 1$ idiosyncratic components $\{\Delta_i^n U_l\}_{l\le p}$, which should mostly pairwise-uncorrelated given the factors. We apply the thresholding estimator of (Bickel and Levina (2008); Fan et al. (2013)) for sparse covariances as follows. Let s_{dl} be the (d,l) th element of $\frac{1}{\Delta_n k_n}\sum_{i\in I_t^n}\widehat{\Delta_i^n\mathbf{U}}\widehat{\Delta_i^n\mathbf{U}}'$. Let the (d,l)-th entry of the estimated covariance be:

$$(\widehat{\mathbf{c}}_{uu,t})_{dl} = \begin{cases} s_{dd}, & \text{if } d = l,\\ \text{th}(s_{dl})1\{|s_{dl}| > \varrho_{dl}\} & \text{if } d \neq l,\end{cases}$$

where th$(\cdot)$ is either the hard-thresholding or soft-thresholding function. Here the threshold value $\varrho_{dl} = \bar{C}(s_{dd}s_{ll})^{1/2}\omega_{np}$, with $\omega_{np} = \sqrt{\frac{\log p}{k_n} + \frac{J}{p}\max_{j,d}\frac{1}{J}\|\boldsymbol{\phi}(\mathbf{x}_{jd})\|^2\sqrt{\log J}}$ for som $\bar{C} > 0$.

The following lemma shows that $\max_{i\in I_t^n}\|\widehat{\mathbf{c}}_{uu,t} - \mathbf{c}_{uu,i}\| = O_P(m_{p,t}\omega_{np})$. Thus the required condition $\max_{i\in I_t^n}\|\widehat{\mathbf{c}}_{uu,t} - \mathbf{c}_{uu,i}\| = o_P(\sqrt{\frac{p}{Jk_n}})$ holds so long as

$$m_{p,t}^2 = o(\frac{p}{J\log p}), \quad m_{p,t}^2 = o(\frac{p^3}{k_n J^3(\log J)\max_{j,d}\frac{1}{J^2}\|\boldsymbol{\phi}(\mathbf{x}_{jd})\|^4}) \tag{E.12}$$

Therefore, (E.12) is viewed as a primitive condition for $\max_{i\in I_t^n}\|\widehat{\mathbf{c}}_{uu,t} - \mathbf{c}_{uu,i}\| = o_P(\sqrt{\frac{p}{Jk_n}})$, which is regarded as the assumption on the level of sparsity on $\mathbf{c}_{uu,i}$.

Lemma E.1. *Suppose* $\min_{d\le p, j\in I_t^n} c_{u,j,dd}$ *is bounded away from zero.*

(i)

$$\max_{i\in I_t^n}\max_{d,l\le p}\frac{|s_{dl} - c_{u,i,dl}|}{\varrho_{dl}} \le 1/2, \tag{E.13}$$

(ii)

$$\max_{i\in I_t^n}\|\widehat{\mathbf{c}}_{uu,t} - \mathbf{c}_{uu,i}\| = O_P(m_{p,t}\omega_{np}).$$

Proof. (i) Let $u_{di} = \Delta_i^n U_d/\sqrt{\Delta_n}$ and $\widehat{u}_{di} = \widehat{\Delta_i^n U}_d/\sqrt{\Delta_n}$. Recall that at time $i \in I_t^n$,

$$\widehat{u}_{di} = \Delta_i^n Y_d \Delta_n^{-1/2} - \widehat{\boldsymbol{\beta}}_{dt}'\widehat{\mathbf{f}}_i, \quad \widehat{\boldsymbol{\beta}}_{dt}' = \frac{1}{k_n}\sum_{j\in I_t^n}\widehat{\mathbf{f}}_j'\Delta_j^n Y_d \Delta_n^{-1/2}.$$

step 1: bound $\max_{d,l\le p}|\frac{1}{k_n}\sum_{i\in I_t^n}(u_{di} - \widehat{u}_{di})u_{li}|$**.**

For each $d \le p$, and any fixed $i \in I_t^n$,

$$\begin{aligned}
\widehat{\boldsymbol{\beta}}_{dt}' &= \boldsymbol{\beta}_{d,i-1}'\mathbf{H}_{nt}^{-1} + \mathbf{m}_{1d} + \mathbf{m}_{2d,i} + \boldsymbol{\beta}_{d,i-1}'\mathbf{m}_3 \\
\mathbf{m}_{1d} &:= \frac{1}{k_n}\sum_{j\in I_t^n}(\Delta_n^{1/2}\alpha_{dj} + u_{dj})\widehat{\mathbf{f}}_j' \\
\mathbf{m}_{2d,i} &:= \frac{1}{k_n}\sum_{j\in I_t^n}(\boldsymbol{\beta}_{d,j-1} - \boldsymbol{\beta}_{d,i-1})'\mathbf{f}_j\widehat{\mathbf{f}}_j' \\
\mathbf{m}_3 &:= \mathbf{H}_{nt}^{-1}\frac{1}{k_n}\sum_{j\in I_t^n}(\mathbf{H}_{nt}\mathbf{f}_j - \widehat{\mathbf{f}}_j)\widehat{\mathbf{f}}_j'.
\end{aligned} \tag{E.14}$$

Here $\mathbf{m}_{1d}$ denotes the drift and the usual statistical error similar to that of OLS; $\mathbf{m}_{2d,i}$ is the time-varying dynamics of betas; $\mathbf{m}_3$ denotes the effect of estimating factors. We have

$$
\begin{aligned}
u_{di}-\widehat{u}_{di} &= -\boldsymbol{\beta}_{d,i-1}'\mathbf{f}_i-\Delta_n^{1/2}\psi_{di}+\widehat{\boldsymbol{\beta}}_{d,t}'\widehat{\mathbf{f}}_i \\
&= -\Delta_n^{1/2}\psi_{di}+\widehat{\boldsymbol{\beta}}_{d,t}'(\widehat{\mathbf{f}}_i-\mathbf{H}_{nt}\mathbf{f}_i)+(\widehat{\boldsymbol{\beta}}_{d,t}'-\boldsymbol{\beta}_{d,i-1}'\mathbf{H}_{nt}^{-1})\mathbf{H}_{nt}\mathbf{f}_i \\
&= -\Delta_n^{1/2}\psi_{di}+\widehat{\boldsymbol{\beta}}_{d,t}'\boldsymbol{\Xi}_{3,t}'\frac{1}{\sqrt{p}}(\mathbf{P}_{i-1}\boldsymbol{\beta}_{i-1}-\mathbf{P}_t\boldsymbol{\beta}_t)\mathbf{f}_i+\mathbf{m}_{1d}\mathbf{H}_{nt}\mathbf{f}_i+\mathbf{m}_{2d,i}\mathbf{H}_{nt}\mathbf{f}_i \\
&\quad +\widehat{\boldsymbol{\beta}}_{d,t}'\boldsymbol{\Xi}_{3,t}'\left(\frac{1}{\sqrt{p}}\Delta_n^{1/2}\mathbf{P}_{i-1}\boldsymbol{\psi}_i+\frac{1}{\sqrt{p}}\mathbf{P}_{i-1}\mathbf{u}_i\right)+\boldsymbol{\beta}_{d,i-1}'\mathbf{m}_3\mathbf{H}_{nt}\mathbf{f}_i. \qquad \text{(E.15)}
\end{aligned}
$$

To bound terms related to $\mathbf{m}_3$, we need the expansion of the estimated factors. Therefore recall (E.7),

$$
\widehat{\mathbf{f}}_i-\mathbf{H}_{nt}\mathbf{f}_i=\boldsymbol{\Xi}_{3,t}'\left(\frac{1}{\sqrt{p}}\Delta_n^{1/2}\mathbf{P}_{i-1}\boldsymbol{\psi}_i+\frac{1}{\sqrt{p}}\mathbf{P}_{i-1}\mathbf{u}_i\right)+\boldsymbol{\Xi}_{3,t}'\frac{1}{\sqrt{p}}(\mathbf{P}_{i-1}\boldsymbol{\beta}_{i-1}-\mathbf{P}_t\boldsymbol{\beta}_t)\mathbf{f}_i.
$$

Lemma E.4 shows $\|\boldsymbol{\Xi}_{3,t}\|=O_P(1)=\|\mathbf{H}_{nt}\|$. So

$$
\begin{aligned}
\|\frac{1}{k_n}\sum_{i\in I_t^n}(\widehat{\mathbf{f}}_i-\mathbf{H}_{nt}\mathbf{f}_i)\mathbf{f}_i'\| &\leq O_P(1)\|\frac{1}{k_n}\sum_{i\in I_t^n}\frac{1}{\sqrt{p}}\Delta_n^{1/2}\mathbf{P}_{i-1}\boldsymbol{\psi}_i\mathbf{f}_i'\| \\
&\quad +O_P(1)\|\frac{1}{k_n}\sum_{i\in I_t^n}\frac{1}{\sqrt{p}}\mathbf{P}_{i-1}\mathbf{u}_i\mathbf{f}_i'\|+O_P(1)\|\frac{1}{k_n}\sum_{i\in I_t^n}\frac{1}{\sqrt{p}}(\mathbf{P}_{i-1}\boldsymbol{\beta}_{i-1}-\mathbf{P}_t\boldsymbol{\beta}_t)\mathbf{f}_i\mathbf{f}_i'\| \\
&= O_P(\sqrt{\frac{J\Delta_n}{k_n}}+\sqrt{\frac{J}{k_np}}+\sqrt{k_n\Delta_n}). \\
\frac{1}{k_n}\sum_{i\in I_t^n}\|\widehat{\mathbf{f}}_i-\mathbf{H}_{nt}\mathbf{f}_i\|^2 &\leq O_P(1)\frac{1}{k_n}\sum_{i\in I_t^n}\|\frac{1}{\sqrt{p}}\Delta_n^{1/2}\mathbf{P}_{i-1}\boldsymbol{\psi}_i\|^2 \\
&\quad +O_P(1)\frac{1}{k_n}\sum_{i\in I_t^n}\|\frac{1}{\sqrt{p}}\mathbf{P}_{i-1}\mathbf{u}_i\|^2+O_P(1)\frac{1}{k_n}\sum_{i\in I_t^n}\|\frac{1}{\sqrt{p}}(\mathbf{P}_{i-1}\boldsymbol{\beta}_{i-1}-\mathbf{P}_t\boldsymbol{\beta}_t)\mathbf{f}_i\|^2 \\
&= O_P(\Delta_nk_n+\frac{J}{p}). \qquad \text{(E.16)}
\end{aligned}
$$

In addition, by Lemma E.6,

$$
\begin{aligned}
&\max_{d\leq p}\|\frac{1}{k_n}\sum_{j\in I_t^n}u_{dj}(\widehat{\mathbf{f}}_j-\mathbf{H}_{nt}\mathbf{f}_j)\|\leq O_P(\frac{J}{p}\max_{j,d}\frac{1}{J}\|\boldsymbol{\phi}(\mathbf{x}_{jd})\|^2)+o_P(\sqrt{\frac{\log p}{k_n}}) \\
&\max_{d\leq p}\|\frac{1}{k_n}\sum_{j\in I_t^n}\Delta_n^{1/2}\alpha_{dj}(\widehat{\mathbf{f}}_j-\mathbf{H}_{nt}\mathbf{f}_j)'\|=o_P(\sqrt{\frac{\log p}{k_n}}). \qquad \text{(E.17)}
\end{aligned}
$$

Therefore, terms of $\mathbf{m}_3$ and $\mathbf{m}_{1d}$ are bounded by

$$
\begin{aligned}
\|\mathbf{m}_3\| &\leq O_P(1)\|\frac{1}{k_n}\sum_{i\in I_t^n}(\widehat{\mathbf{f}}_i - \mathbf{H}_{nt}\mathbf{f}_i)\mathbf{f}_i'\| + O_P(1)\frac{1}{k_n}\sum_{i\in I_t^n}\|\widehat{\mathbf{f}}_i - \mathbf{H}_{nt}\mathbf{f}_i\|^2 \\
&\leq O_P(\sqrt{\frac{J\Delta_n}{k_n}} + \sqrt{\frac{J}{k_n p}} + \sqrt{\Delta_n k_n} + \frac{J}{p}), \\
\max_{d\leq p}\|\mathbf{m}_{1d}\| &\leq O_P(1)\max_{d\leq p}\|\frac{1}{k_n}\sum_{j\in I_t^n}(\Delta_n^{1/2}\alpha_{dj} + u_{dj})\mathbf{f}_j'\| \\
&\quad + \max_{d\leq p}\|\frac{1}{k_n}\sum_{j\in I_t^n}(\Delta_n^{1/2}\alpha_{dj} + u_{dj})(\widehat{\mathbf{f}}_j - \mathbf{H}_{nt}\mathbf{f}_j)'\| \\
&\leq O_P(\frac{J}{p}\max_{j,d}\frac{1}{J}\|\boldsymbol{\phi}(\mathbf{x}_{jd})\|^2) + O_P(\sqrt{\frac{\log p}{k_n}}) \qquad \text{(E.18)}
\end{aligned}
$$

For the term related to $\mathbf{m}_{2d,i}$ in (E.15), we have

$$
\begin{aligned}
\max_{d,l\leq p}|\frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{m}_{2d,i}\mathbf{f}_i u_{li}| &\leq \max_{d,l\leq p}|\frac{1}{k_n}\sum_{i\in I_t^n}\frac{1}{k_n}\sum_{j\in I_t^n}(\boldsymbol{\beta}_{d,j-1} - \boldsymbol{\beta}_{d,i-1})'\mathbf{f}_j\widehat{\mathbf{f}}_j'\mathbf{f}_i u_{li}| \\
&\leq \max_{d\leq p}\max_{j\in I_t^n}\|\boldsymbol{\beta}_{d,j-1} - \boldsymbol{\beta}_{dt}\|\max_{l\leq p}\|\frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{f}_i u_{li}\|O_P(1) \\
&\quad + \max_{d,l\leq p}\|\frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{f}_i u_{li}(\boldsymbol{\beta}_{d,i-1} - \boldsymbol{\beta}_{dt})\|O_P(1).
\end{aligned}
$$

Note that for an large enough $M > 0$, an arbitrarily small $\epsilon > 0$, and $L^2 > \epsilon p\Delta_n k_n M$,

$$
\begin{aligned}
&\mathbb{P}(\max_{d\leq p}\max_{j\in I_t^n}\|\boldsymbol{\beta}_{d,j-1} - \boldsymbol{\beta}_{dt}\| > L) \leq p\max_{d\leq p}\mathbb{P}(\max_{j\in I_t^n}\|\boldsymbol{\beta}_{d,j-1} - \boldsymbol{\beta}_{dt}\| > L) \\
\leq\ & p\max_{d\leq p}\mathbb{E}\max_{j\in I_t^n}\|\boldsymbol{\beta}_{d,j-1} - \boldsymbol{\beta}_{dt}\|^2/L^2 \leq p\Delta_n k_n M/L^2 < \epsilon.
\end{aligned}
$$

So $\max_{d\leq p}\max_{j\in I_t^n}\|\boldsymbol{\beta}_{d,j-1} - \boldsymbol{\beta}_{dt}\| = O_P(\sqrt{pk_n\Delta_n}) = o_P(1)$. In addition, for $x = C\sqrt{\frac{\log p}{k_n}}\sqrt{pk_n\Delta_n\log k_n}$ and large enough $C > 0$,

$$
\begin{aligned}
&\mathbb{P}(\max_{d,l\leq p}\|\frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{f}_i u_{li}(\boldsymbol{\beta}_{d,i-1} - \boldsymbol{\beta}_{dt})\| > x) \leq \mathbb{P}(\max_d\frac{1}{k_n}\sum_{i\in I_t^n}\|\boldsymbol{\beta}_{d,i-1} - \boldsymbol{\beta}_{dt}\|^2 > Mpk_n\Delta_n\log k_n) \\
&+p^2\exp\left(-\frac{k_n x^2}{\max_d\frac{1}{k_n}\sum_{i\in I_t^n}\|\boldsymbol{\beta}_{d,i-1} - \boldsymbol{\beta}_{dt}\|^2}\right)1\{\max_d\frac{1}{k_n}\sum_{i\in I_t^n}\|\boldsymbol{\beta}_{d,i-1} - \boldsymbol{\beta}_{dt}\|^2 < Mpk_n\Delta_n\log k_n\}
\end{aligned}
$$

$$\leq \mathbb{P}(\max_d \max_{i\in I_t^n} \|\boldsymbol{\beta}_{d,i-1} - \boldsymbol{\beta}_{dt}\| > \sqrt{Mpk_n\Delta_n \log k_n}) + p^2 \exp\left(-\frac{k_n x^2}{Mpk_n\Delta_n \log k_n}\right)$$
$$\leq p \max_{d\leq p} \mathbb{E} \max_{j\in I_t^n} \|\boldsymbol{\beta}_{d,j-1} - \boldsymbol{\beta}_{dt}\|^2/(Mpk_n\Delta_n \log k_n) + o(1) = o(1)$$

where we used $\max_{d\leq p} \mathbb{E} \max_{j\in I_t^n} \|\boldsymbol{\beta}_{d,j-1} - \boldsymbol{\beta}_{dt}\|^2 \leq M\Delta_n k_n$. So $\max_{d,l\leq p} \|\frac{1}{k_n}\sum_{i\in I_t^n} \mathbf{f}_i u_{li}(\boldsymbol{\beta}_{d,i-1} - \boldsymbol{\beta}_{dt})\| = o_P(\sqrt{\frac{\log p}{k_n}})$. Together, $\max_{d,l\leq p} |\frac{1}{k_n}\sum_{i\in I_t^n} \mathbf{m}_{2d,i}\mathbf{f}_i u_{li}| = o_P(\sqrt{\frac{\log p}{k_n}})$.

Now we are ready to bound $\max_{d,l\leq p} |\frac{1}{k_n}\sum_{i\in I_t^n}(u_{di} - \widehat{u}_{di})u_{li}|$. Note that

$$\widehat{\boldsymbol{\beta}}'_{d,t} = \boldsymbol{\beta}'_{d,i-1}(\mathbf{H}_{nt}^{-1} + \mathbf{m}_3 - \boldsymbol{\Upsilon}'_{nt}) + \mathbf{m}_{1d} + \frac{1}{k_n}\sum_{j\in I_t^n} \boldsymbol{\beta}'_{d,j}\mathbf{f}_j\widehat{\mathbf{f}}'_j.$$

So $\max_d \|\widehat{\boldsymbol{\beta}}'_{d,t}\| = O_P(1)$. By (E.15),

$$\max_{d,l\leq p} |\frac{1}{k_n}\sum_{i\in I_t^n}(\widehat{u}_{di} - u_{di})u_{li}| \leq \max_{d,l\leq p} |\frac{1}{k_n}\sum_{i\in I_t^n}\Delta_n^{1/2}\alpha_{di}u_{li}| + \max_{d,l\leq p} |\frac{1}{k_n}\sum_{i\in I_t^n}\widehat{\boldsymbol{\beta}}'_{d,t}(\widehat{\mathbf{f}}_i - \mathbf{H}_{nt}\mathbf{f}_i)u_{li}|$$
$$+ \max_{d,l\leq p} |\frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{m}_{1d}\mathbf{H}_{nt}\mathbf{f}_i u_{li}| + \max_{d,l\leq p} |\frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{m}_{2d,i}\mathbf{H}_{nt}\mathbf{f}_i u_{li}| + \max_{d,l\leq p} |\frac{1}{k_n}\sum_{i\in I_t^n}\boldsymbol{\beta}'_{d,i-1}\mathbf{m}_3\mathbf{H}_{nt}\mathbf{f}_i u_{li}|$$
$$\leq \max_{d,l\leq p} |\frac{1}{k_n}\sum_{i\in I_t^n}\Delta_n^{1/2}\alpha_{di}u_{li}| + \max_{d\leq p}\|\widehat{\boldsymbol{\beta}}_{d,t}\| \max_{l\leq p} \|\frac{1}{k_n}\sum_{i\in I_t^n}(\widehat{\mathbf{f}}_i - \mathbf{H}_{nt}\mathbf{f}_i)u_{li}\|$$
$$+ O_P(1)[\max_{d\leq p}\|\mathbf{m}_{1d}\| + \|\mathbf{m}_3\|] \max_{l\leq p} \|\frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{f}_i u_{li}\| + O_P(1) \max_{d,l\leq p} |\frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{m}_{2d,i}\mathbf{f}_i u_{li}|$$
$$\leq o_P(\sqrt{\frac{\log p}{k_n}}) + O_P(\frac{J}{p}\max_{j,d}\frac{1}{J}\|\boldsymbol{\phi}(\mathbf{x}_{jd})\|^2).$$

step 2: bound $\max_{d\leq p} \frac{1}{k_n}\sum_{i\in I_t^n}(u_{di} - \widehat{u}_{di})^2$.

By (E.15) $\max_{d\leq p} \frac{1}{k_n}\sum_{i\in I_t^n}(u_{di} - \widehat{u}_{di})^2$ is upper bounded by

$$\max_{d\leq p} \frac{1}{k_n}\sum_{i\in I_t^n}|\Delta_n^{1/2}\alpha_{di}|^2 + \frac{1}{k_n}\sum_{i\in I_t^n}\|\widehat{\mathbf{f}}_i - \mathbf{H}_{nt}\mathbf{f}_i\|^2 + \max_{d\leq p}\|\mathbf{m}_{1d}\|^2 + \max_{d\leq p}\frac{1}{k_n}\sum_{i\in I_t^n}|\mathbf{m}_{2d,i}\mathbf{H}_{nt}\mathbf{f}_i|^2 + \|\mathbf{m}_3\|^2$$
$$\leq O_P(\Delta_n k_n + \frac{1}{p} + \frac{J^2}{p^2}\max_j \frac{1}{J^2}\|\boldsymbol{\phi}(\mathbf{x}_{jd})\|^4 + \frac{\log p}{k_n}) = o_P(\sqrt{\frac{\log p}{k_n}}) + o_P(\frac{J}{p}\max_{j,d}\frac{1}{J}\|\boldsymbol{\phi}(\mathbf{x}_{jd})\|^2)$$

step 3: bound $\max_{j\in I_t^n}\max_{d,l\leq p}|s_{dl} - c_{u,j,dl}|$.

We have $s_{dl} = \frac{1}{k_n}\sum_{i\in I_t^n}\widehat{u}_{li}\widehat{u}_{di}$.

$$\max_{d,l\leq p} |\frac{1}{k_n}\sum_{i\in I_t^n}(\widehat{u}_{li}\widehat{u}_{di} - u_{li}u_{di})| \leq 2\max_{d,l\leq p} |\frac{1}{k_n}\sum_{i\in I_t^n}(\widehat{u}_{di} - u_{di})u_{li}| + \max_{d\leq p}\frac{1}{k_n}\sum_{i\in I_t^n}(\widehat{u}_{di} - u_{di})^2$$

$$\leq \quad o_P(\sqrt{\frac{\log p}{k_n}}) + O_P(\frac{J}{p}\max_{j,d}\frac{1}{J}\|\boldsymbol{\phi}(\mathbf{x}_{jd})\|^2)$$

Because $\Delta_i^n U_l/\sqrt{\Delta_n}$ is Gaussian with mean $O(\sqrt{\Delta_n})$ and variance $c_{u,i,ll}$, we have with probability approaching one, $\max_{d,l\leq p}|\frac{1}{k_n}\sum_{i\in I_t^n}(u_{di}u_{li}-\mathbb{E}u_{di}u_{li})| < C\sqrt{\frac{\log p}{k_n}}$. Hence

$$\begin{aligned}\max_{j\in I_t^n}\max_{d,l\leq p}|\frac{1}{k_n}\sum_{i\in I_t^n}u_{di}u_{li}-c_{u,j,dl}| &\leq C\sqrt{\frac{\log p}{k_n}}+\max_{d,l\leq p}\max_{i\in I_t^n}|\frac{1}{k_n}\sum_{j\in I_t^n}c_{u,i,dl}-c_{u,j,dl}| \\ &\leq C\sqrt{\frac{\log p}{k_n}}+L\sqrt{\Delta_n k_n}\leq C\sqrt{\frac{\log p}{k_n}}.\end{aligned}$$

Thus $\max_{d,l\leq p}|s_{dl}-c_{u,j,dl}| \leq C\sqrt{\frac{\log p}{k_n}} + C\frac{J}{p}\max_{j,d}\frac{1}{J}\|\boldsymbol{\phi}(\mathbf{x}_{jd})\|^2\sqrt{\log J}$ with probability approaching one. In addition, $\min_{d\leq p, j\in I_t^n} c_{u,j,dd}$ is bounded away from zero, we have

$$\max_{d,l\leq p}\frac{|s_{dl}-c_{u,j,dl}|}{(s_{dd}s_{ll})^{1/2}}\leq C\sqrt{\frac{\log p}{k_n}}+C\frac{J}{p}\max_{j,d}\frac{1}{J}\|\boldsymbol{\phi}(\mathbf{x}_{jd})\|^2\sqrt{\log J}=C\omega_{np}\leq\frac{1}{2}\bar{C}\omega_{np}.$$

This completes the proof.

(ii) Let $\widehat{c}_{u,t,dl}$ denote the (d,l) th entry of $\widehat{\mathbf{c}}_{uu,t}$. Note that $\widehat{c}_{u,t,dl}=0$ if $|s_{dl}|<\varrho_{dl}$. Thus

$$\begin{aligned}\max_{i\in I_t^n}\|\widehat{\mathbf{c}}_{uu,t}-\mathbf{c}_{uu,i}\| &\leq \max_{i\in I_t^n}\max_{l\leq p}\sum_{d\leq p}|\widehat{c}_{u,t,dl}-c_{u,i,dl}| \\ &= \max_{i\in I_t^n}\max_{l\leq p}\sum_{d\leq p}|\widehat{c}_{u,t,dl}1\{|s_{dl}|>\varrho_{dl}\}-c_{u,i,dl}1\{s_{dl}>\varrho_{dl}\}-c_{u,i,dl}1\{s_{dl}<\varrho_{dl}\}| \\ &\leq \max_{i\in I_t^n}\max_{l\leq p}\sum_{d\leq p}|\widehat{c}_{u,t,dl}-s_{dl}|1\{|s_{dl}|>\varrho_{dl}\}+\max_{i\in I_t^n}\max_{l\leq p}\sum_{d\leq p}|c_{u,i,dl}-s_{dl}|1\{|s_{dl}|>\varrho_{dl}\} \\ &\quad +\max_{i\in I_t^n}\max_{l\leq p}\sum_{d\leq p}|c_{u,i,dl}|1\{|s_{dl}|<\varrho_{dl}\}.\end{aligned}$$

Conditioning on the event $|s_{dl}-c_{u,i,dl}|\leq 0.5\varrho_{dl}$, which holds with probability approaching one uniformly in (d,l,i) given part (i) of this lemma, we have

$$1\{|s_{dl}|>\varrho_{dl}\}\leq 1\{|c_{u,i,dl}|>\varrho_{dl}/2\},\quad 1\{|s_{dl}|<\varrho_{dl}\}\leq 1\{|c_{u,i,dl}|<3\varrho_{dl}/2\}.$$

For both hard and soft thresholding, $|\widehat{c}_{u,t,dl} - s_{dl}| = |\text{th}(s_{dl}) - s_{dl}| \leq \varrho_{dl} \leq C\omega_{np}$. Hence

$$\begin{aligned}
\max_{i\in I_t^n}\max_{l\leq p}\sum_{d\leq p}|\widehat{c}_{u,t,dl} - s_{dl}|1\{|s_{dl}|>\varrho_{dl}\} &\leq C\omega_{np}\max_{i\in I_t^n}\max_{l\leq p}\sum_{d\leq p}1\{|c_{u,i,dl}|>\varrho_{dl}/2\}\leq C\omega_{np}m_{p,t}\\
\max_{i\in I_t^n}\max_{l\leq p}\sum_{d\leq p}|c_{u,i,dl} - s_{dl}|1\{|s_{dl}|>\varrho_{dl}\} &\leq C\omega_{np}\max_{i\in I_t^n}\max_{l\leq p}\sum_{d\leq p}1\{|c_{u,i,dl}|>\varrho_{dl}/2\}\leq C\omega_{np}m_{p,t}\\
\max_{i\in I_t^n}\max_{l\leq p}\sum_{d\leq p}|c_{u,i,dl}|1\{|s_{dl}|<\varrho_{dl}\} &\leq \max_{i\in I_t^n}\max_{l\leq p}\sum_{d\leq p}|c_{u,i,dl}|1\{|c_{u,i,dl}|<3\varrho_{dl}/2\}\\
&\leq \max_{i\in I_t^n}\max_{l\leq p}\sum_{d\leq p}C|\varrho_{dl}|1\{|c_{u,i,dl}|\neq 0\}\leq C\omega_{np}m_{p,t}.
\end{aligned}$$

Putting together, $\max_{i\in I_t^n}\|\widehat{\mathbf{c}}_{uu,t} - \mathbf{c}_{uu,i}\| \leq C\omega_{np}m_{p,t}$ with probability approaching one.

E.3 Proof of Theorem 3.5: long-run g

Based on Proposition E.2 below, the proof of the uniform coverage property for $\int_0^T \mathbf{g}_{lt}dt$ follows from the same proof of Theorem 4.2.

Proposition E.2. *Suppose*

$$\frac{1}{p}\sum_{m=1}^{p}\frac{1}{[T/\Delta_n]}\sum_{t=1}^{[T/\Delta_n]-k_n}\mathbb{E}(\|\boldsymbol{\gamma}_{mt}\|^4 h_{t,ml}^4|\{\mathbf{X}_t\}) \leq C\lambda_{\min}^2\left[\frac{1}{p}\sum_{m=1}^{p}\text{Var}(\frac{1}{[T/\Delta_n]}\sum_{t=1}^{[T/\Delta_n]-k_n}\boldsymbol{\gamma}_{mt}h_{t,ml}|\{\mathbf{X}_t\})\right] \tag{E.19}$$

Let $\bar{\boldsymbol{\Sigma}}_n = \frac{\Delta_n}{p}\mathbf{V}_u + \frac{1}{p}\mathbf{V}_\gamma$, *where*

$$\begin{aligned}
\mathbf{V}_u &= \text{Var}\left[\frac{\sqrt{p}}{k_n\sqrt{\Delta_n}}\sum_{t=1}^{[T/\Delta_n]-k_n}\widehat{\mathbf{c}}_{FF,t}^{-1}\sum_{i\in I_t^n}\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{U}'\mathbf{P}_{i-1,l}\middle|\{\mathbf{f}_t,\mathbf{X}_t\}\right]\\
\mathbf{V}_\gamma &= \text{Var}\left[\sum_{t=1}^{[T/\Delta_n]-k_n}\frac{1}{\sqrt{p}}\sum_{m=1}^{p}\boldsymbol{\gamma}_{mt}h_{t,ml}\Delta_n\middle|\{\mathbf{X}_t\}\right].
\end{aligned}$$

Then (i)

$$(\mathbf{v}'\boldsymbol{\Sigma}_n\mathbf{v})^{-1/2}\,\mathbf{v}'\left(\widehat{\int_0^T \mathbf{g}_{lt}dt} - \int_0^T \mathbf{g}_{lt}dt\right)\overset{\mathcal{L}\text{-}s}{\to} N(0,1),$$

(ii) In the bootstrap sampling space

$$\left(\mathbf{v}'\bar{\boldsymbol{\Sigma}}_n\mathbf{v}\right)^{-1/2}\mathbf{v}'\left(\widehat{\int_0^T \mathbf{g}_{lt}dt}^{\,*} - \widehat{\int_0^T \mathbf{g}_{lt}dt}\right) \xrightarrow{d^*} N(0,1).$$

Proof. (i) Recall $\widehat{\mathbf{c}}_{FF,t} = \frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{F}'$ and $\widehat{\int_0^T \mathbf{g}_{lt}dt} = \sum_{t=1}^{[T/\Delta_n]-k_n}\widehat{\mathbf{g}}_{lt}\Delta_n$. Then

$$\begin{aligned}
\widehat{\int_0^T \mathbf{g}_{lt}dt} - \int_0^T \mathbf{g}_{lt}dt &= \sum_{t=1}^{[T/\Delta_n]-k_n}(\widehat{\mathbf{g}}_{lt}-\mathbf{g}_{lt})\Delta_n + \sum_{t=1}^{[T/\Delta_n]-k_n}\mathbf{g}_{lt}\Delta_n - \int_0^T \mathbf{g}_{lt}dt \\
&= \sum_{t=1}^{[T/\Delta_n]-k_n}\Delta_n\boldsymbol{\Gamma}_t'\mathbf{P}_{t,l} + \sum_{t=1}^{[T/\Delta_n]-k_n}\Delta_n\widehat{\mathbf{c}}_{FF,t}^{-1}\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{U}'\mathbf{P}_{t,l} + \sum_{d=1}^{5}\mathbf{A}_d
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{A}_1 &= \sum_{t=1}^{[T/\Delta_n]-k_n}\Delta_n\widehat{\mathbf{c}}_{FF,t}^{-1}\frac{1}{k_n}\sum_{i\in I_t^n}\Delta_i^n\mathbf{F}\boldsymbol{\psi}_i'\mathbf{P}_{t,l} \\
\mathbf{A}_2 &= \sum_{t=1}^{[T/\Delta_n]-k_n}[\mathbf{G}_t'\mathbf{P}_{t,l} - \mathbf{g}_{lt}] \\
\mathbf{A}_3 &= \sum_{t=1}^{[T/\Delta_n]-k_n}\Delta_n\widehat{\mathbf{c}}_{FF,t}^{-1}\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{F}'[\mathbf{G}_{i-1}-\mathbf{G}_t]'\mathbf{P}_{t,l} \\
\mathbf{A}_4 &= \sum_{t=1}^{[T/\Delta_n]-k_n}\Delta_n\widehat{\mathbf{c}}_{FF,t}^{-1}\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{F}'[\boldsymbol{\Gamma}_{i-1}-\boldsymbol{\Gamma}_t]'\mathbf{P}_{t,l} \\
\mathbf{A}_5 &= \sum_{t=1}^{[T/\Delta_n]-k_n}\mathbf{g}_{lt}\Delta_n - \int_0^T \mathbf{g}_{lt}dt
\end{aligned}$$

It follows from Lemma E.10, $\sqrt{p/\Delta_n}(\mathbf{A}_1+\mathbf{A}_2+\mathbf{A}_3+\mathbf{A}_4) = o_P(1)$. As for $\mathbf{A}_5$, it is easy to see that this is the approximation error of a Riemann integral. Since $\{\mathbf{g}_{lt}\}_{t\geq 0}$ is bounded according to the localization argument, $\mathbf{A}_5 = O_P(\Delta_n) = o_P(\sqrt{\Delta_n/p})$. Therefore

$$\begin{aligned}
\widehat{\int_0^T \mathbf{g}_{lt}dt} - \int_0^T \mathbf{g}_{lt}dt &= \mathbf{a}_1 + \mathbf{a}_2 + o_P(\sqrt{\frac{\Delta_n}{p}}), \\
\mathbf{a}_1 &= \frac{1}{p}\sum_{m=1}^{p}\sum_{t=1}^{[T/\Delta_n]-k_n}\Delta_n\widehat{\mathbf{c}}_{FF,t}^{-1}\frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{f}_i u_{mi} h_{i-1,ml}
\end{aligned}$$

$$\mathbf{a}_2 \;=\; \frac{1}{p}\sum_{m=1}^{p}\sum_{t=1}^{[T/\Delta_n]-k_n} \Delta_n \boldsymbol{\gamma}_{t,m} h_{t,ml}$$

Note that $\left[\frac{1}{p}\mathbf{v}'\mathbf{V}_\gamma\mathbf{v}\right]^{-1/2}\mathbf{v}'\mathbf{a}_2 \stackrel{\mathcal{L}\text{-}s}{\to} N(0,1)$, due to cross-sectional independence of $\boldsymbol{\gamma}_{mt}$.

We now verify the CLT for $\mathbf{a}_1$. For $t = 1, \cdots, [T/(k_n\Delta_n)]$ and $q = 1, \cdots, k_n$, let

$$\tilde{\xi}_{t,q} = \frac{1}{p}\sum_{m=1}^{p}\widehat{\mathbf{c}}^{-1}_{FF,tk_n+q}\frac{1}{k_n}\sum_{i=1}^{k_n}\mathbf{f}_{tk_n+q+i}u_{m,tk_n+q+i}h_{tk_n+q+i-1,ml}\, k_n\Delta_n$$

$$\xi_{t,q} = \frac{1}{p}\sum_{m=1}^{p}\mathbf{c}^{-1}_{FF,tk_n+q}\frac{1}{k_n}\sum_{i=1}^{k_n}\mathbf{f}_{tk_n+q+i}u_{m,tk_n+q+i}h_{tk_n+q+i-1,ml}\, k_n\Delta_n$$

Note that

$$\begin{aligned}\mathbf{a}_1 &= \frac{1}{k_n}\sum_{q=1}^{k_n}\sum_{t=1}^{[T/(k_n\Delta_n)]-2k_n}\widehat{\mathbf{c}}^{-1}_{FF,tk_n+q}\frac{1}{k_n}\sum_{i=1}^{k_n}\mathbf{f}_{tk_n+q+i}u_{m,tk_n+q+i}h_{tk_n+q+i-1,ml}\, k_n\Delta_n \\ &= \frac{1}{k_n}\sum_{q=1}^{k_n}\sum_{t=1}^{[T/(k_n\Delta_n)]-2k_n}\tilde{\xi}_{t,q}.\end{aligned}$$

We are going to show that for any constant vector $\mathbf{v}$ and any q that

$$\sqrt{\frac{p}{\Delta_n}}\sum_{t=1}^{[T/(k_n\Delta_n)]-k_n-q}\mathbf{v}'\xi_{t,q}$$

converges stably in law to a random variable, which, conditional on $\mathcal{F}$, is centered Gaussian with variance given by $\mathbf{v}'\mathbf{V}_u\mathbf{v}$.

For notation simplicity, we let the dimension of F to be 1, $q = 0$ and write $\xi^n_{t,0}$ as ξ^n_t. First of all, since $\mathbf{f}$ and u are independent, it is easy to verify by using iterated expectation that

$$\begin{aligned}&\sqrt{\frac{p}{\Delta_n}}\sum_{t=1}^{[T/(k_n\Delta_n)]-k_n}\mathbb{E}\Big(\mathbf{v}'\,\xi^n_t \,\Big|\, \mathcal{F}^n_{tk_n}\Big)\\ =&\sqrt{\frac{p}{\Delta_n}}\sum_{t=1}^{[T/(k_n\Delta_n)]-k_n}\mathbf{v}'\,\mathbf{c}^{-1}_{FF,tk_n}\mathbb{E}\Big(\frac{1}{p}\sum_{m=1}^{p}\frac{1}{k_n}\sum_{i=1}^{k_n}\mathbf{f}_{tk_n+i}u_{m,tk_n+i}h_{tk_n+i-1,ml}\,\Big|\,\mathcal{F}^n_{tk_n}\Big)\,k_n\Delta_n\end{aligned}$$

$$= \sqrt{\frac{p}{\Delta_n}} \sum_{t=1}^{[T/(k_n\Delta_n)]-k_n} \mathbf{v}'\, \mathbf{c}_{FF,tk_n}^{-1} \mathbb{E}\Big(\frac{1}{p} \sum_{m=1}^{p} \frac{1}{k_n} \sum_{i=1}^{k_n} \mathbb{E}\big(\mathbf{f}_{tk_n+i} u_{m,tk_n+i} \,|\, \mathcal{F}_{ik_n+i-1}^n \big) h_{tk_n+i-1,ml} \,\big|\, \mathcal{F}_{tk_n}^n \Big)\, k_n\Delta_n$$

$$= 0.$$

Note that if $j > i$, then we have

$$\mathbb{E}\big(\mathbf{f}_i u_{mi} u'_{mj} \mathbf{f}_i \,|\, \mathcal{F}_{i-1}^n\big) = \mathbb{E}\big(\mathbf{f}_i u_{ni}\, \mathbb{E}(u'_{mj}\mathbf{f}_j \,|\, \mathcal{F}_{j-1}^n) \,|\, \mathcal{F}_{i-1}^n\big) = 0.$$

Following this and the cross-sectional independence of u_m, we can deduce that

$$\begin{aligned}
&\frac{p}{\Delta_n} \sum_{t=1}^{[T/(k_n\Delta_n)]-k_n} \mathbb{E}\Big(\mathbf{v}'\, \xi_t^n\, \xi_t^{n\prime}\, \mathbf{v} \Big|\, \mathcal{F}_{tk_n}^n\Big) \\
&= \frac{p}{\Delta_n} \sum_{t=1}^{[T/(k_n\Delta_n)]-k_n} \mathbf{v}'\, \mathbf{c}_{FF,tk_n}^{-1} \mathbb{E}\Big(\frac{1}{p^2} \sum_{m,m'=1}^{p} \frac{1}{k_n^2} \sum_{i=1}^{k_n} \mathbb{E}\big(\mathbf{f}_{tk_n+i} u_{m,tk_n+i} h_{tk_n+i-1,ml} \\
&\quad \times h_{tk_n+i-1,m'l} u'_{m',tk_n+i} \mathbf{f}_{tk_n+i} \,|\, \mathcal{F}_{ik_n+i-1}^n \big) \,\big|\, \mathcal{F}_{tk_n}^n \Big) \mathbf{c}_{FF,tk_n}^{-1} \mathbf{v}\, k_n^2 \Delta_n^2 \\
&= \sum_{t=1}^{[T/(k_n\Delta_n)]-k_n} \mathbf{v}'\, \mathbf{c}_{FF,tk_n}^{-1} \mathbb{E}\Big(\frac{1}{p} \sum_{m=1}^{p} \frac{1}{k_n} \sum_{i=1}^{k_n} \mathbb{E}\big(\mathbf{f}_{tk_n+i} u_{m,tk_n+i} h_{tk_n+i-1,ml} \\
&\quad \times h_{tk_n+i-1,ml} u'_{m,tk_n+i} \mathbf{f}_{tk_n+i} \,|\, \mathcal{F}_{ik_n+i-1}^n \big) \,\big|\, \mathcal{F}_{tk_n}^n \Big) \mathbf{c}_{FF,tk_n}^{-1} \mathbf{v}\, k_n \Delta_n \\
&\xrightarrow{P} \mathbf{V}_u.
\end{aligned}$$

Similarly, using the independence of $\mathbf{f}$ and u and the cross-sectional independence of u_m, it can also be proved by tedious calculation that

$$\begin{aligned}
&\frac{p^2}{\Delta_n^2} \sum_{t=1}^{[T/(k_n\Delta_n)]-k_n} \mathbb{E}\Big(\big(\mathbf{v}'\, \xi_t^n\big)^4 \Big|\, \mathcal{F}_{tk_n}^n \Big) \xrightarrow{P} 0 \\
&= \frac{p^2}{\Delta_n^2} \sum_{t=1}^{[T/(k_n\Delta_n)]-k_n} \mathbb{E}\Big(\frac{1}{p^4} \sum_{m,m'=1}^{p} \frac{1}{k_n^4} \sum_{i,i'=1}^{k_n} \mathbb{E}\big((\mathbf{v}'\, \mathbf{c}_{FF,tk_n}^{-1} \mathbf{f}_{tk_n+i} u_{m,tk_n+i} h_{tk_n+i-1,ml})^2 \\
&\quad \times (\mathbf{v}'\, \mathbf{c}_{FF,tk_n}^{-1} \mathbf{f}_{tk_n+i'} u_{m',tk_n+i'} h_{tk_n+i'-1,m'l})^2 \,|\, \mathcal{F}_{ik_n+i-1}^n \big) \,\big|\, \mathcal{F}_{tk_n}^n \Big) k_n^4 \Delta_n^4 \\
&\leq \sum_{t=1}^{[T/(k_n\Delta_n)]-k_n} L k_n^2 \Delta_n^2 \leq L k_n \Delta_n \longrightarrow 0,
\end{aligned}$$

where the first inequality comes from the strengthened assumption that all the stochastic processes are bounded on $[0, T]$.

For any continuous martingale M defined on the same filtered probability space. Generally speaking, M can be decomposed into three parts: one correlated with F, one correlated with U, the last one is not correlated with both F and U. For each component, one can check that (here we use M to represent its component for simplicity)

$$\frac{p}{\Delta_n}\sum_{t=1}^{[T/(k_n\Delta_n)]-k_n}\mathbb{E}\Big(\mathbf{v}'\,\xi_t^n\left(M_{(t+1)k_n\Delta_n}-M_{tk_n\Delta_n}\right)\Big|\,\mathcal{F}_{tk_n}^n\Big)=0.$$

Then, according to Theorems 2.2.13 and 2.2.15 in Jacod and Protter (2011), we get the above desired result.

Moreover, note that

$$\sqrt{\frac{p}{\Delta_n}}\sum_{t=1}^{[T/(k_n\Delta_n)]-k_n-q}\mathbf{v}'\xi_{t,q}=\sqrt{\frac{p}{\Delta_n}}\sum_{i=1}^{[T/\Delta_n]-k_n}\mathbf{c}_{FF,[(i-q)/k_n]*k_n+q}\mathbf{f}_i u_i h_{i-1,ml}k_n\Delta_n.$$

That is, for different q, all such sums are subject to the same innovation sequences f and u, with asymptotically same weights. Hence, the following average

$$\sqrt{\frac{p}{\Delta_n}}\frac{1}{k_n}\sum_{q=1}^{k_n}\sum_{t=1}^{[T/(k_n\Delta_n)]-k_n-q}\mathbf{v}'\xi_{t,q}$$

converges to the same limit.

What left to show now is

$$\sqrt{\frac{p}{\Delta_n}}\sum_{t=1}^{[T/(k_n\Delta_n)]-k_n-q}\mathbf{v}'\big(\tilde{\xi}_{t,q}-\xi_{t,q}\big)\xrightarrow{\mathbb{P}}0$$

We leave this to Lemma E.10 (v).

(ii) We have

$$\begin{aligned}\widehat{\int_0^T\mathbf{g}_{lt}dt}^{\,*}-\int_0^T\mathbf{g}_{lt}dt &= \sum_{t=1}^{[T/\Delta_n]-k_n}(\widehat{\mathbf{g}}_{lt}^*-\mathbf{g}_{lt})\Delta_n+\sum_{t=1}^{[T/\Delta_n]-k_n}\mathbf{g}_{lt}\Delta_n-\int_0^T\mathbf{g}_{lt}dt\\ &= \Delta_n\sum_{t=1}^{[T/\Delta_n]-k_n}\widehat{\mathbf{c}}_{FF,t}^{-1}\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{U}^{*\prime}\mathbf{P}_{i-1,l}^*+\Delta_n\sum_{t=1}^{[T/\Delta_n]-k_n}\mathbf{\Gamma}_t^{*\prime}\mathbf{P}_{t,l}^*\end{aligned}$$

$$
+\sum_{d=1}^{5}\mathbf{A}_d^* \tag{E.20}
$$

where

$$
\begin{aligned}
\mathbf{A}_1^* &= \Delta_n \sum_{t=1}^{[T/\Delta_n]-k_n} \widehat{\mathbf{c}}_{FF,t}^{-1} \frac{1}{k_n} \sum_{i\in I_t^n} \Delta_i^n \mathbf{F} \boldsymbol{\psi}_i^{*'} \mathbf{P}_{t,l}^* \\
\mathbf{A}_2^* &= \Delta_n \sum_{t=1}^{[T/\Delta_n]-k_n} [\mathbf{G}_t^{*'} \mathbf{P}_{t,l}^* - \mathbf{g}_{lt}] \\
\mathbf{A}_3^* &= \Delta_n \sum_{t=1}^{[T/\Delta_n]-k_n} \widehat{\mathbf{c}}_{FF,t}^{-1} \frac{1}{k_n\Delta_n} \sum_{i\in I_t^n} \Delta_i^n \mathbf{F} \Delta_i^n \mathbf{F}' [\mathbf{G}_{i-1}^* - \mathbf{G}_t^*]' \mathbf{P}_{t,l} \\
\mathbf{A}_4^* &= \Delta_n \sum_{t=1}^{[T/\Delta_n]-k_n} \widehat{\mathbf{c}}_{FF,t}^{-1} \frac{1}{k_n\Delta_n} \sum_{i\in I_t^n} \Delta_i^n \mathbf{F} \Delta_i^n \mathbf{F}' [\boldsymbol{\Gamma}_{i-1}^* - \boldsymbol{\Gamma}_t^*]' \mathbf{P}_{t,l}^* \\
\mathbf{A}_5^* &= \sum_{t=1}^{[T/\Delta_n]-k_n} \mathbf{g}_{lt}\Delta_n - \int_0^T \mathbf{g}_{lt} dt
\end{aligned}
$$

The last term $\mathbf{A}_5^*$ is the same as $\mathbf{A}_5$ above, hence is also asymptotically negligible as long as $p\Delta_n \to 0$. Lemma E.11 implies $\sqrt{p/\Delta_n}(\mathbf{A}_1^*+\mathbf{A}_2^*+\mathbf{A}_3^*+\mathbf{A}_4^*) = o_{P^*}(1)$. In addition, by Lemma E.12, the first two terms on the right hand side of (E.20) are respectively $\mathbf{a}_1^*+o_{P^*}((p/\Delta_n)^{-1/2})$, and $\mathbf{a}_2^* + o_{P^*}(p^{-1/2})\lambda_{\min}^{1/2}(\mathbf{V}_\gamma)$, where

$$
\begin{aligned}
\mathbf{a}_1^* &= \Delta_n \sum_{t=1}^{[T/\Delta_n]-k_n} \widehat{\mathbf{c}}_{FF,t}^{-1} \frac{1}{k_n\Delta_n} \sum_{i\in I_t^n} \Delta_i^n \mathbf{F} \frac{1}{p} \sum_{m=1}^{p} \Delta_i^n U_m^* \boldsymbol{\phi}_{mt}^{*'} (\frac{1}{p}\boldsymbol{\Phi}_t'\boldsymbol{\Phi}_t)^{-1} \boldsymbol{\phi}_{lt} \\
\mathbf{a}_2^* &= \Delta_n \sum_{t=1}^{[T/\Delta_n]-k_n} \frac{1}{p} \sum_{m=1}^{p} \boldsymbol{\gamma}_{mt}^* \boldsymbol{\phi}_{mt}^{*'} (\frac{1}{p}\boldsymbol{\Phi}_t'\boldsymbol{\Phi}_t)^{-1} \boldsymbol{\phi}_{lt}.
\end{aligned}
$$

Hence $\widehat{\int_0^T \mathbf{g}_{lt}dt}^* - \int_0^T \mathbf{g}_{lt}dt = \mathbf{a}_1^* + \mathbf{a}_2^* + o_{P^*}((p/\Delta_n)^{-1/2}) + o_{P^*}(p^{-1/2})\lambda_{\min}^{1/2}(\mathbf{V}_\gamma)$. We have

$$
\widehat{\int_0^T \mathbf{g}_{lt}dt}^* - \widehat{\int_0^T \mathbf{g}_{lt}dt} = (\mathbf{a}_1^* - \mathbf{a}_1) + (\mathbf{a}_2^* - \mathbf{a}_2) + o_{P^*}(\sqrt{\frac{\Delta_n}{p}}) + o_{P^*}(p^{-1/2})\lambda_{\min}^{1/2}(\mathbf{V}_{\gamma,t}).
$$

In addition, by the CLT for independent data in the bootstrap sampling space, for

$$\begin{aligned}\mathbf{V}_u^* &= \frac{1}{p}\sum_{m=1}^{p}\left[\frac{1}{\sqrt{\Delta_n}}\sum_{t=1}^{[T/\Delta_n]-k_n}\widehat{\mathbf{c}}_{FF,t}^{-1}\frac{1}{k_n}\sum_{i\in I_t^n}\Delta_i^n\mathbf{F}\Delta_i^nU_m h_{i-1,ml}\right]^2\\ \mathbf{V}_\gamma^* &= \frac{1}{p}\sum_{m=1}^{p}\left[\Delta_n\sum_{t=1}^{[T/\Delta_n]-k_n}\boldsymbol{\gamma}_{mt}h_{t,ml}\right]^2,\end{aligned}$$

we have

$$\left[\mathbf{v}'(\frac{\Delta_n}{p}\mathbf{V}_u^*+\frac{1}{p}\mathbf{V}_\gamma^*+\mathbb{E}^*(\mathbf{a}_1^*-\mathbf{a}_1)(\mathbf{a}_2^*-\mathbf{a}_2))\mathbf{v}\right]^{-1/2}\mathbf{v}'[(\mathbf{a}_1^*-\mathbf{a}_1)+(\mathbf{a}_2^*-\mathbf{a}_2)]\xrightarrow{d^*}N(0,1).$$

Now we show:

(1) $\mathbf{V}_u^*=\mathbf{V}_u(1+o_P(1))$;

(2) $\mathbf{V}_\gamma^*=\mathbf{V}_\gamma(1+o_P(1))$;

(3) $\mathbb{E}^*(\mathbf{a}_1^*-\mathbf{a}_1)(\mathbf{a}_2^*-\mathbf{a}_2)=o_P(1)\lambda_{\min}(\frac{\Delta_n}{p}\mathbf{V}_u+\frac{1}{p}\mathbf{V}_\gamma)$.

These then imply $\left[\mathbf{v}'\bar{\boldsymbol{\Sigma}}_n\mathbf{v}\right]^{-1/2}\mathbf{v}'[(\mathbf{a}_1^*-\mathbf{a}_1)+(\mathbf{a}_2^*-\mathbf{a}_2)]\xrightarrow{d^*}N(0,1)$ and thus

$$\left(\mathbf{v}'\bar{\boldsymbol{\Sigma}}_n\mathbf{v}\right)^{-1/2}\mathbf{v}'\left(\widehat{\int_0^T\mathbf{g}_{lt}dt}^{\,*}-\widehat{\int_0^T\mathbf{g}_{lt}dt}\right)\xrightarrow{d^*}N(0,1).$$

To show (1): recall that $u_{mi}:=\Delta_i^nU_m/\sqrt{\Delta_n}$. Because $\{u_{mi}\}$ is cross-sectionally uncorrelated,

$$\begin{aligned}\mathbf{V}_u &= \mathrm{Var}\left[\frac{\sqrt{p}}{k_n\sqrt{\Delta_n}}\sum_{t=1}^{[T/\Delta_n]-k_n}\widehat{\mathbf{c}}_{FF,t}^{-1}\sum_{i\in I_t^n}\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{U}'\mathbf{P}_{t,l}\middle|\{\mathbf{f}_t,\mathbf{X}_t\}\right]\\ &= \frac{\Delta_n}{k_n^2p}\sum_{m=1}^{p}\sum_{t=1}^{[T/\Delta_n]-k_n}\widehat{\mathbf{c}}_{FF,t}^{-1}\sum_{i\in I_t^n}\mathbf{f}_i\mathbf{f}_i'\widehat{\mathbf{c}}_{FF,t}^{-1}h_{t,ml}^2\mathbb{E}u_{mi}^2.\end{aligned}$$

Also,

$$\mathbf{V}_u^* = \frac{\Delta_n}{k_n^2p}\sum_{m=1}^{p}\sum_{t=1}^{[T/\Delta_n]-k_n}\widehat{\mathbf{c}}_{FF,t}^{-1}\sum_{i\in I_t^n}\mathbf{f}_i\mathbf{f}_i'\widehat{\mathbf{c}}_{FF,t}^{-1}h_{t,ml}^2u_{mi}^2$$

$$
\begin{aligned}
& +\frac{\Delta_n}{pk_n^2}\sum_{m=1}^{p}\sum_{\substack{t,s=1\\|t-s|\le k_n}}^{[T/\Delta_n]-k_n}\sum_{\substack{i\in I_t^n,j\in I_s^n\\i\neq j}}\widehat{\mathbf{c}}_{FF,t}^{-1}h_{t,ml}f_iu_{mi}\,\widehat{\mathbf{c}}_{FF,s}^{-1}h_{t,ml}f_iu_{mj} \\
& +\frac{\Delta_n}{pk_n^2}\sum_{m=1}^{p}\sum_{\substack{t,s=1\\|t-s|>k_n}}^{[T/\Delta_n]-k_n}\sum_{i\in I_t^n,j\in I_s^n}\widehat{\mathbf{c}}_{FF,t}^{-1}h_{t,ml}f_iu_{mi}\,\widehat{\mathbf{c}}_{FF,s}^{-1}h_{t,ml}f_iu_{mj} \\
= & \frac{\Delta_n}{k_n^2p}\sum_{m=1}^{p}\sum_{t=1}^{[T/\Delta_n]-k_n}\widehat{\mathbf{c}}_{FF,t}^{-1}\sum_{i\in I_t^n}\mathbf{f}_i\mathbf{f}_i'\widehat{\mathbf{c}}_{FF,t}^{-1}h_{t,ml}^2u_{mi}^2+\underbrace{o_P(1)}_{\text{Lemma E.13}}.
\end{aligned}
$$

Hence $\mathbf{V}_u^*-\mathbf{V}_u=\frac{\Delta_n}{k_n^2p}\sum_{m=1}^p\sum_{t=1}^{[T/\Delta_n]-k_n}\widehat{\mathbf{c}}_{FF,t}^{-1}\sum_{i\in I_t^n}\mathbf{f}_i\mathbf{f}_i'\widehat{\mathbf{c}}_{FF,t}^{-1}h_{t,ml}^2(u_{mi}^2-\mathbb{E}u_{mi}^2)+o_P(1)=o_P(1)$. Note that $\lambda_{\min}(\mathbf{V}_u)$ is bounded away from zero. Hence $\mathbf{V}_u^*-\mathbf{V}_u=o_P(1)\lambda_{\min}(\mathbf{V}_u)$.

To show (2): we have

$$
\begin{aligned}
\mathbf{V}_\gamma &= \operatorname{Var}\left[\Delta_n\sum_{t=1}^{[T/\Delta_n]-k_n}\frac{1}{\sqrt{p}}\sum_{m=1}^{p}\boldsymbol{\gamma}_{mt}h_{t,ml}\Big|\{\mathbf{X}_t\}\right] \\
&= \frac{\Delta_n^2}{p}\sum_{t=1}^{[T/\Delta_n]-k_n}\sum_{m=1}^{p}\sum_{s=1}^{[T/\Delta_n]-k_n}\mathbb{E}(\boldsymbol{\gamma}_{ms}\boldsymbol{\gamma}_{mt}'|\{\mathbf{X}_t\})h_{t,ml}h_{s,ml} \\
&= \frac{1}{p}\sum_{m=1}^{p}\operatorname{var}(\frac{1}{[T/\Delta_n]}\sum_{t=1}^{[T/\Delta_n]-k_n}\boldsymbol{\gamma}_{mt}h_{t,ml}|\{\mathbf{X}_t\}).
\end{aligned}
$$

Hence

$$
\begin{aligned}
& \mathbb{E}[\|\mathbf{V}_\gamma^*-\mathbf{V}_\gamma\|^2|\{\mathbf{X}_t\}] \\
= & \mathbb{E}[\|\frac{\Delta_n^2}{p}\sum_{t=1}^{[T/\Delta_n]-k_n}\sum_{m=1}^{p}\sum_{s=1}^{[T/\Delta_n]-k_n}(\boldsymbol{\gamma}_{ms}\boldsymbol{\gamma}_{mt}'-\mathbb{E}(\boldsymbol{\gamma}_{ms}\boldsymbol{\gamma}_{mt}'|\{\mathbf{X}_t\}))h_{t,ml}h_{s,ml}\|^2|\{\mathbf{X}_t\}] \\
= & \sum_{k_1,k_2\le K}\frac{1}{p^2}\sum_{m=1}^{p}\operatorname{var}(\Delta_n^2\sum_{t=1}^{[T/\Delta_n]-k_n}\sum_{s=1}^{[T/\Delta_n]-k_n}\gamma_{ms,k_2}\gamma_{mt,k_1}h_{t,ml}h_{s,ml}|\{\mathbf{X}_t\}) \\
\le & C\frac{\Delta_n}{p^2}\sum_{t=1}^{[T/\Delta_n]-k_n}\sum_{m=1}^{p}\mathbb{E}(\|\boldsymbol{\gamma}_{mt}\|^4h_{t,ml}^4|\{\mathbf{X}_t\}) \\
\le & C\frac{1}{p}\lambda_{\min}^2\left[\frac{1}{p}\sum_{m=1}^{p}\operatorname{var}(\Delta_n\sum_{t=1}^{[T/\Delta_n]-k_n}\boldsymbol{\gamma}_{mt}h_{t,ml}|\{\mathbf{X}_t\})\right]=C\frac{1}{p}\lambda_{\min}^2(\mathbf{V}_\gamma)
\end{aligned}
$$

where the last inequality is due to Assumption 3.8. So $\|\mathbf{V}_\gamma^*-\mathbf{V}_\gamma\|=o_P(\lambda_{\min}(\mathbf{V}_\gamma))$.

To show (3): note that $\mathbb{E}^*(\mathbf{a}_1^*-\mathbf{a}_1)(\mathbf{a}_2^*-\mathbf{a}_2)'=\frac{2}{p^2}\sum_{m=1}^p\mathbf{a}_{1,m}\mathbf{a}_{2,m}'-\frac{2}{p}\mathbf{a}_1\mathbf{a}_2'$. The first term is

$$\frac{1}{p}\|\frac{1}{p}\sum_{m=1}^{p}\mathbf{a}_{1m}\mathbf{a}'_{2m}\| = \frac{1}{p}\|\frac{1}{p}\sum_{m=1}^{p}\sum_{s=1}^{[T/\Delta_n]-k_n}\Delta_n\boldsymbol{\gamma}_{s,m}h_{s,ml}\sum_{t=1}^{[T/\Delta_n]-k_n}\Delta_n\widehat{\mathbf{c}}_{FF,t}^{-1}\frac{1}{k_n}\sum_{i\in I_t^n}\mathbf{f}_i u_{mi}h_{t,ml}\|$$

Now the goal becomes to prove: $\frac{1}{p}\|\frac{1}{p}\sum_{m=1}^{p}\mathbf{a}_{1m}\mathbf{a}'_{2m}\| = o_P(1)\sqrt{\frac{\Delta_n}{p}\frac{\mathbf{V}_\gamma}{p}}$.

Note that

$$\begin{aligned}
&\frac{1}{p^2}\mathbb{E}\Big(\|\frac{1}{p}\sum_{m=1}a_{1m}a'_{2m}\|^2\Big)\\
&=\frac{1}{p^4}\sum_{m,m'=1}^{p}\sum_{s,s'=1}^{[T/\Delta_n]-k_n}\sum_{t,t'=1}^{[T/\Delta_n]-k_n}\frac{1}{k_n^2}\sum_{i\in I_t^n}\sum_{i'\in I_{t'}^n}\mathbb{E}\Big(\gamma_{s,m}\gamma_{s',m'}h_{s,ml}h_{s',m'l}\widehat{\mathbf{c}}_{FF,t}^{-1}\widehat{\mathbf{c}}_{FF,t'}^{-1}\mathbf{f}_i\mathbf{f}_{i'}u_{mi}u_{m'i'}\\
&\qquad\times h_{t,ml}h_{t',m'l}\Big)\Delta_n^4\\
&=\frac{1}{p^4}\sum_{m=1}^{p}\sum_{s,s'=1}^{[T/\Delta_n]-k_n}\sum_{t,t'=1,|t-t'|<k_n}^{[T/\Delta_n]-k_n}\frac{1}{k_n^2}\sum_{i\in I_t^n\cap I_{t'}^n}\mathbb{E}\Big(\gamma_{s,m}\gamma_{s',m}h_{s,ml}h_{s',ml}\widehat{\mathbf{c}}_{FF,t}^{-1}\widehat{\mathbf{c}}_{FF,t'}^{-1}\mathbf{f}_i^2u_{mi}^2h_{t',ml}^2\Big)\Delta_n^4.
\end{aligned}$$

The last equality holds for the following reasons. First, the cross-sectional independence of u (and γ) imply that the above expectation is zero when $m \neq m'$. Second, the time domain independence of $\mathbf{f}$ (and $\mathbf{u}_m$) further imply that the above expectation is zero when $i \neq i'$. In order to have $i = i'$, it is necessary to have $|t-t'| < k_n$.

Cauchy-Schwartz inequality yields that

$$\begin{aligned}
&\frac{1}{p^4}\sum_{m=1}^{p}\sum_{s,s'=1}^{[T/\Delta_n]-k_n}\sum_{t,t'=1,|t-t'|<k_n}^{[T/\Delta_n]-k_n}\frac{1}{k_n^2}\sum_{i\in I_t^n\cap I_{t'}^n}\mathbb{E}\Big(\gamma_{s,m}\gamma_{s',m}h_{s,ml}h_{s',ml}\widehat{\mathbf{c}}_{FF,t}^{-1}\widehat{\mathbf{c}}_{FF,t'}^{-1}\mathbf{f}_i^2u_{mi}^2h_{t,ml}h_{t',ml}\Big)\Delta_n^4\\
&\leq\frac{1}{p^3}\Bigg(\mathbb{E}\Big(\frac{1}{p}\sum_{m=1}^{p}\sum_{s,s'=1}^{[T/\Delta_n]-k_n}\gamma_{s,m}\gamma_{s',m}h_{s,ml}h_{s',ml}\Delta_n^2\Big)^2\\
&\qquad\times\mathbb{E}\Big(\frac{1}{p}\sum_{m=1}^{p}\sum_{t,t'=1,|t-t'|<k_n}^{[T/\Delta_n]-k_n}\frac{1}{k_n^2}\sum_{i\in I_t^n\cap I_{t'}^n}\widehat{\mathbf{c}}_{FF,t}^{-1}\widehat{\mathbf{c}}_{FF,t'}^{-1}\mathbf{f}_i^2u_{mi}^2h_{t,ml}h_{t',ml}\Delta_n^2\Big)^2\Bigg)^{1/2}\\
&\leq\frac{1}{p^3}\Bigg(\mathbb{E}\Big(\frac{1}{p^2}\sum_{m=1}^{p}\sum_{s=1}^{[T/\Delta_n]-k_n}\mathbb{E}\big(\|\gamma_{s,m}\|^4h_{s,ml}^4|\{X_t\}\big)\Big)Ck_n^2\Delta_n^2\Bigg)^{1/2}\leq C\sqrt{\frac{k_n^2\Delta_n}{p\sqrt{p}}\frac{\Delta_n}{p}\frac{\lambda_{\min}(\mathbf{V}_\gamma)}{p}}.
\end{aligned}$$

It then follows

$$\mathbb{E}\Big(\frac{1}{p}\sum_{m=1}^{p}\sum_{t,t'=1,|t-t'|<k_n}^{[T/\Delta_n]-k_n}\frac{1}{k_n^2}\sum_{i\in I_t^n\cap I_{t'}^n}\widehat{\mathbf{c}}_{FF,t}^{-1}\widehat{\mathbf{c}}_{FF,t'}^{-1}\mathbf{f}_i^2u_{mi}^2h_{t,ml}h_{t',ml}\Delta_n^2\Big)^2$$

$$=\frac{1}{p}\sum_{m,m'=1}^{p}\sum_{t,t'=1,|t-t'|<k_n}^{[T/\Delta_n]-k_n}\frac{1}{k_n^2}\sum_{i\in I_t^n\cap I_{t'}^n}\sum_{s,s'=1,|s-s'|<k_n}^{[T/\Delta_n]-k_n}\frac{1}{k_n^2}\sum_{i\in I_s^n\cap I_{s'}^n}$$
$$\mathbb{E}(\widehat{\mathbf{c}}_{FF,t}^{-1}\widehat{\mathbf{c}}_{FF,t'}^{-1}\widehat{\mathbf{c}}_{FF,s}^{-1}\widehat{\mathbf{c}}_{FF,s'}^{-1}h_{t,ml}h_{t',ml}h_{t,m'l}h_{t',m'l}f_i^2u_{mi}^2u_{m'j}^2)\leq k_n^2\Delta_n^2.$$

Then as long as $k_n^2\Delta_n/(p\sqrt{p})\to 0$, which is implied by $k_n^2\Delta_n\to 0$, we have

$$\frac{1}{p}\|\frac{1}{p}\sum_{m=1}^{p}\mathbf{a}_{1m}\mathbf{a}_{2m}'\|=o_P(1)\sqrt{\frac{\Delta_n}{p}\frac{\mathbf{V}_\gamma}{p}}.$$

On the other hand, $\|\frac{2}{p}\mathbf{a}_1\mathbf{a}_2'\|\leq\frac{C}{p}(\|\mathbf{a}_1\|^2+\|\mathbf{a}_2\|^2)\leq o_P(1)\lambda_{\min}(\frac{\Delta_n}{p}\mathbf{V}_u+\frac{1}{p}\mathbf{V}_\gamma)$. Thus $\mathbb{E}^*(\mathbf{a}_1^*-\mathbf{a}_1)(\mathbf{a}_2^*-\mathbf{a}_2)'=o_P(1)\lambda_{\min}(\frac{\Delta_n}{p}\mathbf{V}_u+\frac{1}{p}\mathbf{V}_\gamma)$.

□

E.4 Technical lemmas

E.4.1 Lemmas for the estimated factors

Lemma E.2. *Suppose $k_npJ^{-2\eta}=o(1)$ and $pk_n^2\Delta_n=o(1)$. Then*
(i) $\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\widehat{\Delta_i^n\mathbf{F}}\boldsymbol{\psi}_i'\mathbf{P}_{t,l}\Delta_n=o_P((k_np)^{-1/2})$
(ii) $\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\widehat{\Delta_i^n\mathbf{F}}\Delta_i^n\mathbf{F}'[\mathbf{G}_{i-1}-\mathbf{G}_t]'\mathbf{P}_{t,l}=o_P((k_np)^{-1/2})$
(iii) $\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\widehat{\Delta_i^n\mathbf{F}}\Delta_i^n\mathbf{F}'[\boldsymbol{\Gamma}_{i-1}-\boldsymbol{\Gamma}_t]\mathbf{P}_{t,l}=o_P((k_np)^{-1/2})$
(iv) $\boldsymbol{\Upsilon}_{nt}(\mathbf{G}_{t,l}'\mathbf{P}_{t,l}-\mathbf{g}_{lt})=o_P((k_np)^{-1/2})$

Proof. For notation simplicity, we assume $K=1$ in the following proofs. The proofs for the general multivariate (but finite) case is essentially the same because one can consider terms element-by-element. Also, write $\widehat{\mathbf{f}}_i=\widehat{\Delta_i^n\mathbf{F}}/\sqrt{\Delta_n}$.

(i) We first show that, for $\boldsymbol{\psi}_j=\frac{1}{\Delta_n}\int_{(j-1)\Delta_n}^{j\Delta_n}\boldsymbol{\alpha}_sds+\frac{1}{\Delta_n}\int_{(j-1)\Delta_n}^{j\Delta_n}(\boldsymbol{\beta}_s-\boldsymbol{\beta}_{(j-1)\Delta_n})d\mathbf{F}_s$,

$$\frac{1}{k_np}\sum_{i\in I_t^n}\|\boldsymbol{\psi}_i\|^2=O_P(1).$$

In fact, $\frac{1}{k_np}\sum_{j\in I_t^n}\|\frac{1}{\Delta_n}\int_{(j-1)\Delta_n}^{j\Delta_n}\boldsymbol{\alpha}_sds\|^2=O_P(1)$, and by using Itô's formula and the Burkholder-

Davis-Gundy inequality, we obtain

$$
\begin{aligned}
&\frac{1}{k_np}\sum_{m=1}^{p}\sum_{j\in I_t^n}\mathbb{E}\Big(\|\frac{1}{\Delta_n}\int_{(j-1)\Delta_n}^{j\Delta_n}(\boldsymbol{\beta}_{ms}-\boldsymbol{\beta}_{m,(j-1)\Delta_n})'d\mathbf{F}_s\|^2\Big)\\
\leq\ &\frac{L}{k_np}\sum_{m=1}^{p}\sum_{j\in I_t^n}\mathbb{E}\Big(\frac{1}{\Delta_n^2}\int_{(j-1)\Delta_n}^{j\Delta_n}\|(\boldsymbol{\beta}_{ms}-\boldsymbol{\beta}_{m,(j-1)\Delta_n})\|^2ds\Big)\\
\leq\ &\frac{L}{k_np}\sum_{m=1}^{p}\sum_{j\in I_t^n}\mathbb{E}\Big(\frac{1}{\Delta_n^2}\int_{(j-1)\Delta_n}^{j\Delta_n}L(s-(j-1)\Delta_n)ds\Big)\leq L.
\end{aligned}
$$

Next, note that $pk_n\Delta_n\to 0$. Also $\mathbb{E}\|\mathbf{P}_{t,l}\|^2=\frac{1}{p}\mathbb{E}h_{t,ll}=O(p^{-1})$. So

$$
\frac{1}{k_n}\sum_{i\in I_t^n}\widehat{\Delta_i^n\mathbf{F}}\boldsymbol{\psi}_i'\mathbf{P}_{t,l}\leq O_P(\Delta_n^{1/2})(\frac{1}{k_np}\sum_{i\in I_t^n}\|\boldsymbol{\psi}_i\|^2)^{1/2}=O_P(\Delta_n^{1/2})=o_P((k_np)^{-1/2}).
$$

(ii)(iii) The Itô semimartingale assumption of $\mathbf{X}_{lt}$ implies that, by the Burkholder-Davis-Grundy inequality (cf. Chapter 2 of Jacod and Protter (2011)), $\mathbb{E}(\|\mathbf{X}_{l,i-1}-\mathbf{X}_{l,t}\|\,|\,\mathcal{F}_{i-1})\leq\big(\mathbb{E}(\|\mathbf{X}_{l,i-1}-\mathbf{X}_{lt}\|^2\,|\,\mathcal{F}_{i-1})\big)^{1/2}\leq L\sqrt{k_n\Delta_n}$ for any $i\in I_t^n$, where L is a positive finite number that does not depend on $i\in I_t^n$. In addition, by the differentiable assumption on $\widetilde{\mathbf{g}}_l(t,\mathbf{x}):=\mathbf{g}_{lt}(\mathbf{x})$, we have:

$$
\begin{aligned}
\|\mathbf{g}_{l,i-1}(\mathbf{X}_{l,i-1})-\mathbf{g}_{l,t}(\mathbf{X}_{lt})\| &\leq \sup_{t,\mathbf{x}}\left|\frac{\partial\widetilde{\mathbf{g}}_l(t,\mathbf{x})}{\partial t}\right||i-t|+\sup_{t,\mathbf{x}}\left\|\frac{\partial\widetilde{\mathbf{g}}_l(t,\mathbf{x})}{\partial\mathbf{x}}\right\|\|\mathbf{X}_{l,i-1}-\mathbf{X}_{lt}\|\\
&\leq Ck_n\Delta_n+C\|\mathbf{X}_{l,i-1}-\mathbf{X}_{lt}\|. \qquad \text{(E.21)}
\end{aligned}
$$

So $\mathbb{E}(|\mathbf{g}_{l,i-1}-\mathbf{g}_{l,t}|^2\,|\,\mathcal{F}_{i-1})\leq Ck_n^2\Delta_n^2+C\mathbb{E}(\|\mathbf{X}_{l,i-1}-\mathbf{X}_{l,t}\|^2\,|\,\mathcal{F}_{i-1})\leq Lk_n\Delta_n$ for some L that is independent of (i,l). It then follows that, due to $pk_n^2\Delta_n=o(1)$,

$$
\begin{aligned}
&\|\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\widehat{\Delta_i^n\mathbf{F}}\Delta_i^n\mathbf{F}'[\mathbf{G}_{i-1}-\mathbf{G}_t]'\mathbf{P}_{t,l}\|^2\leq O_P(1)\frac{1}{k_np}\sum_{m=1}^{p}\sum_{i\in I_t^n}|\mathbf{f}_i'[\mathbf{g}_{m,i-1}-\mathbf{g}_{mt}]|^2\\
\leq\ & O_P(1)\,\mathbb{E}\,\mathbb{E}\left(\frac{1}{k_np}\sum_{m=1}^{p}\sum_{i\in I_t^n}|\mathbf{f}_i'[\mathbf{g}_{m,i-1}-\mathbf{g}_{mt}]|^2|\mathcal{F}_{i-1}\right)=O_P(k_n\Delta_n)=o_P((pk_n)^{-1}).
\end{aligned}
$$

The proof for (iii) is the same because $\boldsymbol{\Gamma}_t$ is also Itô semimartingale.

(iv) Note that $\|\boldsymbol{\Upsilon}_{nt}\| \leq (\frac{1}{k_n}\sum_{i\in I_t^n}\|\widehat{\mathbf{f}}_i\|^2)^{1/2}(\frac{1}{k_n}\sum_{i\in I_t^n}\|\mathbf{f}_i\|^2)^{1/2} = O_P(1)$. So

$$\|\boldsymbol{\Upsilon}_{nt}(\mathbf{G}_{t,l}'\mathbf{P}_{t,l} - \mathbf{g}_{lt})\| \leq O_P(J^{-\eta}) = o_P((pk_n)^{-1}).$$

Lemma E.3. *Suppose $J = o(p^2)$, $(J^2 + k_n J)\Delta_n = o(1)$, and $k_n = o(p^3)$.*

(i) $\boldsymbol{\Xi}_{1,t}'\frac{1}{k_n\sqrt{p}}\sum_{i\in I_t^n}\mathbf{P}_{i-1}\mathbf{c}_{uu,i}\mathbf{P}_{i-1,l} = o_P((k_np)^{-1/2})$

(ii) $\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\boldsymbol{\Xi}_{3,t}'\frac{1}{\sqrt{p}}\Delta_n\mathbf{P}_{i-1}\boldsymbol{\psi}_i\Delta_i^n\mathbf{U}'\mathbf{P}_{i-1,l} = o_P((k_np)^{-1/2})$

(iii) $\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\boldsymbol{\Xi}_{3,t}'\frac{1}{\sqrt{p}}(\mathbf{P}_{i-1}\boldsymbol{\beta}_{i-1} - \mathbf{P}_t\boldsymbol{\beta}_t)\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{U}'\mathbf{P}_{i-1,l} = o_P((k_np)^{-1/2})$

(iv) $\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\boldsymbol{\Xi}_{3,t}'\frac{1}{\sqrt{p}}\mathbf{P}_{i-1}(\Delta_i^n\mathbf{U}\Delta_i^n\mathbf{U}' - \mathbb{E}\Delta_i^n\mathbf{U}\Delta_i^n\mathbf{U}')\mathbf{P}_{i-1,l} = o_P((k_np)^{-1/2})$

Proof. By Lemma E.4, $\|\boldsymbol{\Xi}_{3,t}| = O_P(1)$. Also, $\|\boldsymbol{\Xi}_{1,t}\| = O_P(\sqrt{\frac{J}{k_np}} + \frac{1}{p})$.

(i) $\|\boldsymbol{\Xi}_{1,t}'\frac{1}{k_n\sqrt{p}}\sum_{i\in I_t^n}\mathbf{P}_{i-1}\mathbf{c}_{uu,i}\mathbf{P}_{i-1,l}\| = \frac{1}{k_n}\sum_{i\in I_t}h_{i,ll}^{1/2}\|\boldsymbol{\Xi}_{1,t}\|O_P(p^{-1}) = O_P(\sqrt{\frac{J}{k_np}} + \frac{1}{p})p^{-1}$ which is $o_P((k_np)^{-1/2})$ given that $J = o(p^2)$ and $k_n = o(p^3)$.

(ii) $\mathbb{E}\|\mathbf{P}_{i-1,l}\|^2 = O(Jp^{-1})$.

$$\begin{aligned}
&\mathbb{E}\|\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\frac{1}{\sqrt{p}}\Delta_n\mathbf{P}_{i-1}\boldsymbol{\psi}_i\Delta_i^n\mathbf{U}'\mathbf{P}_{i-1,l}\|^2 = \frac{1}{pk_n^2}\sum_{k\leq p}\mathbb{E}(\sum_{i\in I_t^n}\mathbf{P}_{i-1,k}'\boldsymbol{\psi}_i\Delta_i^n\mathbf{U}'\mathbf{P}_{i-1,l})^2 \\
= \;& \frac{1}{pk_n^2}\sum_{k\leq p}\sum_{i\in I_t^n}\mathbb{E}(\mathbf{P}_{i-1,k}'\boldsymbol{\psi}_i\Delta_i^n\mathbf{U}'\mathbf{P}_{i-1,l})^2 = \frac{1}{pk_n^2}\sum_{k\leq p}\sum_{i\in I_t^n}\mathbb{E}(\mathbf{P}_{i-1,k}'\boldsymbol{\psi}_i)^2\mathbf{P}_{i-1,l}'\operatorname{Var}(\Delta_i^n\mathbf{U}|\mathcal{F}_{i-1})\mathbf{P}_{i-1,l} \\
\leq \;& C\frac{1}{pk_n^2}\sum_{i\in I_t^n}\mathbb{E}\|\mathbf{P}_{i-1}\|_F^2\|\boldsymbol{\psi}_i\|^2\|\mathbf{P}_{i-1,l}\|^2\Delta_n \\
\leq \;& C\frac{1}{k_n^2}\sum_{i\in I_t^n}\mathbb{E}\|\mathbf{P}_{i-1,l}\|^2\left[\max_{i\in i_t^n, s\in[(i-1)\Delta_n, i\Delta_n]}\frac{J}{p}\|\boldsymbol{\beta}_s - \boldsymbol{\beta}_{(i-1)\Delta_n}\|^2 + J\Delta_n\right] \\
= \;& o_P((k_np)^{-1}).
\end{aligned}$$

(iii) We use the same argument to reach

$$\begin{aligned}
&\mathbb{E}\|\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\frac{1}{\sqrt{p}}(\mathbf{P}_{i-1}\boldsymbol{\beta}_{i-1} - \mathbf{P}_t\boldsymbol{\beta}_t)\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{U}'\mathbf{P}_{i-1,l}\|^2 \\
= \;& \frac{1}{p}\sum_{k\leq p}\frac{1}{k_n^2}\mathbb{E}(\sum_{i\in I_t^n}(\mathbf{P}_{i-1}\boldsymbol{\beta}_{i-1} - \mathbf{P}_t\boldsymbol{\beta}_t)_k'\mathbf{f}_i\mathbf{u}_i'\mathbf{P}_{i-1,l})(\sum_{j\in I_t^n}(\mathbf{P}_{j-1}\boldsymbol{\beta}_{j-1} - \mathbf{P}_t\boldsymbol{\beta}_t)_k'\mathbf{f}_j\mathbf{u}_j'\mathbf{P}_{j-1,l}) \\
= \;& \frac{1}{p}\sum_{k\leq p}\frac{1}{k_n^2}\sum_{i\in I_t^n}\mathbb{E}((\mathbf{P}_{i-1}\boldsymbol{\beta}_{i-1} - \mathbf{P}_t\boldsymbol{\beta}_t)_k'\mathbf{f}_i\mathbf{u}_i'\mathbf{P}_{i-1,l})^2 \\
= \;& \frac{1}{p}\sum_{k\leq p}\frac{1}{k_n^2}\sum_{i\in I_t^n}\sum_{d'\leq K}\sum_{d\leq K}\mathbb{E}(\mathbf{P}_{i-1}\boldsymbol{\beta}_{i-1} - \mathbf{P}_t\boldsymbol{\beta}_t)_{kd}\mathbf{P}_{i-1,l}'\mathbb{E}(\mathbf{u}_i\mathbf{u}_i'f_{di}f_{d'i}|\mathcal{F}_{i-1})\mathbf{P}_{i-1,l}(\mathbf{P}_{i-1}\boldsymbol{\beta}_{i-1} - \mathbf{P}_t\boldsymbol{\beta}_t)_{kd'}
\end{aligned}$$

$$\leq \frac{CJ}{k_n p}\max_{k\leq p}\sup_{i\in I_t^n}\mathbb{E}\|\mathbf{P}'_{i-1,k}\boldsymbol{\beta}_{i-1}-\mathbf{P}'_{t,k}\boldsymbol{\beta}_t\|^2 \leq O(\frac{k_n\Delta_n J}{k_n p}) = o((k_n p)^{-1}).$$

(iv) Similar to the proof of Lemma E.2 (iii),

$$\begin{aligned}
&\mathbb{E}\|\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\frac{1}{\sqrt{p}}\mathbf{P}_{i-1}(\Delta_i^n\mathbf{U}\Delta_i^n\mathbf{U}'-\mathbb{E}\Delta_i^n\mathbf{U}\Delta_i^n\mathbf{U}')\mathbf{P}_{i-1,l}\|^2 \\
\leq\ & \frac{1}{pk_n^2}\frac{1}{p^4}\sum_{k\leq p}\sum_{i\in I_t^n}\sum_{m,r\leq p}\sum_{m',r'\leq p}\mathbb{E}h_{i-1,kr}h_{i-1,lm}h_{i-1,kr'}h_{i-1,lm'}\mathrm{Cov}(u_{mi}u_{ri},u_{m'i}u_{r'i}|\mathcal{F}_{i-1}) \\
\leq\ & \frac{1}{pk_n^2}\frac{1}{p^4}\sum_{k\leq p}\sum_{i\in I_t^n}\max_{i,k,r,l,m,r',m'}|\mathbb{E}h_{i-1,kr}h_{i-1,lm}h_{i-1,kr'}h_{i-1,lm'}|\sum_{m,r\leq p}\sum_{m',r'\leq p}|\mathrm{Cov}(u_{mi}u_{ri},u_{m'i}u_{r'i})| \\
\leq\ & \frac{1}{k_n}\frac{1}{p^2}\max_{i,k,r}|\mathbb{E}h_{i-1,kr}^4| = o((k_n p)^{-1}).
\end{aligned}$$

□

Lemma E.4. *Suppose the eigenvalues of $\mathbf{s}_{G,t}^{1/2}\mathbf{c}_f\mathbf{s}_{G,t}^{1/2}$ are distinct, $(J+p)\Delta_n = o(1)$*
(i) $\widehat{\mathbf{V}}_t^{-1} = O_P(1) = \widehat{\mathbf{V}}_t$.
(ii) $\|\boldsymbol{\Xi}_{2,t}\| = O_P(1) = \|\boldsymbol{\Xi}_{3,t}\|$
(iii) $\|\boldsymbol{\Xi}_{1,t}\| = O_P(\sqrt{\frac{J}{k_n p}}+\frac{1}{p})$.
(iv) $\mathbf{H}_{nt}-\boldsymbol{\Upsilon}_{nt}\mathbf{c}_{FF,t}^{-1} = o_P(1)$ *and* $(\boldsymbol{\Upsilon}_{nt}\boldsymbol{\Upsilon}'_{nt})^{-1} = O_P(1)$.
(v) $\boldsymbol{\Upsilon}_{nt}-\boldsymbol{\Upsilon}_t = o_P(1)$ *where* $\boldsymbol{\Upsilon}_t$ *is nonrandom conditional on* $\mathbf{X}_t$.

Proof. (i) Recall that $\widehat{\mathbf{V}}_t$ is the $K\times K$ diagonal matrix of the first K eigenvalues of $\frac{1}{pk_n\Delta_n}\sum_{i\in I_t^n}\mathbf{P}_{i-1}\Delta_i^n\mathbf{Y}\Delta_i^n\mathbf{Y}'\mathbf{P}_{i-1} = \frac{1}{pk_n\Delta_n}\sum_{i\in I_t^n}\mathbf{G}_t\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{F}'\mathbf{G}'_t + o_P(1)$. Let $\mathbf{V}_t$ be the $K\times K$ diagonal matrix of the first K eigenvalues of $\frac{1}{pk_n\Delta_n}\sum_{i\in I_t^n}\mathbf{G}_t\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{F}'\mathbf{G}'_t$. Then $\|\mathbf{V}_t-\widehat{\mathbf{V}}_t\| = o_P(1)$. Note that $\|\mathbf{V}_t\| = O_P(1)$, which implies $\|\widehat{\mathbf{V}}_t\| = O_P(1)$. In addition, the K th largest eigenvalue of $\frac{1}{pk_n\Delta_n}\sum_{i\in I_t^n}\mathbf{G}_t\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{F}'\mathbf{G}'_t$ is the same as the minimum eigenvalue of $\widehat{\mathbf{c}}_{FF,t}^{1/2}\frac{1}{p}\mathbf{G}'_t\mathbf{G}_t\widehat{\mathbf{c}}_{FF,t}^{1/2}$, where $\widehat{\mathbf{c}}_{FF,t} = \frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{F}'$, and is bounded away from zero. Hence $\lambda_{\min}(\widehat{\mathbf{V}}_t)\geq\lambda_{\min}(\mathbf{V}_t)-o_P(1)$, which is bounded away from zero. This implies $\|\widehat{\mathbf{V}}_t^{-1}\| = O_P(1)$.

(ii) By (i) and $\frac{1}{k_n\Delta_n}\sum_{j\in I_t^n}\|\widehat{\Delta_j^n\mathbf{F}}\|^2 = K$, $\|\boldsymbol{\Xi}_{2,t}\| = \|\frac{1}{\sqrt{p}k_n\Delta_n}\sum_{j\in I_t^n}\mathbf{P}_{j-1}\boldsymbol{\beta}_{j-1}\Delta_j^n\mathbf{F}\widehat{\Delta_j^n\mathbf{F}}'\widehat{\mathbf{V}}_t^{-1}\| = O_P(1)$. Also, $\|\boldsymbol{\Xi}_{1,t}\| = O_P(1)$. Hence $\|\boldsymbol{\Xi}_{3,t}\| = O_P(1)$.

(iii) The triangular inequality and the expression for $\widehat{\Delta_i^n\mathbf{F}}-\mathbf{H}_t\Delta_i^n\mathbf{F}$ in (E.7) yields, by Lemma E.2,

$$\|\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\mathbf{P}_{i-1}\Delta_i^n\mathbf{U}\widehat{\Delta_i^n\mathbf{F}}'\| \leq \mathbf{a}_1+...+\mathbf{a}_5,$$

$$
\begin{aligned}
\mathbf{a}_1 &= \|\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\mathbf{P}_{i-1}\Delta_i^n\mathbf{U}\Delta_i^n\mathbf{F}'\|O_P(1) = O_P((J/k_n)^{1/2}) \\
\mathbf{a}_2 &= \|\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\frac{1}{\sqrt{p}}\Delta_n\mathbf{P}_{i-1}\boldsymbol{\psi}_i\Delta_i^n\mathbf{U}'\mathbf{P}_{i-1}\|O_P(1) = O_P(J(\Delta_n/k_n)^{1/2}) \\
\mathbf{a}_3 &= \|\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\frac{1}{\sqrt{p}}(\mathbf{P}_{i-1}\boldsymbol{\beta}_{i-1}-\mathbf{P}_t\boldsymbol{\beta}_t)\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{U}'\mathbf{P}_{i-1}\|O_P(1) = O_P(\sqrt{\Delta_n J}) \\
\mathbf{a}_4 &= \|\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\frac{1}{\sqrt{p}}\mathbf{P}_{i-1}(\Delta_i^n\mathbf{U}\Delta_i^n\mathbf{U}'-\mathbb{E}\Delta_i^n\mathbf{U}\Delta_i^n\mathbf{U}')\mathbf{P}_{i-1}\|O_P(1) = O_P((k_np)^{-1/2}) \\
\mathbf{a}_5 &= \|\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\frac{1}{\sqrt{p}}\mathbf{P}_{i-1}(\mathbb{E}\Delta_i^n\mathbf{U}\Delta_i^n\mathbf{U}')\mathbf{P}_{i-1}\|O_P(1) = O_P(\frac{1}{\sqrt{p}}).
\end{aligned}
$$

So

$$
\|\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\mathbf{P}_{i-1}\Delta_i^n\mathbf{U}\widehat{\Delta_i^n\mathbf{F}}'\| = O_P((J/k_n)^{1/2}+p^{-1/2}). \tag{E.22}
$$

In addition, by Lemma E.2,

$$
\begin{aligned}
&\|\frac{1}{k_n\Delta_n}\sum_{j\in I_t^n}\mathbf{P}_{j-1}\boldsymbol{\psi}_j\Delta_n\widehat{\Delta_j^n\mathbf{F}}'\| \leq \mathbf{b}_1+...+\mathbf{b}_4 \\
\mathbf{b}_1 &= \|\frac{1}{k_n}\sum_{j\in I_t^n}\mathbf{P}_{j-1}\boldsymbol{\psi}_j\Delta_j^n\mathbf{F}'\|O_P(1) = O_P(\sqrt{p\Delta_n/k_n}). \\
\mathbf{b}_2 &= \frac{1}{\sqrt{p}}\Delta_n\|\frac{1}{k_n}\sum_{j\in I_t^n}\mathbf{P}_{j-1}\boldsymbol{\psi}_j\boldsymbol{\psi}_j'\mathbf{P}_{j-1}\|O_P(1) = O_P(\Delta_n\sqrt{p}) \\
\mathbf{b}_3 &= \sqrt{\Delta_n}\|\frac{1}{k_n}\sum_{j\in I_t^n}\mathbf{P}_{j-1}\boldsymbol{\psi}_j\frac{1}{\sqrt{p}}\mathbf{f}_j'(\boldsymbol{\beta}_{j-1}'\mathbf{P}_{j-1}-\boldsymbol{\beta}_t'\mathbf{P}_t)\|O_P(1) \\
&\leq O_P(\sqrt{\Delta_n p})(\max_{j\in I_t^n}\mathbb{E}\|\frac{1}{\sqrt{p}}(\boldsymbol{\beta}_{j-1}'\mathbf{P}_{j-1}-\boldsymbol{\beta}_t'\mathbf{P}_t)\|^2)^{1/2} \leq O_P(\Delta_n\sqrt{pk_n}) \\
\mathbf{b}_4 &= \|\frac{1}{k_n}\sum_{j\in I_t^n}\mathbf{P}_{j-1}\boldsymbol{\psi}_j\frac{1}{\sqrt{p}}\Delta_j^n\mathbf{U}'\mathbf{P}_{j-1}\|O_P(1) = O_P(J(\Delta_n/k_n)^{1/2}).
\end{aligned}
$$

Hence because $\Delta_n k_n^2 = o(1)$ and

$$
\|\frac{1}{k_n\Delta_n}\sum_{j\in I_t^n}\mathbf{P}_{j-1}\boldsymbol{\psi}_j\Delta_n\widehat{\Delta_j^n\mathbf{F}}'\| \leq O_P(\sqrt{(p+J^2)\Delta_n/k_n}+\Delta_n\sqrt{pk_n}) = O_P(\sqrt{(p+J^2)\Delta_n/k_n}).
$$

Together, $\|\boldsymbol{\Xi}_{1,t}\| \leq O_P(\sqrt{(1+J^2/p)\Delta_n/k_n}) + O_P((J/k_n)^{1/2}/\sqrt{p}+p^{-1}) = O_P(\sqrt{\frac{J}{k_np}}+\frac{1}{p})$.

(iv) By definition $\boldsymbol{\Upsilon}_{nt} = \frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\widehat{\Delta_i^n\mathbf{F}}\Delta_i^n\mathbf{F}'$. By (E.7), $\frac{1}{k_n\Delta_n}\sum_{j\in I_t^n}\|\widehat{\Delta_i^n\mathbf{F}}-\mathbf{H}_{nt}\Delta_i^n\mathbf{F}\|^2 =$

$o_P(1)$. Hence $\boldsymbol{\Upsilon}_{nt} = o_P(1) + \mathbf{H}_{nt}\widehat{\mathbf{c}}_{FF,t}$, implying $\boldsymbol{\Upsilon}_{nt}\widehat{\mathbf{c}}_{FF,t}^{-1} = o_P(1) + \mathbf{H}_{nt}$. Also, $\mathbf{H}_{nt} - \boldsymbol{\Upsilon}_{nt}\mathbf{c}_{FF,t}^{-1} = o_P(1)$ since $\widehat{\mathbf{c}}_{FF,t}^{-1} - \mathbf{c}_{FF,t}^{-1} = o_P(1)$. Finally,

$$\frac{1}{k_n\Delta_n}\sum_{j\in I_t^n}\widehat{\Delta_j^n\mathbf{F}}\widehat{\Delta_j^n\mathbf{F}}' = \mathbf{I}_K = \mathbf{H}_{nt}\widehat{\mathbf{c}}_{FF,t}\mathbf{H}_{nt}' + o_P(1).$$

Hence $\lambda_{\min}(\boldsymbol{\Upsilon}_{nt}\boldsymbol{\Upsilon}_{nt}') \geq \lambda_{\min}(\mathbf{H}_{nt}\widehat{\mathbf{c}}_{FF,t}^2\mathbf{H}_{nt}') - o_P(1) \geq \lambda_{\min}(\mathbf{H}_{nt}\widehat{\mathbf{c}}_{FF,t}^2\mathbf{H}_{nt}')C - o_P(1)$, which is bounded away from zero. This finishes the proof.

(v) Define $\mathbf{s}_{B,t} := \frac{1}{p}\boldsymbol{\beta}_t'\mathbf{P}_t\boldsymbol{\beta}_t$. Let $\mathbf{V}^* = \mathrm{diag}(\boldsymbol{\Upsilon}_{nt}\mathbf{s}_{B,t}\boldsymbol{\Upsilon}_{nt}')$ be a $K\times K$ diagonal matrix whose diagonal entries are those of $\boldsymbol{\Upsilon}_{nt}\mathbf{s}_{B,t}\boldsymbol{\Upsilon}_{nt}'$. Also, by part (iv), $\mathbf{V}^{*-1} = O_P(1)$. Let

$$\begin{aligned}
\mathbf{c}_1 &:= \frac{1}{pk_n\Delta_n}\sum_{j\in I_t^n}\widehat{\mathbf{V}}_t^{-1}\widehat{\Delta_j^n\mathbf{F}}\Delta_j^n\mathbf{F}'(\boldsymbol{\beta}_{j-1}'\mathbf{P}_{j-1} - \boldsymbol{\beta}_t'\mathbf{P}_t)\mathbf{P}_t\boldsymbol{\beta}_t = o_P(1)\\
\mathbf{c}_2 &:= \frac{1}{\Delta_n}\widehat{\mathbf{V}}_t\frac{1}{k_n}\sum_{j\in I_t^n}(\mathbf{H}_{nt}\Delta_j^n\mathbf{F} - \widehat{\Delta_j^n\mathbf{F}})\Delta_j^n\mathbf{F}' = o_P(1),\\
\mathbf{C}_n &:= \mathbf{s}_{B,t}^{1/2}\boldsymbol{\Upsilon}_{nt}'\mathbf{V}^{*-1/2}.
\end{aligned}$$

Then the columns of $\mathbf{C}_n$ have unit lengths. We now divide the proof into several steps.

step 1: show that the columns of $\mathbf{C}_n$ are the eigenvectors of $\mathbf{s}_{B,t}^{1/2}\widehat{\mathbf{c}}_{FF,t}\mathbf{s}_{B,t}^{1/2} + o_P(1)$. Note that

$$\begin{aligned}
\mathbf{H}_{nt} &= \frac{1}{\sqrt{p}}\boldsymbol{\Xi}_{3,t}'\mathbf{P}_t\boldsymbol{\beta}_t = o_P(1) + \widehat{\mathbf{V}}_t^{-1}\frac{1}{\sqrt{p}k_n\Delta_n}\sum_{j\in I_t^n}\widehat{\Delta_j^n\mathbf{F}}\Delta_j^n\mathbf{F}'\boldsymbol{\beta}_{j-1}'\mathbf{P}_{j-1}\frac{1}{\sqrt{p}}\mathbf{P}_t\boldsymbol{\beta}_t\\
&= o_P(1) + \mathbf{c}_1 + \widehat{\mathbf{V}}_t^{-1}\boldsymbol{\Upsilon}_{nt}\mathbf{s}_{B,t}.
\end{aligned}$$

This implies $\widehat{\mathbf{V}}_t\mathbf{H}_{nt}\frac{1}{\sqrt{\Delta_n}}\Delta_j^n\mathbf{F} = o_P(1) + \boldsymbol{\Upsilon}_{nt}\mathbf{s}_{B,t}\frac{1}{\sqrt{\Delta_n}}\Delta_j^n\mathbf{F}$. We have

$$\frac{1}{\sqrt{\Delta_n}}\widehat{\mathbf{V}}_t(\mathbf{H}_{nt}\Delta_j^n\mathbf{F} - \widehat{\Delta_j^n\mathbf{F}}) = o_P(1) + \boldsymbol{\Upsilon}_{nt}\mathbf{s}_{B,t}\frac{1}{\sqrt{\Delta_n}}\Delta_j^n\mathbf{F} - \widehat{\mathbf{V}}_t\frac{1}{\sqrt{\Delta_n}}\widehat{\Delta_j^n\mathbf{F}}. \tag{E.23}$$

Right multiply by $\frac{1}{\sqrt{\Delta_n}}\Delta_j^n\mathbf{F}'$ and average over $j\in I_t^n$:

$$\mathbf{c}_2 = \frac{1}{\Delta_n}\widehat{\mathbf{V}}_t\frac{1}{k_n}\sum_{j\in I_t^n}(\mathbf{H}_{nt}\Delta_j^n\mathbf{F} - \widehat{\Delta_j^n\mathbf{F}})\Delta_j^n\mathbf{F}' = o_P(1) + \boldsymbol{\Upsilon}_{nt}\mathbf{s}_{B,t}\widehat{\mathbf{c}}_{FF,t} - \widehat{\mathbf{V}}_t\boldsymbol{\Upsilon}_{nt}$$

We have $\widehat{\mathbf{V}}_t\boldsymbol{\Upsilon}_{nt} = \boldsymbol{\Upsilon}_{nt}\mathbf{s}_{B,t}\widehat{\mathbf{c}}_{FF,t} + o_P(1)$. Left multiply $\mathbf{V}^{*-1/2}$ and right multiply by $\mathbf{s}_{B,t}^{1/2}$, and reach $\mathbf{V}^{*-1/2}\widehat{\mathbf{V}}_t\boldsymbol{\Upsilon}_{nt}\mathbf{s}_{B,t}^{1/2} = \mathbf{V}^{*-1/2}\boldsymbol{\Upsilon}_{nt}\mathbf{s}_{B,t}^{1/2}\mathbf{s}_{B,t}^{1/2}\widehat{\mathbf{c}}_{FF,t}\mathbf{s}_{B,t}^{1/2} + o_P(1)$. It is more convenient to look at its transformation: $\mathbf{s}_{B,t}^{1/2}\boldsymbol{\Upsilon}'_{nt}\widehat{\mathbf{V}}_t\mathbf{V}^{*-1/2} = \mathbf{s}_{B,t}^{1/2}\widehat{\mathbf{c}}_{FF,t}\mathbf{s}_{B,t}^{1/2}\mathbf{s}_{B,t}^{1/2}\boldsymbol{\Upsilon}'_{nt}\mathbf{V}^{*-1/2} + o_P(1)$. Since diagonal matrices $\mathbf{V}^{*-1/2}$ and $\widehat{\mathbf{V}}_t$ are commutable, we have $\mathbf{C}_n\widehat{\mathbf{V}}_t = \mathbf{s}_{B,t}^{1/2}\widehat{\mathbf{c}}_{FF,t}\mathbf{s}_{B,t}^{1/2}\mathbf{C}_n + o_P(1)$, which further implies

$$\mathbf{C}_n\widehat{\mathbf{V}}_t = (\mathbf{s}_{B,t}^{1/2}\widehat{\mathbf{c}}_{FF,t}\mathbf{s}_{B,t}^{1/2} + o_P(1))\mathbf{C}_n.$$

This implies the columns of $\mathbf{C}_n$ are the eigenvectors of $\mathbf{s}_{B,t}^{1/2}\widehat{\mathbf{c}}_{FF,t}\mathbf{s}_{B,t}^{1/2} + o_P(1)$.

step 2: show that $\|\boldsymbol{\Upsilon}'_{nt} - \mathbf{s}_{G,t}^{-1/2}\mathbf{A}\mathbf{V}^{*1/2}\| = o_P(1)$.

Note that $\mathbf{s}_{G,t} - \mathbf{s}_{B,t} = o_P(1)$ and $\mathbf{s}_{B,t}^{1/2}\widehat{\mathbf{c}}_{FF,t}\mathbf{s}_{B,t}^{1/2} - \mathbf{s}_{G,t}^{1/2}\mathbf{c}_f\mathbf{s}_{G,t}^{1/2} = o_P(1)$, and the eigenvalues of $\mathbf{s}_{G,t}^{1/2}\mathbf{c}_f\mathbf{s}_{G,t}^{1/2}$ are distinct by our assumption. Then by Sine-Theta theorem, columns of $\mathbf{C}_n - \mathbf{A} = o_P(1)$. Thus

$$\begin{aligned}\|\boldsymbol{\Upsilon}'_{nt} - \mathbf{s}_{G,t}^{-1/2}\mathbf{A}\mathbf{V}^{*1/2}\| &\leq \|\boldsymbol{\Upsilon}'_{nt} - \mathbf{s}_{B,t}^{-1/2}\mathbf{A}\mathbf{V}^{*1/2}\| + \|\mathbf{s}_{B,t}^{-1/2} - \mathbf{s}_{G,t}^{-1/2}\|\|\mathbf{A}\mathbf{V}^{*1/2}\| \\ &\leq \|(\mathbf{s}_{B,t}^{-1/2}(\mathbf{C}_n - \mathbf{A})\mathbf{V}^{*1/2}\| + o_P(1) = o_P(1).\end{aligned}$$

where we used definition $\boldsymbol{\Upsilon}'_{nt} = \mathbf{s}_{B,t}^{-1/2}\mathbf{C}_n\mathbf{V}^{*1/2}$.

It remains to prove that $\mathbf{V}^*$ converges in probability. By the definition of $\mathbf{V}^*$, it suffices to prove that $\boldsymbol{\Upsilon}_{nt}\mathbf{s}_{B,t}\boldsymbol{\Upsilon}'_{nt}$ converges in probability to a matrix that is non-random conditional on $\mathbf{X}_t$. This is done by step 3 below.

step 3: show that $\boldsymbol{\Upsilon}_{nt}\mathbf{s}_{B,t}\boldsymbol{\Upsilon}'_{nt}$ converges in probability.

On (E.23), right multiply by $\frac{1}{\sqrt{\Delta_n}}\widehat{\Delta_j^n\mathbf{F}}'$ and average over j (note that $\frac{1}{k_n\Delta_n}\sum_j\widehat{\Delta_j^n\mathbf{F}}\widehat{\Delta_j^n\mathbf{F}}' = \mathbf{I}$):

$$\frac{1}{\Delta_n}\widehat{\mathbf{V}}_t\frac{1}{k_n}\sum_{j\in I_t^n}(\mathbf{H}_{nt}\Delta_j^n\mathbf{F} - \widehat{\Delta_j^n\mathbf{F}})\widehat{\Delta_j^n\mathbf{F}}' = o_P(1) + \boldsymbol{\Upsilon}_{nt}\mathbf{s}_{B,t}\boldsymbol{\Upsilon}'_{nt} - \widehat{\mathbf{V}}_t.$$

Note $\frac{1}{\Delta_n}\widehat{\mathbf{V}}_t\frac{1}{k_n}\sum_{j\in I_t^n}(\mathbf{H}_{nt}\Delta_j^n\mathbf{F} - \widehat{\Delta_j^n\mathbf{F}})\widehat{\Delta_j^n\mathbf{F}}' = o_P(1)$. Thus $\boldsymbol{\Upsilon}_{nt}\mathbf{s}_{B,t}\boldsymbol{\Upsilon}'_{nt} - \widehat{\mathbf{V}}_t = o_P(1)$. Moreover, the proof of part (i) implies $\|\mathbf{V}_t - \widehat{\mathbf{V}}_t\| = o_P(1)$, where $\mathbf{V}_t$ is the $K\times K$ diagonal matrix of the first K eigenvalues of $\frac{1}{pk_n\Delta_n}\sum_{i\in I_t^n}\mathbf{G}_t\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{F}'\mathbf{G}'_t$. Also note that

$$\|\frac{1}{pk_n\Delta_n}\sum_{i\in I_t^n}\mathbf{G}_t\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{F}'\mathbf{G}'_t - \frac{1}{p}\mathbf{G}_t\mathbf{c}_f\mathbf{G}'_t\| = o_P(1)$$

and the first K eigenvalues of $\frac{1}{p}\mathbf{G}_t\mathbf{c}_f\mathbf{G}'_t$ are the same as those of $\mathbf{c}_f^{1/2}\mathbf{s}_{G,t}\mathbf{c}_f^{1/2}$. Let $\bar{\mathbf{V}}_t$ be a K by K diagonal matrix, whose entries are the eigenvalues of $\mathbf{s}_{G,t}^{1/2}\mathbf{c}_f\mathbf{s}_{G,t}^{1/2} := \bar{\mathbf{A}}$. Thus $\mathbf{V}_t - \bar{\mathbf{V}}_t = o_P(1)$, implying $\widehat{\mathbf{V}}_t - \bar{\mathbf{V}}_t = o_P(1)$. Thus $\boldsymbol{\Upsilon}_{nt}\mathbf{s}_{B,t}\boldsymbol{\Upsilon}'_{nt} - \bar{\mathbf{V}}_t = o_P(1)$ and $\mathbf{V}^* = \mathrm{diag}(\boldsymbol{\Upsilon}_{nt}\mathbf{s}_{B,t}\boldsymbol{\Upsilon}'_{nt}) - \bar{\mathbf{V}}_t = o_P(1)$. Therefore, $\boldsymbol{\Upsilon}_{nt} - \boldsymbol{\Upsilon}_t = o_P(1)$, where $\boldsymbol{\Upsilon}_t = \bar{\mathbf{V}}_t^{1/2}\mathbf{A}'\mathbf{s}_{G,t}^{-1/2}$.

Recall that $\mathbf{A}$ and $\bar{\mathbf{V}}_t$ respectively denote the eigenvalues and eigenvectors of the matrix $\mathbf{s}_{G,t}^{1/2}\mathbf{c}_f\mathbf{s}_{G,t}^{1/2}$, so $\boldsymbol{\Upsilon}_t$ is a non-random matrix conditional on $\mathbf{X}_t$.

step 4: show that $\mathbf{H}_{nt}\frac{1}{k_n\Delta_n}\sum_{j\in I_t^n}\Delta_j^n\mathbf{F}\Delta_j^n\mathbf{F}'-\boldsymbol{\Upsilon}_t=o_P(1)$.

$\frac{1}{k_n\Delta_n}\sum_j\|\mathbf{H}_{nt}\Delta_j^n\mathbf{F}-\widehat{\Delta_j^n\mathbf{F}}\|^2=o_P(1)$ implies

$$\boldsymbol{\Upsilon}_{nt}=\frac{1}{k_n\Delta_n}\sum_{j\in I_t^n}(\widehat{\Delta_j^n\mathbf{F}}-\mathbf{H}_{nt}\Delta_j^n\mathbf{F})\Delta_j^n\mathbf{F}'+\mathbf{H}_{nt}\frac{1}{k_n\Delta_n}\sum_{j\in I_t^n}\Delta_j^n\mathbf{F}\Delta_j^n\mathbf{F}'=o_P(1)+\mathbf{H}_{nt}\frac{1}{k_n\Delta_n}\sum_{j\in I_t^n}\Delta_j^n\mathbf{F}\Delta_j^n\mathbf{F}'.$$

Then the result follows from $\boldsymbol{\Upsilon}_{nt}-\boldsymbol{\Upsilon}_t=o_P(1)$ Q.E.D.

E.4.2 Lemmas for the bias correction

The following technical lemmas are used to prove that the estimation errors for the bias term are negligible.

Lemma E.5. *Suppose $k_n=o(p^3),k_n^2\Delta_n=o(p)$. Then* $\frac{1}{k_np^2}\sum_{m=1}^p[\sum_{i\in I_t}\frac{1}{p}\sum_{d=1}^p h_{i-1,ld}h_{i-1,md}(\widehat{u}_{di}^2-u_{di}^2)]^2=o_P(1)$

Proof. Let $\Delta_i^nY_d$ denote the d th component of the p dimensional vector $\Delta_i^n\mathbf{Y}$. Then for $i\in I_t^n$, $\widehat{u}_{di}=\Delta_i^nY_d\Delta_n^{-1/2}-\widehat{\boldsymbol{\beta}}_{d,t}'\widehat{\mathbf{f}}_i$, implying

$$\widehat{u}_{di}-u_{di}=\Delta_n^{1/2}\alpha_{d,i-1}-\mathbf{f}_i'\mathbf{H}_{nt}'(\widehat{\boldsymbol{\beta}}_{d,t}-\mathbf{H}_{nt}'^{-1}\boldsymbol{\beta}_{d,i-1})-\widehat{\boldsymbol{\beta}}_{d,i-1}'(\widehat{\mathbf{f}}_i-\mathbf{H}_{nt}\mathbf{f}_i).$$

Also, $\widehat{\boldsymbol{\beta}}_{d,t}=\frac{1}{\Delta_nk_n}\sum_{j\in I_t^n}\widehat{\Delta_j^n\mathbf{F}}\Delta_j^nY_d$ implies

$$\begin{aligned}\widehat{\boldsymbol{\beta}}_{d,t}-\mathbf{H}_{nt}'^{-1}\boldsymbol{\beta}_{d,i-1}&=\mathbf{C}+(\mathbf{D}-\boldsymbol{\Upsilon}_{nt})\boldsymbol{\beta}_{d,i-1},\quad\text{where}\\ \mathbf{C}&=\frac{1}{k_n}\sum_{j\in I_t^n}\widehat{\mathbf{f}}_ju_{jd}+\frac{1}{k_n}\sum_{j\in I_t^n}\widehat{\Delta_j^n\mathbf{F}}\alpha_{d,j-1}+\frac{1}{k_n}\sum_{j\in I_t^n}\widehat{\mathbf{f}}_j\mathbf{f}_j'\boldsymbol{\beta}_{d,j-1}\\ \mathbf{D}&=\frac{1}{k_n}\sum_{j\in I_t^n}\widehat{\mathbf{f}}_j(\mathbf{H}_{nt}\mathbf{f}_j-\widehat{\mathbf{f}}_j)'\mathbf{H}_{nt}'^{-1}.\end{aligned}$$

Thus we have

$$\begin{aligned}\widehat{u}_{di}-u_{di}\quad=\quad&\Delta_n^{1/2}\alpha_{d,i-1}-\mathbf{f}_i'\mathbf{H}_{nt}'[\mathbf{H}_{nt}\frac{1}{k_n}\sum_{j\in I_t^n}\mathbf{f}_ju_{jd}+\frac{1}{k_n}\sum_{j\in I_t^n}\widehat{\Delta_j^n\mathbf{F}}\alpha_{d,j-1}]\\&-\mathbf{f}_i'\mathbf{H}_{nt}'\frac{1}{k_n}\sum_{j\in I_t^n}\widehat{\mathbf{f}}_j\mathbf{f}_j'(\boldsymbol{\beta}_{d,j-1}-\boldsymbol{\beta}_{d,i-1})-\widehat{\boldsymbol{\beta}}_{d,i-1}'(\widehat{\mathbf{f}}_i-\mathbf{H}_{nt}\mathbf{f}_i)\end{aligned}$$

$$+\boldsymbol{\beta}'_{d,i-1}\mathbf{H}_{nt}^{-1}\frac{1}{k_n}\sum_{j\in I_t^n}(\widehat{\mathbf{f}}_j-\mathbf{H}_{nt}\mathbf{f}_j)\widehat{\mathbf{f}}_j'\mathbf{H}_{nt}\mathbf{f}_i-\mathbf{f}_i'\mathbf{H}_{nt}'\frac{1}{k_n}\sum_{j\in I_t^n}(\widehat{\mathbf{f}}_j-\mathbf{H}_{nt}\mathbf{f}_j)u_{jd}. \tag{E.24}$$

We also have

$$\begin{aligned}
&\frac{1}{k_np^2}\sum_{m=1}^p[\sum_{i\in I_t}\frac{1}{p}\sum_{d=1}^p h_{i-1,ld}h_{i-1,md}(\widehat{u}_{di}^2-u_{di}^2)]^2\\
\leq\ &\frac{2}{k_np^2}\sum_{m=1}^p[\sum_{i\in I_t}\frac{1}{p}\sum_{d=1}^p h_{i-1,ld}h_{i-1,md}u_{di}(\widehat{u}_{di}-u_{di})]^2+\frac{1}{k_np^2}\sum_{m=1}^p[\sum_{i\in I_t}\frac{1}{p}\sum_{d=1}^p h_{i-1,ld}h_{i-1,md}(\widehat{u}_{di}-u_{di})^2]^2.
\end{aligned}$$

Substitute (E.24) in the first term on the right hand side, it is straightforward to show the first term is $o_P(1)$ and we omit the details. Also $\frac{1}{k_np^2}\sum_{m=1}^p[\sum_{i\in I_t}\frac{1}{p}\sum_{d=1}^p h_{i-1,ld}h_{i-1,md}(\widehat{u}_{di}-u_{di})^2]^2$ is bounded by $b_1+...+b_7$, with each term defined and bounded below:

$$\begin{aligned}
b_1 &= \frac{1}{k_np^2}\sum_{m=1}^p[\sum_{i\in I_t}\frac{1}{p}\sum_{d=1}^p h_{i-1,ld}h_{i-1,md}\Delta_n\alpha_{d,i-1}^2]^2=O_P(\frac{k_n\Delta_n^2}{p})=o_P(1)\\
b_2 &= \frac{1}{k_np^2}\sum_{m=1}^p[\sum_{i\in I_t}\frac{1}{p}\sum_{d=1}^p h_{i-1,ld}h_{i-1,md}(\mathbf{f}_i'\mathbf{H}_{nt}'\mathbf{H}_{nt}\frac{1}{k_n}\sum_{j\in I_t^n}\mathbf{f}_ju_{jd})^2]^2=O_P(\frac{1}{pk_n})=o_P(1)\\
b_3 &= \frac{1}{k_np^2}\sum_{m=1}^p[\sum_{i\in I_t}\frac{1}{p}\sum_{d=1}^p h_{i-1,ld}h_{i-1,md}(\frac{1}{k_n}\sum_{j\in I_t^n}\widehat{\Delta_j^n\mathbf{F}}\alpha_{d,j-1})^2]^2=O_P(\frac{k_n\Delta_n^2}{p})=o_P(1)\\
b_4 &= \frac{1}{k_np^2}\sum_{m=1}^p[\sum_{i\in I_t}\frac{1}{p}\sum_{d=1}^p h_{i-1,ld}h_{i-1,md}(\mathbf{f}_i'\mathbf{H}_{nt}'\frac{1}{k_n}\sum_{j\in I_t^n}\widehat{\mathbf{f}}_j\mathbf{f}_j'(\boldsymbol{\beta}_{d,j-1}-\boldsymbol{\beta}_{d,i-1}))^2]^2\\
&\leq O_P(\frac{k_n}{p})\frac{1}{k_n^2p}\sum_{d=1}^p\sum_{i,j\in I_t^n}\|\boldsymbol{\beta}_{d,j-1}-\boldsymbol{\beta}_{d,i-1}\|^4\\
&\leq O_P(\frac{k_n}{p})\frac{1}{k_np}\sum_{d=1}^p\sum_{j\in I_t^n}\|\boldsymbol{\beta}_{d,j-1}-\boldsymbol{\beta}_{d,t}\|^4
\end{aligned}$$

Write $\boldsymbol{\beta}_{d,j-1}-\boldsymbol{\beta}_{d,t}=\mathbf{M}_{d,j-1}^{BM}+\mathbf{M}_{d,j-1}^{Jump}+\mathbf{M}_{d,j-1}^{df}$, which are the Brownian motion, jump and drift decomposition; see Chapter 2 of Jacod and Protter (2011) for detailed definitions of each component. Then by the Burkholder-Davis-Grundy inequality (cf. Chapter 2 of Jacod and Protter (2011)), $\max_{d\leq p}\mathbb{E}\max_{j\in I_t^n}\|\mathbf{M}_{d,j-1}^{BM}\|^4=O(k_n^2\Delta_n^2)$, $\max_{d\leq p}\mathbb{E}\max_{j\in I_t^n}\|\mathbf{M}_{d,j-1}^{jump}\|^4=O(k_n\Delta_n)$, and $\max_{d\leq p}\mathbb{E}\max_{j\in I_t^n}\|\mathbf{M}_{d,j-1}^{df}\|^4=O(k_n^4\Delta_n^4)$, so $\max_{d\leq p}\mathbb{E}\max_{j\in I_t^n}\|\boldsymbol{\beta}_{d,j-1}-\boldsymbol{\beta}_{d,t}\|^4=O(k_n\Delta_n)$, implying

$$b_4=O_P(k_n^2\Delta_n/p)=o_P(1).$$

Also, note that, by the Burkholder-Davis-Grundy inequality (cf. Chapter 2 of Jacod and Protter (2011)),

$$\begin{aligned}\frac{1}{k_n}\sum_{j\in I_t^n}\|\widehat{\mathbf{f}}_j-\mathbf{H}_{nt}\mathbf{f}_j\|^4 &\leq O_P(\Delta_n^2+\frac{1}{p^2})+\frac{1}{k_n}\sum_{j\in I_t^n}\|\frac{1}{\sqrt{p}}(\mathbf{P}_{i-1}\boldsymbol{\beta}_{i-1}-\mathbf{P}_t\boldsymbol{\beta}_t)\|^4\\ &\leq O_P(\Delta_n^2+\frac{1}{p^2})+[\frac{1}{p}\sum_{l=1}^{p}\max_{i\in I_t^n}\|(\mathbf{P}'_{i-1,l}\boldsymbol{\beta}_{i-1}-\mathbf{P}'_{t,l}\boldsymbol{\beta}_t)\|^2]^2\\ &\leq O_P(\Delta_n^2+\frac{1}{p^2})+O_P(1)[\max_{l\leq p}\mathbb{E}\max_{i\in I_t^n}\|(\mathbf{P}'_{i-1,l}\boldsymbol{\beta}_{i-1}-\mathbf{P}'_{t,l}\boldsymbol{\beta}_t)\|^2]^2\\ &\leq O_P(k_n^2\Delta_n^2+\frac{1}{p^2}). \qquad \text{(E.25)}\end{aligned}$$

Now by (E.25),

$$\begin{aligned}b_5 &= \frac{1}{k_np^2}\sum_{m=1}^{p}[\sum_{i\in I_t}\frac{1}{p}\sum_{d=1}^{p}h_{i-1,ld}h_{i-1,md}(\widehat{\boldsymbol{\beta}}'_{d,i-1}(\widehat{\mathbf{f}}_i-\mathbf{H}_{nt}\mathbf{f}_i))^2]^2\\ &\leq O_P(\frac{k_n}{p})\frac{1}{k_n}\sum_{j\in I_t^n}\|\widehat{\mathbf{f}}_j-\mathbf{H}_{nt}\mathbf{f}_j\|^4=O_P(\frac{k_n}{p})O_P(k_n^2\Delta_n^2+\frac{1}{p^2}).\\ b_6 &= \frac{1}{k_np^2}\sum_{m=1}^{p}[\sum_{i\in I_t}\frac{1}{p}\sum_{d=1}^{p}h_{i-1,ld}h_{i-1,md}(\boldsymbol{\beta}'_{d,i-1}\mathbf{H}_{nt}^{-1}\frac{1}{k_n}\sum_{j\in I_t^n}(\widehat{\mathbf{f}}_j-\mathbf{H}_{nt}\mathbf{f}_j)\widehat{\mathbf{f}}_j'\mathbf{H}_{nt}\mathbf{f}_i)^2]^2\\ &\leq O_P(\frac{k_n}{p})(\frac{1}{k_n}\sum_{j\in I_t^n}\|\widehat{\mathbf{f}}_j-\mathbf{H}_{nt}\mathbf{f}_j\|^2)^2=o_P(1)\\ b_7 &= \frac{1}{k_np^2}\sum_{m=1}^{p}[\sum_{i\in I_t}\frac{1}{p}\sum_{d=1}^{p}h_{i-1,ld}h_{i-1,md}(\mathbf{f}_i'\mathbf{H}_{nt}'\frac{1}{k_n}\sum_{j\in I_t^n}(\widehat{\mathbf{f}}_j-\mathbf{H}_{nt}\mathbf{f}_j)u_{jd})^2]^2\\ &\leq O_P(\frac{k_n}{p})(\frac{1}{k_n}\sum_{j\in I_t^n}\|\widehat{\mathbf{f}}_j-\mathbf{H}_{nt}\mathbf{f}_j\|^2)^2=o_P(1).\end{aligned}$$

Putting together, $\frac{1}{k_np^2}\sum_{m=1}^{p}[\sum_{i\in I_t}\frac{1}{p}\sum_{d=1}^{p}h_{i-1,ld}h_{i-1,md}(\widehat{u}_{di}^2-u_{di}^2)]^2=o_P(1)$. Q.E.D.

The following lemma serves as an intermediate step to prove Lemma E.1.

Lemma E.6. *(i)* $\max_{d\leq p}\|\frac{1}{k_n}\sum_{j\in I_t^n}\Delta_n^{1/2}\psi_{dj}(\widehat{\mathbf{f}}_j-\mathbf{H}_{nt}\mathbf{f}_j)'\|=O_P(\Delta_n\sqrt{k_n}+\sqrt{\frac{\Delta_nJ}{pk_n}})$.
(ii) $\max_{d\leq p}\|\frac{1}{k_n}\sum_{j\in I_t^n}u_{dj}(\widehat{\mathbf{f}}_j-\mathbf{H}_{nt}\mathbf{f}_j)\|\leq O_P(\frac{J}{p}\max_{j,d}\frac{1}{J}\|\boldsymbol{\phi}(\mathbf{x}_{jd})\|^2)+o_P(\sqrt{\frac{\log p}{k_n}})$.

Proof. Note that

$$\widehat{\mathbf{f}}_i-\mathbf{H}_{nt}\mathbf{f}_i=\boldsymbol{\Xi}'_{3,t}\left(\frac{1}{\sqrt{p}}\Delta_n^{1/2}\mathbf{P}_{i-1}\boldsymbol{\psi}_i+\frac{1}{\sqrt{p}}\mathbf{P}_{i-1}\mathbf{u}_i\right)+\boldsymbol{\Xi}'_{3,t}\frac{1}{\sqrt{p}}(\mathbf{P}_{i-1}\boldsymbol{\beta}_{i-1}-\mathbf{P}_t\boldsymbol{\beta}_t)\mathbf{f}_i.$$

So (i)

$$\max_{d\le p}\|\frac{1}{k_n}\sum_{j\in I_t^n}\Delta_n^{1/2}\psi_{dj}(\widehat{\mathbf{f}}_j-\mathbf{H}_{nt}\mathbf{f}_j)'\|\le O_P(1)\max_{d\le p}\|\frac{1}{k_n}\sum_{j\in I_t^n}\Delta_n^{1/2}\psi_{dj}\frac{1}{\sqrt{p}}\Delta_n^{1/2}\mathbf{P}_{j-1}\boldsymbol{\psi}_j\|$$
$$+O_P(1)\max_{d\le p}|\psi_{dt}|\|\frac{1}{k_n}\sum_{j\in I_t^n}\Delta_n^{1/2}\frac{1}{\sqrt{p}}\mathbf{P}_{j-1}\mathbf{u}_j\|$$
$$+O_P(1)\max_{d\le p}\|\frac{1}{k_n}\sum_{j\in I_t^n}\Delta_n^{1/2}\psi_{dj}\frac{1}{\sqrt{p}}(\mathbf{P}_{j-1}\boldsymbol{\beta}_{j-1}-\mathbf{P}_t\boldsymbol{\beta}_t)\mathbf{f}_j\|$$
$$+O_P(\Delta_n^{1/2})\max_{j\in I_t^n}\max_{d\le p}|\psi_{dj}-\psi_{dt}|[\frac{1}{k_n}\sum_{j\in I_t^n}\|\frac{1}{\sqrt{p}}\mathbf{P}_{j-1}\mathbf{u}_j\|^2]^{1/2}\le O_P(\Delta_n\sqrt{k_n}+\sqrt{\frac{\Delta_nJ}{pk_n}}).$$

(E.26)

(ii)

$$\max_{d\le p}\|\frac{1}{k_n}\sum_{j\in I_t^n}u_{dj}(\widehat{\mathbf{f}}_j-\mathbf{H}_{nt}\mathbf{f}_j)\|\le O_P(1)(a_1+a_2+a_3)$$

$$\begin{aligned}
a_1 &= \max_{d\le p}\|\frac{1}{k_n}\sum_{j\in I_t^n}\frac{1}{\sqrt{p}}\mathbf{P}_{j-1}\mathbb{E}\mathbf{u}_ju_{dj}\|\le\left(\max_{d\le p}\sum_{l=1}^{p}\|\frac{1}{k_n}\sum_{j\in I_t^n}\frac{1}{\sqrt{p}}|\mathbf{P}'_{j-1,l}\mathbb{E}\mathbf{u}_ju_{dj}|\|^2\right)^{1/2}\\
&\le O_P(1)\max_{l,d,j}|\mathbf{P}_{j,ld}|\max_d\|\mathbb{E}\mathbf{u}_ju_{dj}\|_1=O_P(1)\max_{l,d,j}|\mathbf{P}_{j,ld}|=O_P(\frac{J}{p}\max_{j,d}\frac{1}{J}\|\boldsymbol{\phi}(\mathbf{x}_{jd})\|^2)\\
a_2 &= \max_{d\le p}\|\frac{1}{k_n}\sum_{j\in I_t^n}u_{dj}\mathbf{C}_j\|,\quad \mathbf{C}_j=\frac{1}{\sqrt{p}}(\mathbf{P}_{j-1}\boldsymbol{\beta}_{j-1}-\mathbf{P}_t\boldsymbol{\beta}_t)\mathbf{f}_j+\frac{1}{\sqrt{p}}\Delta_n^{1/2}\mathbf{P}_{j-1}\boldsymbol{\psi}_j\\
a_3 &= \max_{d\le p}\|\frac{1}{k_n}\sum_{j\in I_t^n}\frac{1}{\sqrt{p}}\mathbf{P}_{j-1}(\mathbf{u}_ju_{dj}-\mathbb{E}\mathbf{u}_ju_{dj})\|.
\end{aligned}$$

Note that for a_2, conditional on $\{\mathbf{C}_j\}$, $u_{dj}C_{j,l}\sim\mathcal{N}(0,\sigma^2_{u,dj}C^2_{j,l})$ where $C_{j,l}$ denotes the l th component of $\mathbf{C}_j$. Then

$$\begin{aligned}
\mathbb{E}\max_l\frac{1}{k_n}\sum_{j\in I_t^n}\sigma^2_{u,dj}C^2_{j,l} &\le 2\mathbb{E}\max_l\frac{1}{k_n}\sum_{j\in I_t^n}\sigma^2_{u,dj}[\frac{1}{\sqrt{p}}(\mathbf{P}_{j-1}\boldsymbol{\beta}_{j-1}-\mathbf{P}_t\boldsymbol{\beta}_t)'_l\mathbf{f}_j]^2\\
&\quad+2\mathbb{E}\max_l\frac{1}{k_n}\sum_{j\in I_t^n}\sigma^2_{u,dj}[\frac{1}{\sqrt{p}}\Delta_n^{1/2}\mathbf{P}'_{j-1,l}\boldsymbol{\psi}_j]^2\\
&\le C\max_{j\in I_t^n}\|\mathbf{f}_j\|^2\sum_{l\le p}\mathbb{E}\max_{j\in I_t^n}\|\mathbf{P}'_{j-1,l}\boldsymbol{\beta}_{j-1}-\mathbf{P}'_{tl}\boldsymbol{\beta}_t\|^2p^{-1}\\
&\quad+C\mathbb{E}\max_l\frac{1}{k_n}\sum_{j\in I_t^n}\|\boldsymbol{\phi}_{j-1,l}\|^2\frac{\Delta_n}{p}
\end{aligned}$$

$$\leq \quad C(\Delta_n k_n \log k_n + \Delta_n J) := \varphi_n \to 0.$$

Let $x = \sqrt{\frac{3\log p}{k_n}}\sqrt{p\psi_n \log(k_n)} = o(\sqrt{\frac{\log p}{k_n}})$ since $\Delta_n p k_n^2 = o(1)$. As $k_n \to \infty$,

$$\begin{aligned}
\mathbb{P}\left(a_2^2 > x^2\right) &= \mathbb{E}\mathbb{P}\left(a_2^2 > x^2 | \{\mathbf{C}_j\}\right) \leq \mathbb{E}\mathbb{P}\left(a_2^2 > x^2 | \{\mathbf{C}_j\}\right) 1\{\max_l \frac{1}{k_n}\sum_{j\in I_t^n} \sigma_{u,dj}^2 C_{j,l}^2 < \varphi_n \log(k_n)\} \\
&\quad + \mathbb{P}(\max_l \frac{1}{k_n}\sum_{j\in I_t^n} \sigma_{u,dj}^2 C_{j,l}^2 \geq \varphi_n \log(k_n)) \\
&\leq \sum_{d,l\leq p} \mathbb{E}\mathbb{P}\left(|\frac{1}{k_n}\sum_{j\in I_t^n} u_{dj} C_{j,l}| > xp^{-1/2} | \{\mathbf{C}_j\}\right) 1\{\max_l \frac{1}{k_n}\sum_{j\in I_t^n} \sigma_{u,dj}^2 C_{j,l}^2 < \varphi_n \log(k_n)\} \\
&\quad + \mathbb{E}\max_{l\leq p} \frac{1}{k_n}\sum_{j\in I_t^n} \sigma_{u,dj}^2 C_{j,l}^2 / [\varphi_n \log(k_n)] \\
&\leq \sum_{d,l\leq p} \mathbb{E}\exp\left(-\frac{(xp^{-1/2})^2 k_n}{\frac{1}{k_n}\sum_{j\in I_t^n} \sigma_{u,dj}^2 C_{j,l}^2}\right) 1\{\max_l \frac{1}{k_n}\sum_{j\in I_t^n} \sigma_{u,dj}^2 C_{j,l}^2 < \varphi_n \log(k_n)\} + \varphi_n \log(k_n) \\
&\leq \exp\left(2\log p - \frac{x^2 k_n}{p\varphi_n \log(k_n)}\right) + (\log k_n)^{-1} = o(1).
\end{aligned}$$

Hence $a_2 = o_P(\sqrt{\frac{\log p}{k_n}})$.

In addition, $a_3^2 \leq \max_{d,l\leq p}(\frac{1}{k_n}\sum_{j\in I_t^n} z_{jd})^2$, where $z_{jd} := \mathbf{P}'_{j-1,l}(\mathbf{u}_j u_{dj} - \mathbb{E}\mathbf{u}_j u_{dj})$. Conditional on $\{\mathbf{P}_{j-1}\}$, z_{jd} are independent across j, and

$$\begin{aligned}
&\max_l \frac{1}{k_n}\sum_{j\in I_t^n} \mathbb{E}z_{jd}^2 | \{\mathbf{P}_{j-1}\} = \max_l \frac{1}{k_n}\sum_{j\in I_t^n} \mathbf{P}_{j-1,l} \operatorname{Var}(\mathbf{u}_j u_{dj}) \mathbf{P}_{j-1,l} \leq \max_l \frac{1}{k_n}\sum_{j\in I_t^n} \|\mathbf{P}_{j-1,l}\|^2 \\
\leq \ &\frac{1}{p}\max_l \frac{1}{k_n}\sum_{j\in I_t^n} h_{j-1,ll}.
\end{aligned}$$

So for any fixed $M > 0$,

$$\begin{aligned}
&\mathbb{P}(\max_l \frac{1}{k_n}\sum_{j\in I_t^n} \mathbb{E}z_{jd}^2 | \{\mathbf{P}_{j-1}\} > M^{-1}p^{-1/2}) \leq \mathbb{P}(\max_l \frac{1}{k_n}\sum_{j\in I_t^n} h_{j-1,ll} > M^{-1}p^{1/2}) \\
\leq \ &M^4 p^{-2}\mathbb{E}\max_l (\frac{1}{k_n}\sum_{j\in I_t^n} h_{j-1,ll})^4 \leq M^4 p^{-1}\frac{1}{p}\sum_l \mathbb{E}(\frac{1}{k_n}\sum_{j\in I_t^n} h_{j-1,ll})^4 = o(1).
\end{aligned}$$

Therefore for $x = \sqrt{\frac{3M^{-1}\log p}{k_n}}p^{-1/4}$,

$$\begin{aligned}
\mathbb{P}(a_3 > x) &\leq \mathbb{E}\mathbb{P}(\max_{d,l\leq p}|\frac{1}{k_n}\sum_{j\in I_t^n} z_{jd}| > x|\{\mathbf{P}_{j-1}\}) \leq \mathbb{P}(\max_l \frac{1}{k_n}\sum_{j\in I_t^n}\mathbb{E}z_{jd}^2|\{\mathbf{P}_{j-1}\} > M^{-1}p^{-1/2}) \\
&\quad +\mathbb{E}\mathbb{P}(\max_{d,l\leq p}|\frac{1}{k_n}\sum_{j\in I_t^n} z_{jd}| > x|\{\mathbf{P}_{j-1}\})1\{\max_l \frac{1}{k_n}\sum_{j\in I_t^n}\mathbb{E}z_{jd}^2|\{\mathbf{P}_{j-1}\} < M^{-1}p^{-1/2}\} \\
&\leq \mathbb{E}\exp\left(2\log p - \frac{x^2 k_n}{\frac{1}{k_n}\sum_{j\in I_t^n}\mathbb{E}z_{jd}^2|\{\mathbf{P}_{j-1}\}}\right)1\{\max_l \frac{1}{k_n}\sum_{j\in I_t^n}\mathbb{E}z_{jd}^2|\{\mathbf{P}_{j-1}\} < M^{-1}p^{-1/2}\} + o(1) \\
&\leq \exp(2\log p - x^2 M k_n) + o(1) = o(1).
\end{aligned}$$

Hence $a_3 = o_P(\sqrt{\frac{\log p}{k_n}})$.

E.4.3 Lemmas for the estimated factors: bootstrap

Lemma E.7. *Suppose $k_n p J^{-2\eta} = o(1)$ and $pk_n^2\Delta_n = o(1)$. Then*
(i) $\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\widehat{\Delta_i^n\mathbf{F}}\boldsymbol{\psi}_i^{*'}\mathbf{P}_{t,l}^*\Delta_n = o_{P^*}((k_n p)^{-1/2})$
(ii) $\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\widehat{\Delta_i^n\mathbf{F}}\Delta_i^n\mathbf{F}'[\mathbf{G}_{i-1} - \mathbf{G}_t]^{*'}\mathbf{P}_{t,l}^* = o_{P^*}((k_n p)^{-1/2})$
(iii) $\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\widehat{\Delta_i^n\mathbf{F}}\Delta_i^n\mathbf{F}'[\boldsymbol{\Gamma}_{i-1} - \boldsymbol{\Gamma}_t]^{*'}\mathbf{P}_{t,l}^* = o_{P^*}((k_n p)^{-1/2})$
(iv) $\boldsymbol{\Upsilon}_{nt}(\mathbf{G}_{t,l}^{*'}\mathbf{P}_{t,l}^* - \mathbf{g}_{lt}) = o_{P^*}((k_n p)^{-1/2})$

Proof. The proof is the same as that of Lemma E.2, so is omitted.

Lemma E.8. *(i)* $\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\boldsymbol{\Xi}_{3,t}'\frac{1}{\sqrt{p}}\Delta_n\mathbf{P}_{i-1}\boldsymbol{\psi}_i\Delta_i^n\mathbf{U}^{*'}\mathbf{P}_{i-1,l}^* = o_{P^*}((k_n p)^{-1/2})$
(ii) $\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\boldsymbol{\Xi}_{3,t}'\frac{1}{\sqrt{p}}(\mathbf{P}_{i-1}\boldsymbol{\beta}_{i-1} - \mathbf{P}_t\boldsymbol{\beta}_t)\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{U}^{*'}\mathbf{P}_{i-1,l}^* = o_{P^*}((k_n p)^{-1/2})$
(iii) $\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\boldsymbol{\Xi}_{3,t}'\frac{1}{\sqrt{p}}\mathbf{P}_{i-1}\Delta_i^n\mathbf{U}[\Delta_i^n\mathbf{U}^{*'}\mathbf{P}_{i-1,l}^* - \frac{1}{p}\sum_{m=1}^p\Delta_i^n U_m^*\boldsymbol{\phi}_{i-1,m}^{*'}(\frac{1}{p}\boldsymbol{\Phi}_{i-1}'\boldsymbol{\Phi}_{i-1})^{-1}\boldsymbol{\phi}_{i-1,l}] = o_{P^*}((k_n p)^{-1/2})$
(iv) $\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\boldsymbol{\Xi}_{3,t}'\frac{1}{\sqrt{p}}\mathbf{P}_{i-1}\Delta_i^n\mathbf{U}\frac{1}{p}\sum_{m=1}^p[\Delta_i^n U_m^*\boldsymbol{\phi}_{i-1,m}^{*'}(\frac{1}{p}\boldsymbol{\Phi}_{i-1}'\boldsymbol{\Phi}_{i-1})^{-1}\boldsymbol{\phi}_{i-1,l} - \Delta_i^n U_m h_{i-1,ml}] = o_{P^*}((k_n p)^{-1/2})$

Proof. (i) Straightforward calculations yield

$$\begin{aligned}
&\|\frac{1}{k_n}\sum_{i\in I_t^n}\frac{1}{\sqrt{p}}\mathbf{P}_{i-1}\boldsymbol{\psi}_i\Delta_i^n\mathbf{U}^{*'}\mathbf{P}_{i-1,l}^*\| = \|\frac{1}{k_n p}\sum_{i\in I_t^n}\frac{1}{\sqrt{p}}\mathbf{P}_{i-1}\boldsymbol{\psi}_i\Delta_i^n\mathbf{U}^{*'}\boldsymbol{\Phi}_{i-1}^*(\frac{1}{p}\boldsymbol{\Phi}_{i-1}^{*'}\boldsymbol{\Phi}_{i-1}^*)^{-1}\boldsymbol{\phi}_{i-1,l}\| \\
&\leq O_{P^*}(\sqrt{\Delta_n}\sqrt{\Delta_n k_n}) + [\sum_{m=1}^p\|\frac{1}{k_n p}\sum_{i\in I_t^n}\frac{1}{\sqrt{p}}\boldsymbol{\phi}_{i-1,l}\mathbf{P}_{i-1,m}'\boldsymbol{\psi}_i\Delta_i^n\mathbf{U}^{*'}\boldsymbol{\Phi}_{i-1}^*\|_F^2\|(\frac{1}{p}\boldsymbol{\Phi}_t^{*'}\boldsymbol{\Phi}_t^*)^{-1}\|^2]^{1/2}J \\
&\leq o_{P^*}((k_n p)^{-1/2}) + [\sum_{m=1}^p\|\frac{1}{k_n p}\sum_{i\in I_t^n}\frac{1}{\sqrt{p}}\boldsymbol{\phi}_{i-1,l}\mathbf{P}_{i-1,m}'\boldsymbol{\psi}_i\Delta_i^n\mathbf{U}^{*'}\boldsymbol{\Phi}_{i-1}^*\|_F^2]^{1/2}J.
\end{aligned}$$

We now show the second term on the right is $o_{P^*}((k_n p)^{-1/2})$.

$$\begin{aligned}
& \mathbb{E} k_n p J^2 \sum_{m=1}^p \| \sum_{d=1}^p \frac{1}{k_n p} \sum_{i \in I_t^n} \frac{1}{\sqrt{p}} \boldsymbol{\phi}_{i-1,l} \mathbf{P}'_{i-1,m} \boldsymbol{\psi}_i \Delta_i^n U_d^* \boldsymbol{\phi}^{*'}_{i-1,d} \|_F^2 \\
\leq \;& \Delta_n k_n p J^2 \sum_{m=1}^p \sum_{q_1,q_2 \leq J} \mathbb{E} | \frac{1}{k_n p} \sum_{d=1}^p \sum_{i \in I_t^n} \frac{1}{\sqrt{p}} \mathbf{P}'_{i-1,m} \boldsymbol{\psi}_i u_{di} \phi_{i-1,d,q_2} \phi_{i-1,l,q_1} |^2 \\
& + \Delta_n k_n p J^2 \sum_{m=1}^p \sum_{q_1,q_2 \leq J} \frac{1}{p^2} \sum_{d=1}^p \mathbb{E} | \frac{1}{k_n} \sum_{i \in I_t^n} \frac{1}{\sqrt{p}} \mathbf{P}'_{i-1,m} \boldsymbol{\psi}_i u_{di} \phi_{i-1,d,q_2} \phi_{i-1,l,q_1} |^2 \\
\leq \;& C \Delta_n J^3 \frac{1}{k_n p} \sum_{d=1}^p \sum_{i \in I_t^n} \mathbb{E} \| \boldsymbol{\phi}_{i-1,d} \|^2 \| \boldsymbol{\phi}_{i-1,l} \|^2 \leq C \Delta_n J^3 \max_{i \in I_t^n, d \leq p} \mathbb{E} h_{i-1,dd} h_{i-1,ll} \\
\leq \;& O(\Delta_n J^3) \leq o(1).
\end{aligned}$$

(ii) The object is bounded by

$$O_{P^*}(1) [\frac{1}{k_n} \sum_{i \in I_t^n} \| \frac{1}{\sqrt{p}} (\mathbf{P}_{i-1} \boldsymbol{\beta}_{i-1} - \mathbf{P}_t \boldsymbol{\beta}_t) \|^2]^{1/2} = O_{P^*}(\sqrt{\Delta_n k_n}) = o_{P^*}((k_n p)^{-1/2}).$$

(iii) The target is bounded by

$$\begin{aligned}
& O_P(1) \| \frac{1}{k_n \Delta_n} \sum_{i \in I_t^n} \frac{1}{\sqrt{p}} \mathbf{P}_{i-1} \Delta_i^n \mathbf{U} \frac{1}{p} \sum_{m=1}^p \Delta_i^n U_m^* \boldsymbol{\phi}^{*'}_{i-1,m} [(\frac{1}{p} \boldsymbol{\Phi}^{*'}_{i-1} \boldsymbol{\Phi}^*_{i-1})^{-1} - (\frac{1}{p} \boldsymbol{\Phi}'_{i-1} \boldsymbol{\Phi}_{i-1})^{-1}] \boldsymbol{\phi}_{i-1,l} \| \\
\leq \;& a_1 + a_2 + a_3 \quad \text{where} \\
a_1 = \;& O_P(1) \| \frac{1}{k_n \Delta_n} \sum_{i \in I_t^n} \frac{1}{\sqrt{p}} \mathbf{P}_{i-1} \Delta_i^n \mathbf{U} \frac{1}{p} \sum_{m=1}^p \Delta_i^n U_m^* \boldsymbol{\phi}^{*'}_{i-1,m} [(\frac{1}{p} \boldsymbol{\Phi}^{*'}_{i-1} \boldsymbol{\Phi}^*_{i-1})^{-1} - (\frac{1}{p} \boldsymbol{\Phi}^{*'}_t \boldsymbol{\Phi}^*_t)^{-1}] \boldsymbol{\phi}_{i-1,l} \| \\
\leq \;& O_{P^*}(\sqrt{\Delta_n k_n}) = o_{P^*}((k_n p)^{-1/2}). \\
a_2 = \;& O_P(1) \| \frac{1}{k_n \Delta_n} \sum_{i \in I_t^n} \frac{1}{\sqrt{p}} \mathbf{P}_{i-1} \Delta_i^n \mathbf{U} \frac{1}{p} \sum_{m=1}^p \Delta_i^n U_m^* \boldsymbol{\phi}^{*'}_{i-1,m} [(\frac{1}{p} \boldsymbol{\Phi}'_{i-1} \boldsymbol{\Phi}_{i-1})^{-1} - (\frac{1}{p} \boldsymbol{\Phi}'_t \boldsymbol{\Phi}_t)^{-1}] \boldsymbol{\phi}_{i-1,l} \| \\
\leq \;& O_{P^*}(\sqrt{\Delta_n k_n}) = o_{P^*}((k_n p)^{-1/2}). \\
a_3 = \;& O_P(1) \| \frac{1}{k_n \Delta_n} \sum_{i \in I_t^n} \frac{1}{\sqrt{p}} \mathbf{P}_{i-1} \Delta_i^n \mathbf{U} \frac{1}{p} \sum_{m=1}^p \Delta_i^n U_m^* \boldsymbol{\phi}^{*'}_{i-1,m} [(\frac{1}{p} \boldsymbol{\Phi}'_t \boldsymbol{\Phi}_t)^{-1} - (\frac{1}{p} \boldsymbol{\Phi}^{*'}_t \boldsymbol{\Phi}^*_t)^{-1}] \boldsymbol{\phi}_{i-1,l} \| \\
\leq \;& O_{P^*}(1) [\frac{J^2}{p} \sum_{d=1}^p \| \frac{1}{k_n \Delta_n} \sum_{i \in I_t^n} \boldsymbol{\phi}_{i-1,l} \frac{1}{\sqrt{p}} \mathbf{P}'_{i-1,d} \Delta_i^n \mathbf{U} \frac{1}{p} \sum_{m=1}^p \Delta_i^n U_m^* \boldsymbol{\phi}^{*'}_{i-1,m} \|_F^2]^{1/2}
\end{aligned}$$

It suffices to show $k_n p \frac{J^2}{p} \sum_{d=1}^p \| \frac{1}{k_n\Delta_n} \sum_{i\in I_t^n} \boldsymbol{\phi}_{i-1,l} \frac{1}{\sqrt{p}} \mathbf{P}'_{i-1,d} \Delta_i^n \mathbf{U} \frac{1}{p} \sum_{m=1}^p \Delta_i^n U_m^* \boldsymbol{\phi}^{*'}_{i-1,m} \|_F^2 = o_{P^*}(1)$:

$$
\begin{aligned}
& \mathbb{E} k_n J^2 \sum_{d=1}^p \| \frac{1}{k_n\Delta_n} \sum_{i\in I_t^n} \boldsymbol{\phi}_{i-1,l} \frac{1}{\sqrt{p}} \mathbf{P}'_{i-1,d} \Delta_i^n \mathbf{U} \frac{1}{p} \sum_{m=1}^p \Delta_i^n U_m^* \boldsymbol{\phi}^{*'}_{i-1,m} \|_F^2 \\
= \; & \mathbb{E} k_n J^2 \sum_{d_1,d_2\le J} \sum_{d=1}^p \mathbb{E}^* | \frac{1}{k_n} \sum_{i\in I_t^n} \frac{1}{\sqrt{p}} \mathbf{P}'_{i-1,d} \mathbf{u}_i \frac{1}{p} \sum_{m=1}^p u^*_{mi} \phi^*_{i-1,m,d_2} \phi_{i-1,l,d_1} |^2 \\
\le \; & k_n J^2 \sum_{d_1,d_2\le J} \sum_{d=1}^p \mathbb{E} | \frac{1}{p} \sum_{m=1}^p \frac{1}{k_n} \sum_{i\in I_t^n} \frac{1}{\sqrt{p}} \mathbf{P}'_{i-1,d} \mathbf{u}_i u_{mi} \phi_{i-1,m,d_2} \phi_{i-1,l,d_1} |^2 \\
& + k_n J^2 \sum_{d_1,d_2\le J} \sum_{d=1}^p \frac{1}{p^2} \sum_{m=1}^p \mathbb{E} | \frac{1}{k_n} \sum_{i\in I_t^n} \frac{1}{\sqrt{p}} \mathbf{P}'_{i-1,d} \mathbf{u}_i u_{mi} \phi_{i-1,m,d_2} \phi_{i-1,l,d_1} |^2 \\
\le \; & C \frac{J^2}{p^4} \max_{i\in I_t^n} \sum_{m,m'\le p} \sum_{q_1,q_2\le p} |\mathrm{Cov}(u_{q_1,i} u_{mi}, u_{q_2,i} u_{m'i})| \\
& + C k_n J^2 \frac{1}{p^2} \max_{m,d\le p, i\in I_t^n} \mathbb{E} h^2_{i-1,dm} h_{i-1,mm} h_{i-1,ll} = o(1)
\end{aligned}
$$

given that $k_n J^2 = o(p^2)$. The last inequality follows from computing the expectations with respect to the "u" terms, given the assumption that $\Delta_i^n \mathbf{U}$ is independent of $\{\mathbf{X}_{lt}\}$.

(iv) Note that $\mathbb{E}^* \frac{1}{p} \sum_{m=1}^p [u^*_{mi} \boldsymbol{\phi}^{*'}_{i-1,m} - u_{mi} \boldsymbol{\phi}'_{i-1,m}] = 0$. It then follows from the assumption that $\Delta_i^n \mathbf{U}$ is independent of $\{\mathbf{X}_{lt}\}$,

$$
\begin{aligned}
& \mathbb{E}\mathbb{E}^* \| \frac{1}{k_n\Delta_n} \sum_{i\in I_t^n} \frac{1}{\sqrt{p}} \mathbf{P}_{i-1} \Delta_i^n \mathbf{U} \frac{1}{p} \sum_{m=1}^p [\Delta_i^n U_m^* \boldsymbol{\phi}^{*'}_{i-1,m} (\frac{1}{p} \boldsymbol{\Phi}'_{i-1} \boldsymbol{\Phi}_{i-1})^{-1} \boldsymbol{\phi}_{i-1,l} - \Delta_i^n U_m h_{i-1,ml}] \|^2 \\
= \; & \mathbb{E} \sum_{k=1}^p \frac{1}{p^3} \sum_{m=1}^p \mathrm{Var}^* (\frac{1}{k_n} \sum_{i\in I_t^n} \mathbf{P}'_{i-1,k} \mathbf{u}_i u^*_{mi} \boldsymbol{\phi}^{*'}_{i-1,m} (\frac{1}{p} \boldsymbol{\Phi}'_{i-1} \boldsymbol{\Phi}_{i-1})^{-1} \boldsymbol{\phi}_{i-1,l}) \\
\le \; & \mathbb{E} \sum_{k=1}^p \frac{1}{p^3} \sum_{m=1}^p (\frac{1}{k_n} \sum_{i\in I_t^n} \mathbf{P}'_{i-1,k} \mathbf{u}_i u_{mi} \boldsymbol{\phi}'_{i-1,m} (\frac{1}{p} \boldsymbol{\Phi}'_{i-1} \boldsymbol{\Phi}_{i-1})^{-1} \boldsymbol{\phi}_{i-1,l})^2 \\
\le \; & \frac{C}{p^2} \frac{1}{k_n} \max_{i,m,lk,q,v} \mathbb{E} | h^2_{i-1,ml} h_{i-1,kq} h_{i-1,kv} | \max_{m\le p} \frac{1}{p} \sum_{v=1}^p \sum_{q=1}^p |\mathrm{Cov}(u_{qi} u_{mi}, u_{vi} u_{mi})| \\
& + \frac{C}{p^3} \max_{m,i} \sum_{q=1}^p |\mathbb{E} u_{qi} u_{mi}| \max_{m,i} \sum_{q'=1}^p |\mathbb{E} u_{q'i} u_{mi}| \max_{i,j,k,q,m,l,q'} \mathbb{E} | h_{i-1,kq} h_{i-1,ml} h_{j-1,kq'} h_{s,ml} | \\
\le \; & o((pk_n)^{-1}) + O(p^{-3}) = o((pk_n)^{-1}).
\end{aligned}
$$

E.4.4 Technical lemmas for Theorem 3.5

Theorem 3.5 is regarding the inference about the long-term instrumental beta $\int_0^T \mathbf{g}_{lt} dt$. While the estimator has a similar expansion:

$$\widehat{\int_0^T \mathbf{g}_{lt} dt} - \int_0^T \mathbf{g}_{lt} dt = \mathbf{a}_1 + \mathbf{a}_2 + o_P(\sqrt{\Delta_n/p})$$

and similar to the spot case that we need to show that the remainder term is negligible, the major difference in the technical argument from the spot case is that, instead of focusing on a fixed window I_t^n, we should be concerned about the behavior and the dependence structure of the blocking schemes over the entire horizon $[0, T]$. We rely on the following lemma in this case.

Lemma E.9. *Let Θ_t^n be a random variable defined on the block I_t^n, hence $\Theta_t^n \in \mathcal{F}_{t+k_n}^n$. Assume Θ_t^n is uniformly bounded in probability over time t. Consider a deterministic sequence $R_n \to +\infty$. If, for any finite T, either*

$$\sup_{t\in[0,T]} \mathbb{E}\big(R_n|\Theta_t^n|\big) = o(1), \tag{E.27}$$

or

$$\begin{aligned} &\sup_{t\in[0,T]} \mathbb{E}\big(R_n^2|\Theta_t^n|^2 k_n\Delta_n\big) = o(1), \\ &\sup_{t\in[0,T]} \big|\mathbb{E}\big(R_n\Theta_t^n \mid \mathcal{F}_t^n\big)\big| = o_P(1). \end{aligned} \tag{E.28}$$

is satisfied, then we have the following result:

$$\Bigg|\sum_{t=0}^{\lfloor T/\Delta_n\rfloor-k_n-1} R_n\Theta_t^n\Delta_n\Bigg| = \Bigg|\frac{1}{k_n}\sum_{t_0=1}^{k_n}\sum_{t=0}^{\lfloor(\frac{T}{\Delta_n}-t_0)/k_n\rfloor-1} R_n\Theta_{tk_n+t_0}^n k_n\Delta_n\Bigg| = o_P(1).$$

Proof. Consider a non-overlapping blocking scheme that has block size k_n and starts from the interval $t_0 = 1, \cdots, k_n$. For any finite T, let

$$V_{t_0}^T := \sum_{t=0}^{\lfloor(\frac{T}{\Delta_n}-t_0)/k_n\rfloor-1} R_n\Theta_{tk_n+t_0}^n k_n\Delta_n.$$

Then we have $\sum_{t=0}^{\lfloor T/\Delta_n \rfloor - k_n - 1} R_n \Theta_t^n \Delta_n = \frac{1}{k_n} \sum_{t_0=1}^{k_n} V_{t_0}^T$.

First, in the case of (E.27), by Markov's inequality, we have

$$\mathbb{P}\big(|V_{t_0}^T| > \epsilon\big) \leq \frac{1}{\epsilon} \mathbb{E}\big(|V_{t_0}^T|\big) \leq \frac{1}{\epsilon} \mathbb{E}\Big(\sum_{t=0}^{\lfloor (\frac{T}{\Delta_n} - t_0)/k_n \rfloor - 1} R_n |\Theta_t^n| k_n \Delta_n \Big) = o(1).$$

This holds for any t_0 and finite T. Then $\mathbb{P}\Big(\big| \frac{1}{k_n} \sum_{t_0=1}^{k_n} V_{t_0}^T \big| > \epsilon \Big) \leq \frac{1}{\epsilon} \frac{1}{k_n} \sum_{t_0=1}^{k_n} \mathbb{E}\big(|V_{t_0}^T|\big) = o(1)$.

Next in the case of (E.28), it follows that

$$\begin{aligned}
&\mathbb{E}\big(|V_{t_0}^T|^2\big) = \mathbb{E}\Bigg(\Bigg| \sum_{t=0}^{\lfloor (\frac{T}{\Delta_n} - t_0)/k_n \rfloor - 1} R_n \Theta_t^n k_n \Delta_n \Bigg|^2 \Bigg) \\
=& \mathbb{E}\Bigg(\sum_{t=0}^{\lfloor (\frac{T}{\Delta_n} - t_0)/k_n \rfloor - 1} \big(R_n \Theta_t^n k_n \Delta_n \big)^2 + \sum_{s \neq t, s, t=0}^{\lfloor (\frac{T}{\Delta_n} - t_0)/k_n \rfloor - 1} R_n^2 \Theta_t^n \Theta_{s k_n + t_0}^n (k_n \Delta_n)^2 \Bigg) \\
=& \sum_{t=0}^{\lfloor (\frac{T}{\Delta_n} - t_0)/k_n \rfloor - 1} \mathbb{E}\Big(\mathbb{E}\big(R_n^2 |\Theta_t^n|^2 k_n \Delta_n \mid \mathcal{F}_t^n \big) \Big) k_n \Delta_n + 2 \sum_{t > s = 0}^{\lfloor (\frac{T}{\Delta_n} - t_0)/k_n \rfloor - 1} \mathbb{E}\Big(\mathbb{E}\big(R_n \Theta_t^n \mid \mathcal{F}_t^n \big) R_n \Theta_s^n \Big) (k_n \Delta_n)^2 \\
=& \sum_{t=0}^{\lfloor (\frac{T}{\Delta_n} - t_0)/k_n \rfloor - 1} o(1) k_n \Delta_n + 2 \sum_{t > s = 0}^{\lfloor (\frac{T}{\Delta_n} - t_0)/k_n \rfloor - 1} o(1) (k_n \Delta_n)^2 = o(1).
\end{aligned}$$

Then by Markov's inequality, we obtain $\mathbb{P}\big(\big| V_{t_0}^T \big| > \epsilon \big) \leq \frac{1}{\epsilon^2} \mathbb{E}\big(\big| V_{t_0}^T \big|^2 \big) = o(1)$. Again, this holds for any t_0 and any finite T.

Consequently, Markov's inequality and Cauchy-Schwartz inequality imply that

$$\begin{aligned}
&\mathbb{P}\Big(\big| \frac{1}{k_n} \sum_{t_0=1}^{k_n} V_{t_0}^T \big| > \epsilon \Big) \leq \frac{1}{\epsilon^2} \mathbb{E}\Big(\big| \frac{1}{k_n} \sum_{t_0=1}^{k_n} V_{t_0}^T \big|^2 \Big) \\
&\leq \frac{1}{\epsilon^2} \mathbb{E}\Big(\sum_{t_0=1}^{k_n} \frac{1}{k_n^2} \sum_{t_0=1}^{k_n} |V_{t_0}^T|^2 \Big) \leq \frac{1}{\epsilon^2} \frac{1}{k_n} \sum_{t_0=1}^{k_n} \mathbb{E}\big(|V_{t_0}^T|^2 \big) = o(1).
\end{aligned}$$

□

Lemma E.10. *(i)* $\sqrt{p/\Delta_n} \sum_{t=1}^{[T/\Delta_n] - k_n} \Delta_n \widehat{\mathbf{c}}_{FF,t}^{-1} \frac{1}{k_n} \sum_{i \in I_t^n} \Delta_i^n \mathbf{F} \boldsymbol{\psi}_i' \mathbf{P}_{t,l} = o_P(1)$

(ii) $\sqrt{p/\Delta_n} \sum_{t=1}^{[T/\Delta_n] - k_n} [\mathbf{P}_t \mathbf{G}_t - \mathbf{G}_t]_l = o_P(1)$

(iii) $\sqrt{p/\Delta_n} \sum_{t=1}^{[T/\Delta_n] - k_n} \Delta_n \widehat{\mathbf{c}}_{FF,t}^{-1} \frac{1}{k_n \Delta_n} \sum_{i \in I_t^n} \Delta_i^n \mathbf{F} \Delta_i^n \mathbf{F}' [\mathbf{G}_{i-1} - \mathbf{G}_t]' \mathbf{P}_{t,l} = o_P(1)$

(iv) $\sqrt{p/\Delta_n} \sum_{t=1}^{[T/\Delta_n] - k_n} \Delta_n \widehat{\mathbf{c}}_{FF,t}^{-1} \frac{1}{k_n \Delta_n} \sum_{i \in I_t^n} \Delta_i^n \mathbf{F} \Delta_i^n \mathbf{F}' [\boldsymbol{\Gamma}_{i-1} - \boldsymbol{\Gamma}_t]' \mathbf{P}_{t,l} = o_P(1)$

(v) For $q = 1, \cdots, k_n$,

$$\sqrt{\frac{p}{\Delta_n}} \sum_{t=1}^{[T/(k_n\Delta_n)]-k_n-q} \mathbf{v}'(\widehat{\mathbf{c}}_{FF,tk_n+q}^{-1} - \mathbf{c}_{FF,tk_n+q}^{-1}) \frac{1}{k_n} \sum_{i=1}^{k_n} \mathbf{f}_{tk_n+q+i} u_{m,tk_n+q+i} h_{tk_n+q+i-1,ml}\, k_n\Delta_n = o_P(1)$$

Proof. (i) Recall

$$\boldsymbol{\psi}_i = \frac{1}{\Delta_n}\int_{(i-1)\Delta_n}^{j\Delta_n} \boldsymbol{\alpha}_s ds + \frac{1}{\Delta_n}\int_{(i-1)\Delta_n}^{j\Delta_n} (\boldsymbol{\beta}_s - \boldsymbol{\beta}_{(i-1)\Delta_n}) d\mathbf{F}_s.$$

For any $t_0 = 1, \cdots, k_n$, define

$$\Xi(1)_t^n := \frac{1}{k_n}\sum_{i\in I_t^n} \Delta_i^n \mathbf{F}\boldsymbol{\alpha}_{i-1}' \mathbf{P}_{t,l}$$

where we recall, writing $\boldsymbol{\alpha}_{i-1} := \boldsymbol{\alpha}_t$ at $t = \Delta_n(i-1)$,

$$\begin{aligned}
\boldsymbol{\psi}_i &= \boldsymbol{\alpha}_{i-1} + \bar{\boldsymbol{\psi}}_i + \bar{\bar{\boldsymbol{\psi}}}_i, \quad \text{where} \\
\bar{\boldsymbol{\psi}}_i &= \frac{1}{\Delta_n}\int_{(i-1)\Delta_n}^{i\Delta_n} (\boldsymbol{\alpha}_s - \boldsymbol{\alpha}_{i-1}) ds \\
\bar{\bar{\boldsymbol{\psi}}}_i &= \frac{1}{\Delta_n}\int_{(i-1)\Delta_n}^{i\Delta_n} (\boldsymbol{\beta}_s - \boldsymbol{\beta}_{(i-1)\Delta_n}) d\mathbf{F}_s
\end{aligned}$$

We now show that

$$\begin{aligned}
&\sqrt{p/\Delta_n} \sum_{t=1}^{[T/\Delta_n]-k_n} \Delta_n \widehat{\mathbf{c}}_{FF,t}^{-1} \Xi(1)_t^n = o_P(1) \\
&\sqrt{p/\Delta_n} \sum_{t=1}^{[T/\Delta_n]-k_n} \Delta_n \widehat{\mathbf{c}}_{FF,t}^{-1} \frac{1}{k_n}\sum_{i\in I_t^n} \Delta_i^n \mathbf{F} m_i' \mathbf{P}_{t,l} = o_P(1), \quad m_i = \bar{\boldsymbol{\psi}}_i, \bar{\bar{\boldsymbol{\psi}}}_i
\end{aligned}$$

In the above three terms, only the term involving $\bar{\boldsymbol{\psi}}_i$ can be simply proved by applying the Cauchy-Schwarz inequality. Proving the other two terms (involving $\boldsymbol{\alpha}_{i-1}$ and $\boldsymbol{\beta}_s - \boldsymbol{\beta}_{(i-1)\Delta_n}$) are more involved. The proof for the $\boldsymbol{\alpha}_{i-1}$ term relies on the fact that $\boldsymbol{\alpha}_{i-1}$ is $\mathcal{F}_{i-1}$-measurable.

We first analyze $\sqrt{p/\Delta_n}\sum_{t=1}^{[T/\Delta_n]-k_n} \Delta_n \widehat{\mathbf{c}}_{FF,t}^{-1} \Xi(1)_t^n$. Note that $\Xi(1)_t^n$ is bounded in probability and is measurable to $\mathcal{F}_{t+k_n}^n$. Using the localization technique, we can strengthen that $\Xi(1)_t^n$ is bounded.

Note that

$$\mathbb{E}(\|\Delta_i^n \mathbf{F}\boldsymbol{\alpha}'_{i-1}\mathbf{P}_{t,l}\|^2 \mid \mathcal{F}_{i-1}^n) \leq L\Delta_n$$

One can verify that for any $t = 0, \cdots, \lfloor(\frac{T}{\Delta_n} - t_0)/k_n\rfloor - 1$ that

$$\Big|\mathbb{E}\Big(\mathbf{c}_{FF,t}^{-1}\Xi(1)_t^n \mid \mathcal{F}_t^n\Big)\Big| \leq \frac{1}{k_n}\sum_{i\in I_t^n}\big\|\mathbf{c}_{FF,t}^{-1}\,\mathbb{E}(\Delta_i^n\boldsymbol{\alpha}_{i-1}\mathbf{P}_{t,l} \mid \mathcal{F}_{i-1}^n)\big\| \leq L\Delta_n,$$

Moreover, we have

$$\begin{aligned}
&\mathbb{E}\Big(\big(\mathbf{c}_{FF,t}^{-1}\Xi(1)_t^n\big)^2 \mid \mathcal{F}_t^n\Big) = \mathbb{E}\Big(\frac{\mathbf{c}_{FF,t}^{-2}}{k_n^2}\sum_{i,j\in I_t^n}\Delta_i^n\mathbf{F}\boldsymbol{\alpha}'_{i-1}\mathbf{P}_{t,l}\,\Delta_j^n\mathbf{F}\boldsymbol{\psi}'_j\mathbf{P}_{t,l} \mid \mathcal{F}_t^n\Big)\\
&= \mathbb{E}\Big(\frac{\mathbf{c}_{FF,t}^{-2}}{k_n^2}\sum_{i\in I_t^n}\mathbb{E}\big((\Delta_i^n\mathbf{F}\boldsymbol{\alpha}'_{i-1}\mathbf{P}_{t,l})^2 \mid \mathcal{F}_{i-1}^n\big) \mid \mathcal{F}_t^n\Big)\\
&\quad + 2\,\mathbb{E}\Big(\frac{\mathbf{c}_{FF,t}^{-2}}{k_n^2}\sum_{i>j\in I_t^n}\mathbb{E}\Big(\Delta_j^n\mathbf{F}\,\mathbb{E}\big(\Delta_i^n\mathbf{F} \mid \mathcal{F}_{i-1}^n\big)\,\boldsymbol{\alpha}'_{i-1}\mathbf{P}_{t,l} \mid \mathcal{F}_{j-1}^n\Big)\boldsymbol{\psi}'_j\mathbf{P}_{t,l} \mid \mathcal{F}_t^n\Big)\\
&\leq L\Big(\frac{\Delta_n}{k_n} + \frac{\Delta_n^2}{k_n}\Big) \leq L\frac{\Delta_n}{k_n}.
\end{aligned}$$

Let $R_n = \sqrt{p/\Delta_n}$. Then, as long as $p\Delta_n \to 0$, we have (cf. (E.28))

$$\sup_{t\in[0,T]}\Big|\mathbb{E}\Big(R_n\Xi(1)_t^n \mid \mathcal{F}_t^n\Big)\Big| \leq L\sqrt{p\Delta_n} = o(1),$$

$$\sup_{t\in[0,T]}\mathbb{E}\Big(\big(R_n\Xi(1)_t^n\big)^2 k_n\Delta_n \mid \mathcal{F}_t^n\Big) \leq L\frac{p}{\Delta_n}\frac{\Delta_n}{k_n}k_n\Delta_n = Lp\Delta_n = o(1).$$

According to Lemma E.9, it follows that $\sqrt{p/\Delta_n}\sum_{t=0}^{\lfloor T/\Delta_n\rfloor - k_n - 1}\Xi(1)_t^n\Delta_n = o_P(1)$.

It remains to show that

$$\sqrt{p/\Delta_n}\sum_{t=0}^{\lfloor T/\Delta_n\rfloor - k_n - 1}(\widehat{\mathbf{c}}_{FF,t}^{-1} - \mathbf{c}_{FF,t}^{-1})\Xi(1)_t^n\Delta_n = o_P(1).$$

Note that $(\widehat{\mathbf{c}}_{FF,t}^{-1} - \mathbf{c}_{FF,t}^{-1})\Xi(1)_t^n$ is also bounded (according to the localization technique) and measurable to $\mathcal{F}_t^n$. For any t, $\widehat{\mathbf{c}}_{FF,t}$ is a consistent estimator of $\mathbf{c}_{FF,t}$. Since $\mathbf{c}_{FF,t}$ is bounded above and below on $[0,T]$, continuous mapping theorem implies that $\widehat{\mathbf{c}}_{FF,t}^{-1} - \mathbf{c}_{FF,t}^{-1} = o_P(1)$ for any t. There are two main sources of error in $\widehat{\mathbf{c}}_{FF,t} - \mathbf{c}_{FF,t}$ (see, e.g. Jacod and Rosenbaum

(2013) and Aït-Sahalia et al. (2017)):

$$\widehat{\mathbf{c}}_{FF,t} - \mathbf{c}_{FF,t} = \frac{\mathbf{c}_{FF,t}}{k_n} \sum_{i \in I_t^n} \Big(\Big(\frac{\Delta_i^n W^F}{\sqrt{\Delta_n}} \Big)^2 - 1 \Big) + \frac{1}{k_n} \sum_{i \in I_t^n} (\mathbf{c}_{FF,i-1} - \mathbf{c}_{FF,t}) + \text{higher order terms}, \tag{E.29}$$

where W^F is the Brownian motion that drives the factors. Here with abuse of notation, we write $\mathbf{c}_{FF,i-1} := \mathbf{c}_{FF,(i-1)\Delta_n}$. Recall that $\mathbf{c}_{FF,t}$ is bounded from below on the interval $[0, T]$. Denote this lower bound by $\underline{\mathbf{c}}$. Since $\mathbf{c}_{FF,t}$ is right continuous almost surely, with an overwhelming probability, $\mathbf{c}_{FF,t}$ has small jumps over the local window. Formally, $\mathbb{P}\Big(|\frac{1}{k_n} \sum_{i \in I_t^n} (\mathbf{c}_{FF,i-1} - \mathbf{c}_{FF,t})| > \underline{\mathbf{c}}/4 \Big) = o(1)$. Following the argument used in Jacod and Protter (2011), it is sufficient to prove the desired result on the set $\{ |\frac{1}{k_n} \sum_{i \in I_t^n} (\mathbf{c}_{FF,i-1} - \mathbf{c}_{FF,t})| < \underline{\mathbf{c}}/4 \}$. Conditional on this set, Taylor expansion implies that

$$\widehat{\mathbf{c}}_{FF,t}^{-1} - \mathbf{c}_{FF,t}^{-1} = \sum_{m=1}^{\infty} \frac{(-1)^m}{\mathbf{c}_{FF,t}^{m+1}} (\widehat{\mathbf{c}}_{FF,t} - \mathbf{c}_{FF,t})^m. \tag{E.30}$$

Therefore, to study $(\widehat{\mathbf{c}}_{FF,t}^{-1} - \mathbf{c}_{FF,t}^{-1}) \Xi(1)_t^n$, we need to analyze $(\widehat{\mathbf{c}}_{FF,t} - \mathbf{c}_{FF,t})^m \frac{1}{k_n} \sum_{i \in I_t^n} \Delta_i^n \mathbf{F} \boldsymbol{\alpha}'_{i-1} \mathbf{P}_{t,l}$ using (E.29). We focus on the non-overlapping blocking schemes, and the overlapping blocking scheme then follows from Lemma E.9.

We shall first study its mean. Consider $m = 1$. With the first term of (E.29), we have

$$\frac{\mathbf{c}_{FF,t}}{k_n} \Big(\Big(\frac{\Delta_i^n W^F}{\sqrt{\Delta_n}} \Big)^2 - 1 \Big) \frac{1}{k_n} \sum_{i \in I_t^n} \Delta_i^n \mathbf{F} \boldsymbol{\alpha}'_{i-1} \mathbf{P}_{t,l},$$

whose mean is at most $O(\Delta_n)$. As for the second term of (E.29) multiplied by $\frac{1}{k_n} \sum_{i \in I_t^n} \Delta_i^n \mathbf{F} \boldsymbol{\alpha}'_{i-1} \mathbf{P}_{t,l}$, its conditional mean is given by

$$\Big| \mathbb{E} \Big(\frac{1}{k_n^2} \sum_{i,j \in I_t^n} (\mathbf{c}_{FF,j-1} - \mathbf{c}_{FF,t}) \Delta_i^n \mathbf{F} \boldsymbol{\alpha}'_{i-1} \mathbf{P}_{t,l} \mid \mathcal{F}_t^n \Big) \Big|.$$

Since we are studying the continuous factor $\mathbf{F}$, so $\Delta_i^n \mathbf{F}$ is a Brownian increment, hence it has zero correlation with the jump increment of $\mathbf{c}_{FF,j-1} - \mathbf{c}_{FF,t}$. Therefore, it suffices to consider the Brownian part of $\mathbf{c}_{FF,j-1} - \mathbf{c}_{FF,t}$.

When $i \geq j$, the factor increment $\Delta_i^n \mathbf{F}$ does not overlap with $\mathbf{c}_{FF,j-1} - \mathbf{c}_{FF,t}$. Therefore, one can use the Law of total expectation and obtain that

$$\Big|\mathbb{E}\Big(\frac{1}{k_n^2}\sum_{i\geq j\in I_t^n}(\mathbf{c}_{FF,j-1}-\mathbf{c}_{FF,t})\Delta_i^n\mathbf{F}\boldsymbol{\alpha}'_{i-1}\mathbf{P}_{t,l}\mid\mathcal{F}_t^n\Big)\Big|\leq Lk_n\Delta_n^2.$$

When $i<j$, the factor increment $\Delta_i^n\mathbf{F}$ may co-move with its volatility $\mathbf{c}_{FF,t}$ over this small interval. However, the order of this co-movement is bounded by $O(\Delta_n)$, uniformly over all such intervals within $[0,T]$. Following this argument, tedious calculation yields that

$$\Big|\mathbb{E}\Big(\frac{1}{k_n^2}\sum_{i<j\in I_t^n}(\mathbf{c}_{FF,j-1}-\mathbf{c}_{FF,t})\Delta_i^n\mathbf{F}\boldsymbol{\alpha}'_{i-1}\mathbf{P}_{t,l}\mid\mathcal{F}_t^n\Big)\Big|\leq Lk_n\Delta_n^2.$$

Now consider the case with $(\widehat{\mathbf{c}}_{FF,t}-\mathbf{c}_{FF,t})^m$. Again, the cross product of Δ_i^nF and the first term's any integer power has zero expectation. For the second term, following similar argument, we can obtain

$$\Big|\mathbb{E}\Big(\frac{1}{k_n^2}\sum_{i<j\in I_t^n}(\mathbf{c}_{FF,j-1}-\mathbf{c}_{FF,t})^m\Delta_i^n\mathbf{F}\boldsymbol{\alpha}'_{i-1}\mathbf{P}_{t,l}\mid\mathcal{F}_t^n\Big)\Big|\leq L(k_n\Delta_n)^m\Delta_n.$$

Since $\sum_{m=1}^{\infty}\frac{(-1)^m}{\mathbf{c}_{FF,t}^{m+1}}(k_n\Delta_n)^m<Lk_n\Delta_n$, we get, almost surely,

$$\sup_{t\in[0,T]}\sqrt{p/\Delta_n}\,\Big|\mathbb{E}\Big((\widehat{\mathbf{c}}_{FF,t}^{-1}-\mathbf{c}_{FF,t}^{-1})\Xi(1)_t^n\mid\mathcal{F}_t^n\Big)\Big|\leq L\sqrt{pk_n\Delta_n^3}=o(1).$$

Next, we consider the variance. As we shall use the Cauchy-Schwarz inequality, considering the conditional variance is the same as considering the unconditional variance. We have

$$\mathbb{E}\Big((\widehat{\mathbf{c}}_{FF,t}^{-1}-\mathbf{c}_{FF,t}^{-1})^2(\Xi(1)_t^n)^2k_n\Delta_n\Big)\leq\Big(\mathbb{E}\big((\widehat{\mathbf{c}}_{FF,t}^{-1}-\mathbf{c}_{FF,t}^{-1})^4\big)\mathbb{E}\big((\Xi(1)_t^n)^4\big)\Big)^{1/2}k_n\Delta_n.$$

The analysis of spot volatility estimator and continuous mapping theorem suggest that $\mathbb{E}\big((\widehat{\mathbf{c}}_{FF,t}^{-1}-\mathbf{c}_{FF,t}^{-1})^4\big)=O(1/k_n^2)$. The leading term in the expansion of $\mathbb{E}\big((\Xi(1)_t^n)^4\big)$ is given by

$$\begin{aligned}&\mathbb{E}\Big(\frac{1}{k_n^4}\sum_{i\in I_t^n}\big(\Delta_i^n\mathbf{F}\boldsymbol{\alpha}'_{i-1}\mathbf{P}_{t,l}\big)^4+\frac{1}{k_n^4}\sum_{i,j\in I_t^n}\big(\Delta_i^n\mathbf{F}\boldsymbol{\alpha}'_{i-1}\mathbf{P}_{t,l}\big)^2\big(\Delta_j^n\mathbf{F}\boldsymbol{\alpha}'_{i-1}\mathbf{P}_{t,l}\big)^2\Big)\\ \leq\,&O\big(\frac{\Delta_n^2}{k_n^3}+\frac{\Delta_n^2}{k_n^2}\big)=O\big(\frac{\Delta_n^2}{k_n^2}\big).\end{aligned}$$

It then follows that, for $R_n = \sqrt{p/\Delta_n}$,

$$\sup_{t\in[0,T]} \mathbb{E}\Big(R_n^2(\widehat{\mathbf{c}}_{FF,t}^{-1} - \mathbf{c}_{FF,t}^{-1})^2(\Xi(1)_t^n)^2 k_n\Delta_n\Big) \leq O(p\Delta_n/k_n) = o(1).$$

Therefore, under the same conditions as above, we have

$$\sqrt{p/\Delta_n}\sum_{t=0}^{\lfloor T/\Delta_n\rfloor - k_n - 1}(\widehat{\mathbf{c}}_{FF,t}^{-1} - \mathbf{c}_{FF,t}^{-1})\Xi(1)_t^n\Delta_n = o_P(1).$$

These yields: $\sqrt{p/\Delta_n}\sum_{t=0}^{\lfloor T/\Delta_n\rfloor-k_n-1}\widehat{\mathbf{c}}_{FF,t}^{-1}\frac{1}{k_n}\sum_{i\in I_t^n}\Delta_i^n\mathbf{F}\boldsymbol{\alpha}_{i-1}'\mathbf{P}_{t,l}\Delta_n = o_P(1)$.

Next, we show $\sqrt{p/\Delta_n}\sum_{t=1}^{[T/\Delta_n]-k_n}\Delta_n\widehat{\mathbf{c}}_{FF,t}^{-1}\frac{1}{k_n}\sum_{i\in I_t^n}\Delta_i^n\mathbf{F}m_i'\mathbf{P}_{t,l} = o_P(1), \quad m_i = \bar{\boldsymbol{\psi}}_i, \bar{\bar{\boldsymbol{\psi}}}_i$.

By Itô's formula, one can show that

$$\begin{aligned}
&\sqrt{p/\Delta_n}\,\|\mathbb{E}\big(\mathbf{c}_{FF,t}^{-1}\,\Delta_i^n F\bar{\bar{\boldsymbol{\psi}}}_i\mathbf{P}_{t,l} \mid \mathcal{F}_t^n\big)\| \\
\leq{}& \sqrt{p/\Delta_n}\,\|\mathbb{E}\big(\mathbf{c}_{FF,t}^{-1}\frac{1}{\Delta_n}\int_{(i-1)\Delta_n}^{i\Delta_n}(\boldsymbol{\beta}_s - \boldsymbol{\beta}_{(i-1)\Delta_n})ds\mathbf{P}_{t,l} \mid \mathcal{F}_t^n\big)\| \leq L\sqrt{p\Delta_n}\to 0, \\
&p/\Delta_n\,\|\mathbb{E}\big(\|\mathbf{c}_{FF,t}^{-1}\,\Delta_i^n F\bar{\bar{\boldsymbol{\psi}}}_i\|^2 \mid \mathcal{F}_t^n\big)\| \leq Lp\Delta_n \to 0.
\end{aligned}$$

According to Lemma E.9, it follows that $\sqrt{p/\Delta_n}\sum_{t=0}^{\lfloor T/\Delta_n\rfloor-k_n-1}\mathbf{c}_{FF,t}^{-1}\Delta_i^n F\bar{\bar{\boldsymbol{\psi}}}_i\mathbf{P}_{t,l}\Delta_n = o_P(1)$. To get

$$\sqrt{p/\Delta_n}\sum_{t=0}^{\lfloor T/\Delta_n\rfloor-k_n-1}(\widehat{\mathbf{c}}_{FF,t}^{-1} - \mathbf{c}_{FF,t}^{-1})\Delta_i^n F\bar{\bar{\boldsymbol{\psi}}}_i\mathbf{P}_{t,l}\Delta_n = o_P(1),$$

one just note that although $\Delta_i^n F\bar{\bar{\boldsymbol{\psi}}}_i$ is an even function of the Brownian motion driving F, the increment $(\boldsymbol{\beta}_s - \boldsymbol{\beta}_{(i-1)\Delta_n}) = O_P(\sqrt{\Delta_n})$, and its mean is of order $O(\Delta_n)$, uniformly in s and i. Therefore, the mean of

$$\frac{\mathbf{c}_{FF,t}}{k_n}\Big(\Big(\frac{\Delta_i^n W^F}{\sqrt{\Delta_n}}\Big)^2 - 1\Big)^m\frac{1}{k_n}\sum_{i\in I_t^n}\Delta_i^n\mathbf{F}\bar{\bar{\boldsymbol{\psi}}}_i'\mathbf{P}_{t,l},$$

is at most $O(\Delta_n)$. For those terms involving $(\mathbf{c}_{FF,j-1} - \mathbf{c}_{FF,t})$, note that it also shrinks to zero uniformly in j and t. Either case, one can replace $\boldsymbol{\alpha}_{i-1}'$ by $\bar{\bar{\boldsymbol{\psi}}}_i$, and the above conclusions regarding $\boldsymbol{\alpha}_{i-1}$ also holds true for $\bar{\bar{\boldsymbol{\psi}}}_i$. The desired results readily follows.

Finally, Cauchy-Schwartz inequality and Burkholder-Davis-Gundy inequality imply that

$$
\begin{aligned}
&\big(\mathbb{E}(\|\widehat{\mathbf{c}}_{FF,t}^{-1}\Delta_i^n\mathbf{F}\bar{\boldsymbol{\psi}}_i'\mathbf{P}_{t,l}\| \mid \mathcal{F}_{i-1}^n)\big)^2 \le \mathbb{E}(\|\widehat{\mathbf{c}}_{FF,t}^{-1}\Delta_i^n\mathbf{F}\|^2 \mid \mathcal{F}_{i-1}^n)\,\mathbb{E}(\|\bar{\boldsymbol{\psi}}_i'\mathbf{P}_{t,l}\|^2 \mid \mathcal{F}_{i-1}^n)\\
&\le L\Delta_n\,\mathbb{E}(\max_{s\in[0,\Delta_n]}\|(\boldsymbol{\alpha}_{(i-1)\Delta_n+s}-\boldsymbol{\alpha}_{(i-1)\Delta_n})\|^2 \mid \mathcal{F}_{i-1}^n)\|\mathbf{P}_{t,l}\|^2 \le L\Delta_n^2.
\end{aligned}
$$

It then follows that

$$
\begin{aligned}
&\mathbb{E}\Big(\Big\|\sqrt{p/\Delta_n}\sum_{t=1}^{[T/\Delta_n]-k_n}\Delta_n\widehat{\mathbf{c}}_{FF,t}^{-1}\frac{1}{k_n}\sum_{i\in I_t^n}\Delta_i^n\mathbf{F}\bar{\boldsymbol{\psi}}_i'\mathbf{P}_{t,l}\Big\|\Big)\\
&\le\sqrt{p/\Delta_n}\sum_{t=1}^{[T/\Delta_n]-k_n}\Delta_n\frac{1}{k_n}\sum_{i\in I_t^n}\mathbb{E}\Big(\mathbb{E}\big(\big\|\widehat{\mathbf{c}}_{FF,t}^{-1}\Delta_i^n\mathbf{F}\bar{\boldsymbol{\psi}}_i'\mathbf{P}_{t,l}\big\| \mid \mathcal{F}_{i-1}^n\big)\Big)\\
&\le\sqrt{p/\Delta_n}\sum_{t=1}^{[T/\Delta_n]-k_n}\Delta_n\frac{1}{k_n}\sum_{i\in I_t^n}L\Delta_n\le L\sqrt{p\Delta_n}\to 0.
\end{aligned}
$$

(ii) Define

$$
\Xi(2)_t^n=[\mathbf{P}_t\mathbf{G}_t-\mathbf{G}_t]_l.
$$

The sum $\sum_{t=1}^{\lfloor T/\Delta_n\rfloor-k_n}\Xi(2)_t^n\Delta_n$ is the average of all such $\Xi(2)_t^n$. Therefore, with $\sqrt{p/\Delta_n}J^{-\eta}=o(1)$, $\sqrt{p/\Delta_n}\sum_{t=1}^{\lfloor T/\Delta_n\rfloor-k_n}\Xi(2)_t^n\Delta_n=o_P(1)$.

(iii) Define

$$
\Xi(3)_t^n=\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{F}'[\mathbf{G}_{i-1}-\mathbf{G}_t]'\mathbf{P}_{t,l}.
$$

Write $\mathbf{w}_{i-1,t}=[\mathbf{G}_{i-1}-\mathbf{G}_t]'\mathbf{P}_{t,l}$. One can verify that

$$
\begin{aligned}
&\Big|\mathbb{E}\Big(\Xi(3)_t^n \mid \mathcal{F}_t^n\Big)\Big|\\
=&\Big|\mathbb{E}\Big(\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{F}'\mathbf{w}_{i-1,t} \mid \mathcal{F}_t^n\Big)\Big|\\
\le&\Big|\mathbb{E}\Big(\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}\Delta_t^n\mathbf{F}\Delta_t^n\mathbf{F}'\mathbf{w}_{i-1,t} \mid \mathcal{F}_t^n\Big)\Big|+\Big|\mathbb{E}\Big(\frac{1}{k_n\Delta_n}\sum_{i\in I_t^n}(\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{F}'-\Delta_t^n\mathbf{F}\Delta_t^n\mathbf{F}')\mathbf{w}_{i-1,t} \mid \mathcal{F}_t^n\Big)\Big|\\
&\le Lk_n\Delta_n,
\end{aligned}
$$

and

$$\begin{aligned}
&\mathbb{E}\Big(\big(\Xi(3)_t^n\big)^2 \mid \mathcal{F}_t^n\Big) \\
=&\mathbb{E}\Big(\frac{1}{(k_n\Delta_n)^2}\sum_{i\in I_t^n}\big(\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{F}'\mathbf{w}_{i-1,t}\big)^2 \mid \mathcal{F}_t^n\Big)+ \\
&\mathbb{E}\Big(\frac{1}{(k_n\Delta_n)^2}\sum_{i\neq j\in I_t^n}\big(\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{F}'\mathbf{w}_{i-1,t}\big)\big(\Delta_j^n\mathbf{F}\Delta_j^n\mathbf{F}'\mathbf{w}_{j-1,t}\big) \mid \mathcal{F}_t^n\Big) \\
\leq& L\Delta_n + Lk_n\Delta_n.
\end{aligned}$$

Since $\mathbf{c}_{FF,t}^{-1}$ is $\mathcal{F}_t^n$-measurable and is bounded uniformly in time because of the localization argument, we obtain the following results

$$\begin{aligned}
\sup_{t\in[0,T]}\Big|\mathbb{E}\Big(\sqrt{p/\Delta_n}\mathbf{c}_{FF,t}^{-1}\Xi(3)_t^n \mid \mathcal{F}_t^n\Big)\Big| &\leq L\sqrt{pk_n^2\Delta_n}, \\
\sup_{t\in[0,T]}\mathbb{E}\Big(\big(\sqrt{p/\Delta_n}\mathbf{c}_{FF,t}^{-1}\Xi(3)_t^n\big)^2 k_n\Delta_n \mid \mathcal{F}_t^n\Big) &\leq Lpk_n^2\Delta_n.
\end{aligned}$$

Hence we have condition (E.28) verified. It then follows that as long as $pk_n^2\Delta_n \to 0$,

$$\sqrt{p/\Delta_n}\sum_{t=0}^{\lfloor T/\Delta_n\rfloor-k_n-1}\mathbf{c}_{FF,t}^{-1}\Xi(3)_t^n\Delta_n = o_P(1).$$

It remains to show that

$$\sqrt{p/\Delta_n}\sum_{t=0}^{\lfloor T/\Delta_n\rfloor-k_n-1}(\widehat{\mathbf{c}}_{FF,t}^{-1}-\mathbf{c}_{FF,t}^{-1})\Xi(3)_t^n\Delta_n = o_P(1),$$

Recall the discussion therein in part (i) of Lemma E.10. Similarly, we restrict our attention on the set $\{|\frac{1}{k_n}\sum_{i\in I_t^n}(\mathbf{c}_{FF,i-1}-\mathbf{c}_{FF,t})| < \underline{c}/4\}$. Then we can use the Taylor expansion (E.30). Recall the decomposition (E.29). We begin with the first component of $\widehat{\mathbf{c}}_{FF,t}-\mathbf{c}_{FF,t}$ in (E.29). We need to consider the expansion of

$$\sum_{j\in I_t^n}\Big(\Big(\frac{\Delta_j^n W^F}{\sqrt{\Delta_n}}\Big)^2-1\Big)^m.$$

When $m = 2r$ is an even number, the leading term of the above conditional expectation is

$$\Big|\mathbb{E}\Big(\frac{\mathbf{c}_{FF,t}^m}{k_n^m}\sum_{j_1\neq\cdots\neq j_r\in I_t^n}\Big(\Big(\frac{\Delta_{j_1}^n W^F}{\sqrt{\Delta_n}}\Big)^2-1\Big)^2\cdots\Big(\Big(\frac{\Delta_{j_r}^n W^F}{\sqrt{\Delta_n}}\Big)^2-1\Big)^2$$
$$\times\frac{1}{k_n}\sum_{i\in I_t^n}\frac{\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{F}'}{\Delta_n}[\mathbf{G}_{i-1}-\mathbf{G}_t]_l\mid\mathcal{F}_t^n\Big)\Big|\leq L\frac{1}{k_n^r}k_n\Delta_n=L\frac{\Delta_n}{k_n^{r-1}}.$$

When $m = 2r + 1$ is an odd number, the leading term becomes

$$\Big|\mathbb{E}\Big(\frac{\mathbf{c}_{FF,t}^m}{k_n^{m+1}}\sum_{j_1\neq\cdots\neq j_r\neq i\in I_t^n}\Big(\Big(\frac{\Delta_{j_1}^n W^F}{\sqrt{\Delta_n}}\Big)^2-1\Big)^2\cdots\Big(\Big(\frac{\Delta_{j_r}^n W^F}{\sqrt{\Delta_n}}\Big)^2-1\Big)^2$$
$$\times\Big(\Big(\frac{\Delta_i^n W^F}{\sqrt{\Delta_n}}\Big)^2-1\Big)\frac{\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{F}'}{\Delta_n}[\mathbf{G}_{i-1}-\mathbf{G}_t]_l\mid\mathcal{F}_t^n\Big)\Big|\leq L\frac{1}{k_n^{r+1}}k_n\Delta_n=L\frac{\Delta_n}{k_n^r}.$$

In general, we have

$$\sup_{t\in[0,T]}\Big|\mathbb{E}\Big(\Big[\frac{\mathbf{c}_{FF,t}}{k_n}\sum_{j\in I_t^n}\Big(\Big(\frac{\Delta_j^n W^F}{\sqrt{\Delta_n}}\Big)^2-1\Big)\Big]^m\frac{1}{k_n}\sum_{i\in I_t^n}\frac{\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{F}'}{\Delta_n}[\mathbf{G}_{i-1}-\mathbf{G}_t]_l\mid\mathcal{F}_t^n\Big)\Big|\leq L\frac{\Delta_n\mathbf{c}_{FF,t}^m}{k_n^{\lceil m/2\rceil}},$$

where $\lceil\cdot\rceil$ is the ceiling function.

As for the second component of $\widehat{\mathbf{c}}_{FF,t}-\mathbf{c}_{FF,t}$ (recall (E.29)), when $\mathbf{c}_{FF,t}$ and $\mathbf{G}$ have co-jumps, we have

$$\Big|\mathbb{E}\Big(\frac{1}{k_n}\sum_{j\in I_t^n}\big(\mathbf{c}_{FF,j-1}-\mathbf{c}_{FF,t}\big)^m\frac{1}{k_n}\sum_{i\in I_t^n}\frac{\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{F}'}{\Delta_n}[\mathbf{G}_{i-1}-\mathbf{G}_t]_l\mid\mathcal{F}_t^n\Big)\Big|\leq L(\underline{\mathbf{c}}/4)^m(k_n\Delta_n).$$

Otherwise, the bound will be replaced by $L(k_n\Delta_n)^{(m+1)/2}$. In either case (whether co-jump is present or not), the second component dominates the first one. Since $\mathbf{c}_{FF,t}$ is uniformly bounded from both above and below, we have $\sum_{m=1}^{\infty}\frac{(-\underline{\mathbf{c}}/2)^m}{\mathbf{c}_{FF,t}^{m+1}}<\infty$. It then follows that (note $(a+b)^m\leq 2^{m-1}(a^m+b^m)$)

$$\sup_{t\in[0,T]}\sqrt{p/\Delta_n}\Big|\sum_{t=0}^{\lfloor(\frac{S}{\Delta_n}-t_0)/k_n\rfloor-1}\mathbb{E}\Big((\widehat{\mathbf{c}}_{FF,t}^{-1}-\mathbf{c}_{FF,t}^{-1})\Xi(2)_t^n k_n\Delta_n\mid\mathcal{F}_t^n\Big)\Big|$$
$$\leq\sup_{t\in[0,T]}\sqrt{p/\Delta_n}\sum_{m=1}^{\infty}\frac{(-1)^m2^{m-1}}{\mathbf{c}_{FF,t}^{m+1}}\Big(\Big|\mathbb{E}\Big(\Big[\frac{\mathbf{c}_{FF,t}}{k_n}\sum_{j\in I_t^n}\Big(\Big(\frac{\Delta_j^n W^F}{\sqrt{\Delta_n}}\Big)^2-1\Big)\Big]^m\frac{1}{k_n}\sum_{i\in I_t^n}\frac{\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{F}'}{\Delta_n}[\mathbf{G}_{i-1}-\mathbf{G}_t]_l\mid\mathcal{F}_t^n\Big)\Big|$$

$$+\Big|\mathbb{E}\Big(\frac{1}{k_n}\sum_{j\in I_t^n}\big(\mathbf{c}_{FF,j-1}-\mathbf{c}_{FF,t}\big)^m\frac{1}{k_n}\sum_{i\in I_t^n}\frac{\Delta_i^n\mathbf{F}\Delta_i^n\mathbf{F}'}{\Delta_n}[\mathbf{G}_{i-1}-\mathbf{G}_t]_l\mid\mathcal{F}_t^n\Big)\Big|\Big)$$

$$\leq L\sqrt{p\Delta_n}\sum_{m=1}^{\infty}\frac{(-1)^m}{k_n^{\lceil m/2\rceil}}+L\sqrt{pk_n^2\Delta_n}\sum_{m=1}^{\infty}\frac{(-\underline{\mathbf{c}}/2)^m}{\mathbf{c}_{FF,t}^{m+1}}\leq L\sqrt{pk_n^2\Delta}\to 0.$$

Next, we have

$$\mathbb{E}\Big((\widehat{\mathbf{c}}_{FF,t}^{-1}-\mathbf{c}_{FF,t}^{-1})^2(\Xi(3)_t^n)^2\Big)\leq\Big(\mathbb{E}\big((\widehat{\mathbf{c}}_{FF,t}^{-1}-\mathbf{c}_{FF,t}^{-1})^4\big)\mathbb{E}\big((\Xi(3)_t^n)^4\big)\Big)^{1/2}\leq\Delta_n.$$

The last inequality comes from that $\mathbb{E}\big((\Xi(3)_t^n)^4\big)\leq L(k_n\Delta_n)^2$. We omit the detailed analysis of $\mathbb{E}\big((\Xi(3)_t^n)^4\big)$, which follows from a similar argument as above. Hence

$$\sup_{t\in[0,T]}\mathbb{E}\Big(\big(\sqrt{p/\Delta_n}(\widehat{\mathbf{c}}_{FF,t}^{-1}-\mathbf{c}_{FF,t}^{-1})\Xi(3)_t^n\big)^2k_n\Delta_n\mid\mathcal{F}_t^n\Big)\leq Lpk_n\Delta_n.$$

Then as long as $pk_n^2\Delta_n\to 0$, we have $\sqrt{p/\Delta_n}\sum_{t=0}^{\lfloor T/\Delta_n\rfloor-k_n-1}\widehat{\mathbf{c}}_{FF,t}^{-1}\Xi(3)_t^n\Delta_n=o_P(1)$.

(iv) The proof is the same as that of (iii) so is omitted for brevity.

(v) Recall the following definitions we introduced in the proof of Proposition E.2

$$\tilde{\xi}_{t,q}=\frac{1}{p}\sum_{m=1}^{p}\widehat{\mathbf{c}}_{FF,tk_n+q}^{-1}\frac{1}{k_n}\sum_{i=1}^{k_n}\mathbf{f}_{tk_n+q+i}u_{m,tk_n+q+i}h_{tk_n+q+i-1,ml}\,k_n\Delta_n$$

$$\xi_{t,q}=\frac{1}{p}\sum_{m=1}^{p}\mathbf{c}_{FF,tk_n+q}^{-1}\frac{1}{k_n}\sum_{i=1}^{k_n}\mathbf{f}_{tk_n+q+i}u_{m,tk_n+q+i}h_{tk_n+q+i-1,ml}\,k_n\Delta_n.$$

What we want to prove is equivalent to

$$\sqrt{\frac{p}{\Delta_n}}\sum_{t=1}^{[T/(k_n\Delta_n)]-k_n-q}\mathbf{v}'(\tilde{\xi}_{t,q}-\xi_{t,q})k_n\Delta_n=o_P(1).$$

Since $\widehat{\mathbf{c}}_{FF,tk_n+q}^{-1}-\mathbf{c}_{FF,tk_n+q}^{-1}$ only depends on F, not U, it is easy to show using the above argument that

$$\sqrt{\frac{p}{\Delta_n}}\sum_{t=1}^{[T/(k_n\Delta_n)]-k_n-q}\mathbb{E}\big(\mathbf{v}'(\tilde{\xi}_{t,q}-\xi_{t,q})\mid\mathcal{F}_{tk_n}^n\big)k_n\Delta_n=0.$$

Following a similar argument and the cross-sectional independence of u_m, we can deduce that

$$
\begin{aligned}
&\frac{p}{\Delta_n} \sum_{t=1}^{[T/(k_n\Delta_n)]-k_n-q} \mathbb{E}\big([\mathbf{v}'(\tilde{\xi}_{t,q} - \xi_{t,q})]^2 \,|\, \mathcal{F}^n_{tk_n}\big) k_n\Delta_n \\
= & \sum_{t=1}^{[T/(k_n\Delta_n)]-k_n-q} \mathbb{E}\Big(\frac{1}{p}\sum_{m=1}^{p} \frac{1}{k_n} \sum_{i=1}^{k_n} \big[\mathbf{v}'(\widehat{\mathbf{c}}^{-1}_{FF,tk_n+q} - \mathbf{c}^{-1}_{FF,tk_n+q})\mathbf{f}_{ik_n+i} u_{m,tk_n+i} h_{tk_n+q_i-1,ml}\big]^2 \,|\, \mathcal{F}^n_{tk_n}\Big) k_n\Delta_n.
\end{aligned}
$$

Then using the above expansion, we can show that the the left hand side term converges in probability to zero.

□

Lemma E.11. *(i)* $\sqrt{p/\Delta_n}\sum_{t=1}^{[T/\Delta_n]-k_n} \Delta_n \widehat{\mathbf{c}}^{-1}_{FF,t} \frac{1}{k_n}\sum_{i\in I^n_t} \Delta^n_i \mathbf{F} \boldsymbol{\psi}^{*'}_{i-1} \mathbf{P}^*_{t,l} = o_{P^*}(1)$
(ii) $\sqrt{p/\Delta_n}\sum_{t=1}^{[T/\Delta_n]-k_n} \Delta_n \widehat{\mathbf{c}}^{-1}_{FF,t} \frac{1}{k_n\Delta_n}\sum_{i\in I^n_t} \Delta^n_i \mathbf{F} \Delta^n_i \mathbf{F}' [\mathbf{P}^*_t \mathbf{G}^*_t - \mathbf{G}^*_t]_l = o_{P^*}(1)$
(iii) $\sqrt{p/\Delta_n}\sum_{t=1}^{[T/\Delta_n]-k_n} \Delta_n \widehat{\mathbf{c}}^{-1}_{FF,t} \frac{1}{k_n\Delta_n}\sum_{i\in I^n_t} \Delta^n_i \mathbf{F} \Delta^n_i \mathbf{F}' [\mathbf{G}^*_{i-1} - \mathbf{G}^*_t]' \mathbf{P}^*_{t,l} = o_{P^*}(1)$
(iv) $\sqrt{p/\Delta_n}\sum_{t=1}^{[T/\Delta_n]-k_n} \Delta_n \widehat{\mathbf{c}}^{-1}_{FF,t} \frac{1}{k_n\Delta_n}\sum_{i\in I^n_t} \Delta^n_i \mathbf{F} \Delta^n_i \mathbf{F}' [\boldsymbol{\Gamma}^*_{i-1} - \boldsymbol{\Gamma}^*_t]' \mathbf{P}^*_{t,l} = o_{P^*}(1)$

Proof. These are the cross-sectional bootstrap version of those term in Lemma E.10. Since the proof of Lemma E.10 mainly concerns the time domain behaviors of these terms, the proof here is the same as in Lemma E.10, so is omitted for brevity. □

Lemma E.12. *Let*

$$
\begin{aligned}
\mathbf{a}^{**}_1 &= \Delta_n \sum_{t=1}^{[T/\Delta_n]-k_n} \widehat{\mathbf{c}}^{-1}_{FF,t} \frac{1}{k_n\Delta_n} \sum_{i\in I^n_t} \Delta^n_i \mathbf{F} \frac{1}{p} \sum_{m=1}^{p} \Delta^n_i U^*_m \boldsymbol{\phi}^{*'}_{mt} (\frac{1}{p} \boldsymbol{\Phi}^{*'}_t \boldsymbol{\Phi}^*_t)^{-1} \boldsymbol{\phi}_{lt} \\
\mathbf{a}^{*}_1 &= \Delta_n \sum_{t=1}^{[T/\Delta_n]-k_n} \widehat{\mathbf{c}}^{-1}_{FF,t} \frac{1}{k_n\Delta_n} \sum_{i\in I^n_t} \Delta^n_i \mathbf{F} \frac{1}{p} \sum_{m=1}^{p} \Delta^n_i U^*_m \boldsymbol{\phi}^{*'}_{mt} (\frac{1}{p} \boldsymbol{\Phi}'_t \boldsymbol{\Phi}_t)^{-1} \boldsymbol{\phi}_{lt} \\
\mathbf{a}^{**}_2 &= \Delta_n \sum_{t=1}^{[T/\Delta_n]-k_n} \frac{1}{p} \sum_{m=1}^{p} \boldsymbol{\gamma}^*_{mt} \boldsymbol{\phi}^{*'}_{mt} (\frac{1}{p} \boldsymbol{\Phi}^{*'}_t \boldsymbol{\Phi}^*_t)^{-1} \boldsymbol{\phi}_{lt} \\
\mathbf{a}^{*}_2 &= \Delta_n \sum_{t=1}^{[T/\Delta_n]-k_n} \frac{1}{p} \sum_{m=1}^{p} \boldsymbol{\gamma}^*_{mt} \boldsymbol{\phi}^{*'}_{mt} (\frac{1}{p} \boldsymbol{\Phi}'_t \boldsymbol{\Phi}_t)^{-1} \boldsymbol{\phi}_{lt}.
\end{aligned}
$$

Then $\mathbf{a}^{**}_1 - \mathbf{a}^*_1 = o_{P^*}((p/\Delta_n)^{-1/2})$ *and* $\mathbf{a}^{**}_2 - \mathbf{a}^*_2 = o_{P^*}(p^{-1/2})\lambda^{1/2}_{\min}(\mathbf{V}_\gamma)$.

Proof. (i) Note that $\mathbf{a}_1^{**} - \mathbf{a}_1^* = \sum_{t=1}^{[T/\Delta_n]-k_n} \Theta_{t\Delta_n}^n$, where

$$\Theta_t^n = \widehat{\mathbf{c}}_{FF,t}^{-1} \frac{1}{k_n \Delta_n} \sum_{i \in I_t^n} \Delta_i^n \mathbf{F} \frac{1}{p} \sum_{m=1}^{p} \Delta_i^n U_m^* \boldsymbol{\phi}_{mt}^{*'} \Big((\frac{1}{p} \boldsymbol{\Phi}_t^{*'} \boldsymbol{\Phi}_t^*)^{-1} - (\frac{1}{p} \boldsymbol{\Phi}_t' \boldsymbol{\Phi}_t)^{-1} \Big) \boldsymbol{\phi}_{lt}.$$

Following Lemma E.9, it is sufficient to show that

$$\sup_{t \in [0,T]} |\mathbb{E}(\sqrt{p/\Delta_n} \Theta_t^n | \mathcal{F}_t^n)| = o_P(1) \quad \text{and} \quad \sup_{t \in [0,T]} |\mathbb{E}\big((p/\Delta_n) \|\Theta_t^n\|^2 k_n \Delta_n\big)| = o(1).$$

First, because $\Delta_i^n \mathbf{U}/\sqrt{\Delta_n}$ is normally distributed with $\mathcal{F}_{i-1}^n$-conditional mean being identically zero. Moreover, it is not correlated (hence independent) with $\Delta_i^n \mathbf{F}/\sqrt{\Delta_n}$, which is also normally distributed. It then readily follows that

$$\Big| \mathbb{E}(\sqrt{p/\Delta_n} \mathbf{c}_{FF,t}^{-1} \frac{1}{k_n \Delta_n} \sum_{i \in I_t^n} \Delta_i^n \mathbf{F} \frac{1}{p} \sum_{m=1}^{p} \Delta_i^n U_m^* \boldsymbol{\phi}_{mt}^{*'} \Big((\frac{1}{p} \boldsymbol{\Phi}_t^{*'} \boldsymbol{\Phi}_t^*)^{-1} - (\frac{1}{p} \boldsymbol{\Phi}_t' \boldsymbol{\Phi}_t)^{-1} \Big) \boldsymbol{\phi}_{lt} | \mathcal{F}_t^n) \Big| = 0.$$

Next note that the estimation error $\widehat{\mathbf{c}}_{FF,t}^{-1} - \mathbf{c}_{FF,t}^{-1}$ does not correlated with $\Delta_i^n U/\sqrt{\Delta_n}$. Hence, following a similar procedure as in the proof of Lemma E.10, we can show that

$$\Big| \mathbb{E}(\sqrt{p/\Delta_n} (\widehat{\mathbf{c}}_{FF,t}^{-1} - \mathbf{c}_{FF,t}^{-1}) \frac{1}{k_n \Delta_n} \sum_{i \in I_t^n} \Delta_i^n \mathbf{F} \frac{1}{p} \sum_{m=1}^{p} \Delta_i^n U_m^* \boldsymbol{\phi}_{mt}^{*'} \Big((\frac{1}{p} \boldsymbol{\Phi}_t^{*'} \boldsymbol{\Phi}_t^*)^{-1} - (\frac{1}{p} \boldsymbol{\Phi}_t' \boldsymbol{\Phi}_t)^{-1} \Big) \boldsymbol{\phi}_{lt} | \mathcal{F}_t^n) \Big| = 0.$$

The above two equalities hold for all $t \in [0,T]$. Hence, we get $\sup_{t \in [0,T]} |\mathbb{E}(\sqrt{p/\Delta_n} \Theta_t^n | \mathcal{F}_t^n)| = o_P(1)$.

For the second order condition, we need to show that $\sup_{t \in [0,T]} |\mathbb{E}\big(p k_n \|\Theta_t^n\|^2\big)| = o_P(1)$. Using iterated conditioning, it is easy to show that

$$\begin{aligned}
& p k_n \, \mathbb{E}\Big[\widehat{\mathbf{c}}_{FF,t}^{-1} \frac{1}{k_n \Delta_n} \sum_{i \in I_t^n} \Delta_i^n \mathbf{F} \frac{1}{p} \sum_{m=1}^{p} \Delta_i^n U_m^* \boldsymbol{\phi}_{mt}^{*'} \Big((\frac{1}{p} \boldsymbol{\Phi}_t^{*'} \boldsymbol{\Phi}_t^*)^{-1} - (\frac{1}{p} \boldsymbol{\Phi}_t' \boldsymbol{\Phi}_t)^{-1} \Big) \boldsymbol{\phi}_{lt} \Big]^2 \\
= & \, p \, \mathbb{E}\Big(\widehat{\mathbf{c}}_{FF,t}^{-2} \frac{1}{k_n} \sum_{i \in I_t^n} \mathbf{c}_{FF,(i-1)\Delta_n} \big[\frac{1}{p} \sum_{m=1}^{p} \frac{\Delta_i^n U_m^*}{\sqrt{\Delta_n}} \boldsymbol{\phi}_{mt}^{*'} \Big((\frac{1}{p} \boldsymbol{\Phi}_t^{*'} \boldsymbol{\Phi}_t^*)^{-1} - (\frac{1}{p} \boldsymbol{\Phi}_t' \boldsymbol{\Phi}_t)^{-1} \Big) \boldsymbol{\phi}_{lt} \big]^2 \Big) \\
\leq & \, p \, \mathbb{E} \left[\frac{1}{k_n} \sum_{i \in I_t^n} A_{i-1} \|\mathbf{c}_{FF,(i-1)\Delta_n}\| \| \Big((\frac{1}{p} \boldsymbol{\Phi}_t^{*'} \boldsymbol{\Phi}_t^*)^{-1} - (\frac{1}{p} \boldsymbol{\Phi}_t' \boldsymbol{\Phi}_t)^{-1} \Big) \boldsymbol{\phi}_{lt} \|^2 \|\widehat{\mathbf{c}}_{FF,t}^{-2}\| \right]
\end{aligned}$$

$$
\begin{aligned}
&\leq CJ\,\mathbb{E}\left[\mathbb{E}(\frac{1}{k_n}\sum_{i\in I_t^n}\|\mathbf{c}_{FF,(i-1)\Delta_n}\|\|\widehat{\mathbf{c}}_{FF,t}^{-2}\||\mathcal{F}_t)\Big((\frac{1}{p}\boldsymbol{\Phi}_t^{*'}\boldsymbol{\Phi}_t^*)^{-1}-(\frac{1}{p}\boldsymbol{\Phi}_t'\boldsymbol{\Phi}_t)^{-1}\Big)^2\|\boldsymbol{\phi}_{lt}\|^2\right]\\
&\leq CJ\,\mathbb{E}\|\boldsymbol{\phi}_{lt}\|^2\mathbb{E}^*\left\|\frac{1}{p}\boldsymbol{\Phi}_t^{*'}\boldsymbol{\Phi}_t^*-\frac{1}{p}\boldsymbol{\Phi}_t'\boldsymbol{\Phi}_t\right\|^2\leq CJ\,\mathbb{E}h_{t,ll}\frac{1}{p^2}\sum_{m=1}^p\|\boldsymbol{\phi}_{mt}\|^4\leq CJ\,\frac{1}{p^2}\sum_{m=1}^p\mathbb{E}h_{t,mm}^2h_{t,ll}\\
&\leq O(J/p)=o(1),
\end{aligned}
$$

where, for $\mathbb{E}_{i-1}(.):=\mathbb{E}(.|\mathcal{F}_{i-1})$, and $\mathbb{E}_{i-1}^*$ be the conditional expectation with respect to the bootstrap sampling, given $\{(\mathbf{u}_i,\mathbf{X}_t)\}$ and $\mathcal{F}_{i-1}$,

$$
\begin{aligned}
A_{i-1} &:= \mathbb{E}_{i-1}\Big(\|\frac{1}{p}\sum_{m=1}^p\frac{\Delta_i^nU_m^*}{\sqrt{\Delta_n}}\boldsymbol{\phi}_{mt}^{*'}\|^2\Big)=\sum_{j=1}^J\mathbb{E}_{i-1}\mathbb{E}_{i-1}^*(|\frac{1}{p}\sum_{m=1}^p u_{mi}^*\phi_{mt,j}^*|^2)\\
&\leq \sum_{j=1}^J\frac{1}{p^2}\boldsymbol{\Phi}_{t,j}'\operatorname{Var}(\mathbf{u}_i|\mathcal{F}_{i-1})\boldsymbol{\Phi}_{t,j}+\frac{1}{p^2}\sum_{m=1}^p(\mathbb{E}u_{mi}^2)\|\boldsymbol{\phi}_{mt}\|^2\\
&\leq \frac{C}{p^2}\|\boldsymbol{\Phi}_t\|_F^2\leq\frac{JC}{p}\|\frac{1}{p}\boldsymbol{\Phi}_t'\boldsymbol{\Phi}_t\|\leq\frac{CJ}{p},\quad \text{almost surely.}
\end{aligned}
$$

(ii) First,

$$
\begin{aligned}
B_1 &:= \Delta_n\sum_{t=1}^{[T/\Delta_n]-k_n}\|(\frac{1}{p}\boldsymbol{\Phi}_t^{*'}\boldsymbol{\Phi}_t^*)^{-1}-(\frac{1}{p}\boldsymbol{\Phi}_t'\boldsymbol{\Phi}_t)^{-1}\|^2\\
&\leq C\Delta_n\sum_{t=1}^{[T/\Delta_n]-k_n}\|\frac{1}{p}\boldsymbol{\Phi}_t^{*'}\boldsymbol{\Phi}_t^*-\frac{1}{p}\boldsymbol{\Phi}_t'\boldsymbol{\Phi}_t\|^2\\
&\leq O_{P^*}(1)\Delta_n\sum_{t=1}^{[T/\Delta_n]-k_n}\frac{1}{p^2}\sum_{m=1}^p h_{t,mm}^2=O_{P^*}(\frac{1}{p}).
\end{aligned}
$$

Secondly, by Assumption 3.8,

$$
\begin{aligned}
B_2 &:= \Delta_n\sum_{t=1}^{[T/\Delta_n]-k_n}\|\frac{1}{p}\sum_{m=1}^p\boldsymbol{\gamma}_{mt}^*\boldsymbol{\phi}_{mt}^{*'}\|^2\|\boldsymbol{\phi}_{lt}\|^2\\
&\leq O_{P^*}(\frac{1}{p})\frac{1}{p}\sum_{m=1}^p\Delta_n\sum_{t=1}^{[T/\Delta_n]-k_n}\mathbb{E}(\|\boldsymbol{\gamma}_{mt}\|^2h_{t,ml}^2|\mathbf{X}_t)\|+O_{P^*}(1)\Delta_n\sum_{t=1}^{[T/\Delta_n]-k_n}\frac{1}{p^2}\sum_{m=1}^p\|\boldsymbol{\gamma}_{mt}\|^2h_{t,mm}h_{t,ll}\\
&\leq O_{P^*}(\frac{1}{p})\lambda_{\min}(\mathbf{V}_\gamma).
\end{aligned}
$$

So $\|\mathbf{a}_2^{**}-\mathbf{a}_2^*\|\leq\sqrt{B_1B_2}\leq\lambda_{\min}^{1/2}(\mathbf{V}_\gamma)O_{P^*}(p^{-1})\leq\lambda_{\min}^{1/2}(\mathbf{V}_\gamma)o_{P^*}(p^{-1})$.

□

The lemma below is an intermediate step to show that for estimating $\int_0^T \mathbf{g}_{lt} dt$, the bootstrap variance "mimicks" the asymptotic variance of the estimator.

Lemma E.13. *We have*

$$\frac{\Delta_n}{pk_n^2} \sum_{m=1}^{p} \sum_{\substack{t,s=1 \\ |t-s|\le k_n}}^{[T/\Delta_n]-k_n} \sum_{\substack{i\in I_t^n, j\in I_s^n \\ i\neq j}} \widehat{\mathbf{c}}_{FF,t}^{-1} h_{i-1,ml} \mathbf{f}_i u_{mi}\, \widehat{\mathbf{c}}_{FF,s}^{-1} h_{s,ml} \mathbf{f}_i' u_{mj}$$
$$+ \frac{\Delta_n}{pk_n^2} \sum_{m=1}^{p} \sum_{\substack{t,s=1 \\ |t-s|> k_n}}^{[T/\Delta_n]-k_n} \sum_{i\in I_t^n, j\in I_s^n} \widehat{\mathbf{c}}_{FF,t}^{-1} h_{i-1,ml} \mathbf{f}_i u_{mi}\, \widehat{\mathbf{c}}_{FF,s}^{-1} h_{s,ml} \mathbf{f}_i' u_{mj} = o_P(1).$$

Proof. For notational simplicity, we assume $\dim(\mathbf{f}_i) = 1$, as one can always work with the terms element-by-elements. According to the assumption, for $m \neq q$, u_{mi} and u_{qi} are independent. For $i \neq j$, f_i and f_j are independent, same for u_{mi} and u_{mj}. Define $\xi_{t,i,m} = \widehat{\mathbf{c}}_{FF,t}^{-1} h_{i-1,ml} f_i u_{mi} 1_{\{i\in I_t^n\}}$. Then the above sum writes as

$$\frac{\Delta_n}{pk_n^2} \sum_{m=1}^{p} \sum_{t,s=1}^{[T/\Delta_n]-k_n} \sum_{i\in I_t^n} \sum_{j\in I_s^n} \xi_{t,i,m}\, \xi_{s,j,m}\, 1_{\{i\neq j\}}$$

Note that $\mathbb{E}(\mathbf{f}_i|\mathcal{F}_{i-1}^n) = 0$ and $\mathbb{E}(u_{mi}|\mathcal{F}_{f-1}^n) = 0$. In addition $\widehat{\mathbf{c}}_{FF,t}^{-1}$ only depends on $\{\mathbf{f}_i\}_{i\in I_t^n}$, hence is independent with the sequence $\{u_{mi}\}$ for any m. Then we have

$$\mathbb{E}\Big(\xi_{t,i,m}\, \xi_{s,j,m}\, 1_{\{i>j\}}\Big) = \mathbb{E}\Big(\mathbb{E}\big(\xi_{t,i,m}\, \xi_{s,j,m}\, 1_{\{i>j\}} | \mathcal{F}_{i-1}^n\big)\Big)$$
$$= \mathbb{E}\Big(h_{s,ml} f_j u_{mj} h_{i-1,ml}\, \mathbb{E}\big(\widehat{\mathbf{c}}_{FF,s}^{-1} \widehat{\mathbf{c}}_{FF,t}^{-1} f_i u_{mi} | \mathcal{F}_{i-1}^n\big)\, 1_{\{i>j\}}\Big) = 0.$$

Similarly, $\mathbb{E}\big(\xi_{t,i,m}\, \xi_{s,j,m}\, 1_{\{i<j\}}\big) = 0$. Consequently, we obtain

$$\mathbb{E}\Big(\frac{\Delta_n}{pk_n^2} \sum_{m=1}^{p} \sum_{t,s=1}^{[T/\Delta_n]-k_n} \sum_{i\in I_t^n} \sum_{j\in I_s^n} \xi_{t,i,m}\, \xi_{s,j,m}\, 1_{\{i\neq j\}}\Big)$$
$$= \frac{\Delta_n}{pk_n^2} \sum_{m=1}^{p} \sum_{t,s=1}^{[T/\Delta_n]-k_n} \sum_{i\in I_t^n} \sum_{j\in I_s^n} \mathbb{E}\Big(\xi_{t,i,m}\, \xi_{s,j,m}\, 1_{\{i\neq j\}}\Big) = 0.$$

Next, we show the variance of the above sum converges to zero.

$$
\begin{aligned}
&\mathbb{E}\Big(\frac{\Delta_n}{pk_n^2}\sum_{m=1}^{p}\sum_{t,s=1}^{[T/\Delta_n]-k_n}\sum_{i\in I_t^n}\sum_{j\in I_s^n}\xi_{t,i,m}\,\xi_{s,j,m}\,1_{\{i\neq j\}}\Big)^2\\
&=\mathbb{E}\Big(\frac{\Delta_n^2}{p^2k_n^4}\sum_{m,m'=1}^{p}\sum_{t,s,t',s'=1}^{[T/\Delta_n]-k_n}\sum_{i\in I_t^n}\sum_{j\in I_s^n}\sum_{i'\in I_{t'}^n}\sum_{j'\in I_{s'}^n}\xi_{t,i,m}\,\xi_{s,j,m}\,\xi_{t',i',m'}\,\xi_{s',j',m'}\,1_{\{i\neq j,i'\neq j'\}}\Big).
\end{aligned}
$$

The situation here is very similar to that in the proof of Lemma D.4. Following similar argument, the cases that the above expectation is non-zero are when $m=m', i=i', j=j'$ and when $m=m', i=j', j=i'$. In such cases, we have

$$
\mathbb{E}(\widehat{\mathbf{c}}_{FF,t}^{-1}\widehat{\mathbf{c}}_{FF,t'}^{-1}\widehat{\mathbf{c}}_{FF,s}^{-1}\widehat{\mathbf{c}}_{FF,s'}^{-1}h_{i-1,ml}^2h_{s,ml}^2f_i^2u_{mi}^2u_{mj}^21_{\{i\neq j\}})\leq L.
$$

However, it is only possible to have $i=i'$ and $j=j'$ (or $i=j'$ and $j=i'$) when $|t-t'|<k_n$ and $|s-s'|<k_n$ (or $|t-s'|<k_n$ and $|s-t'|<k_n$). Therefore, we get

$$
\begin{aligned}
&\mathbb{E}\Big(\frac{\Delta_n^2}{p^2k_n^4}\sum_{m,m'=1}^{p}\sum_{t,s,t',s'=1}^{[T/\Delta_n]-k_n}\sum_{i\in I_t^n}\sum_{j\in I_s^n}\sum_{i'\in I_{t'}^n}\sum_{j'\in I_{s'}^n}\xi_{t,i,m}\,\xi_{s,j,m}\,\xi_{t',i',m'}\,\xi_{s',j',m'}\,1_{\{i\neq j,i'\neq j'\}}\Big)\\
&=2\,\frac{\Delta_n^2}{p^2k_n^4}\sum_{m=1}^{p}\sum_{|t-t'|<k_n,|s-s'|<t_n,t,s,t',s'=1}^{[T/\Delta_n]-k_n}\sum_{i\in I_t^n\cap I_{t'}^n}\sum_{j\in I_s^n\cap I_{s'}^n}\mathbb{E}\Big(\xi_{t,i,m}\,\xi_{s,j,m}\,\xi_{t',i,m}\,\xi_{s',j,m}\,1_{\{i\neq j,i'\neq j'\}}\Big)\\
&\leq L\,\frac{\Delta_n^2}{p^2k_n^4}\,p\,\frac{1}{\Delta_n^2}\,k_n^2=\frac{L}{pk_n^2}=o(1).
\end{aligned}
$$

Since both the mean and the variance of the above sum are $o(1)$.

Q.E.D.

□

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Table 3: Averaged cross-sectional and time-series standard deviations of $\mathbf{G}$ and $\mathbf{\Gamma}$

	Mkt		HML		SMB		RMW	
	$\mathbf{G}$	$\mathbf{\Gamma}$	$\mathbf{G}$	$\mathbf{\Gamma}$	$\mathbf{G}$	$\mathbf{\Gamma}$	$\mathbf{G}$	$\mathbf{\Gamma}$
	Averaged (over time) cross-sectional std							
	grouped by size							
small	0.221	0.499	0.409	1.140	0.144	0.613	0.336	1.405
medium	0.179	0.446	0.352	1.063	0.112	0.524	0.281	1.290
large	0.165	0.411	0.333	0.994	0.142	0.466	0.260	1.245
	grouped by volatility							
small	0.082	0.352	0.267	0.799	0.182	0.407	0.186	0.949
medium	0.085	0.425	0.271	1.018	0.178	0.503	0.206	1.219
large	0.178	0.550	0.412	1.310	0.204	0.668	0.325	1.670
	Averaged (over firms) times-series std							
	grouped by size							
small	0.195	0.471	0.407	1.095	0.175	0.590	0.431	1.332
medium	0.152	0.416	0.324	0.993	0.140	0.511	0.353	1.214
large	0.148	0.389	0.315	0.908	0.133	0.460	0.336	1.124
	grouped by volatility							
small	0.142	0.368	0.294	0.861	0.131	0.447	0.324	1.042
medium	0.155	0.403	0.324	0.946	0.139	0.490	0.347	1.158
large	0.194	0.503	0.426	1.187	0.177	0.622	0.445	1.467

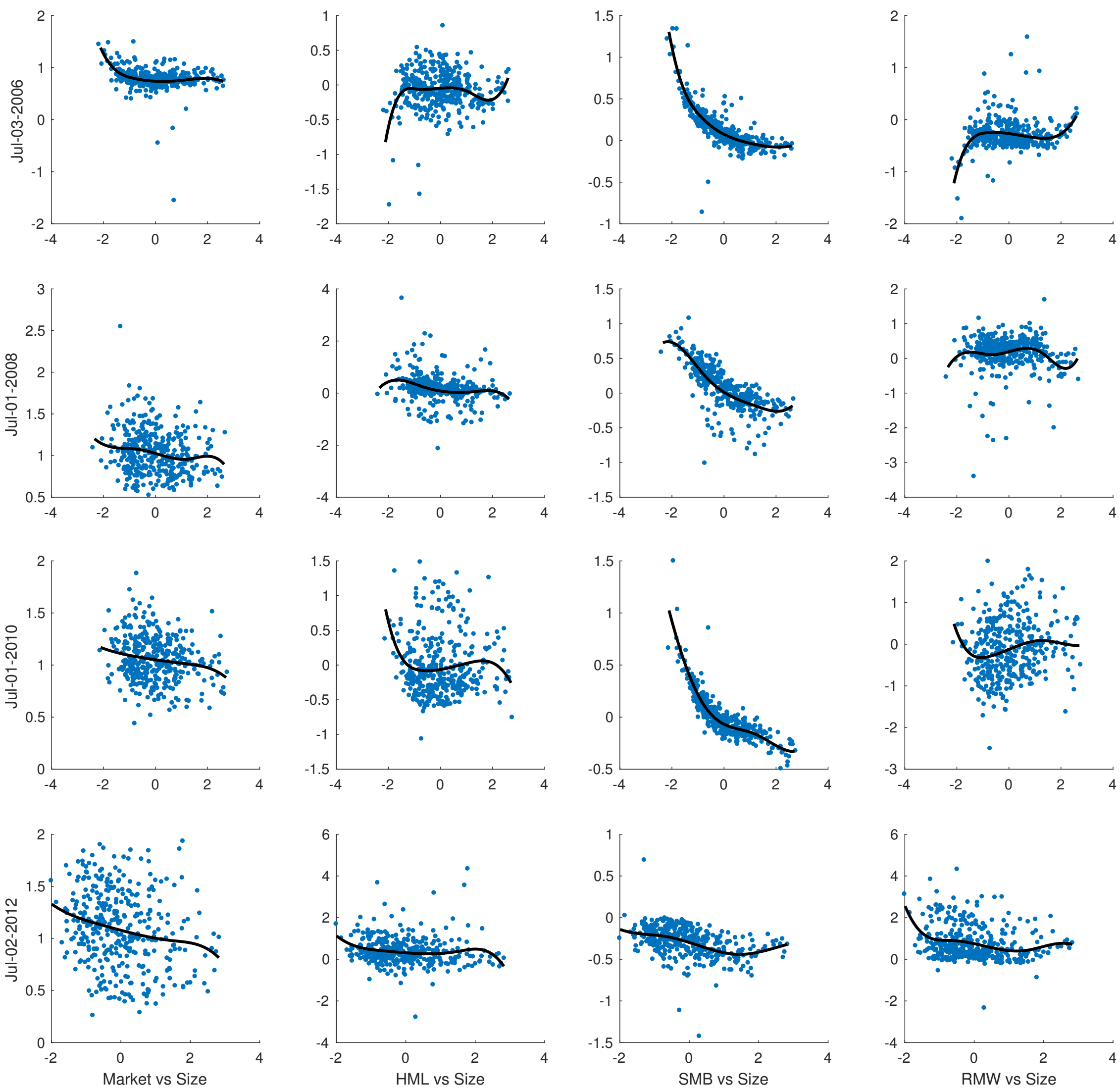

Figure 1: Different factors' G versus size at representative days

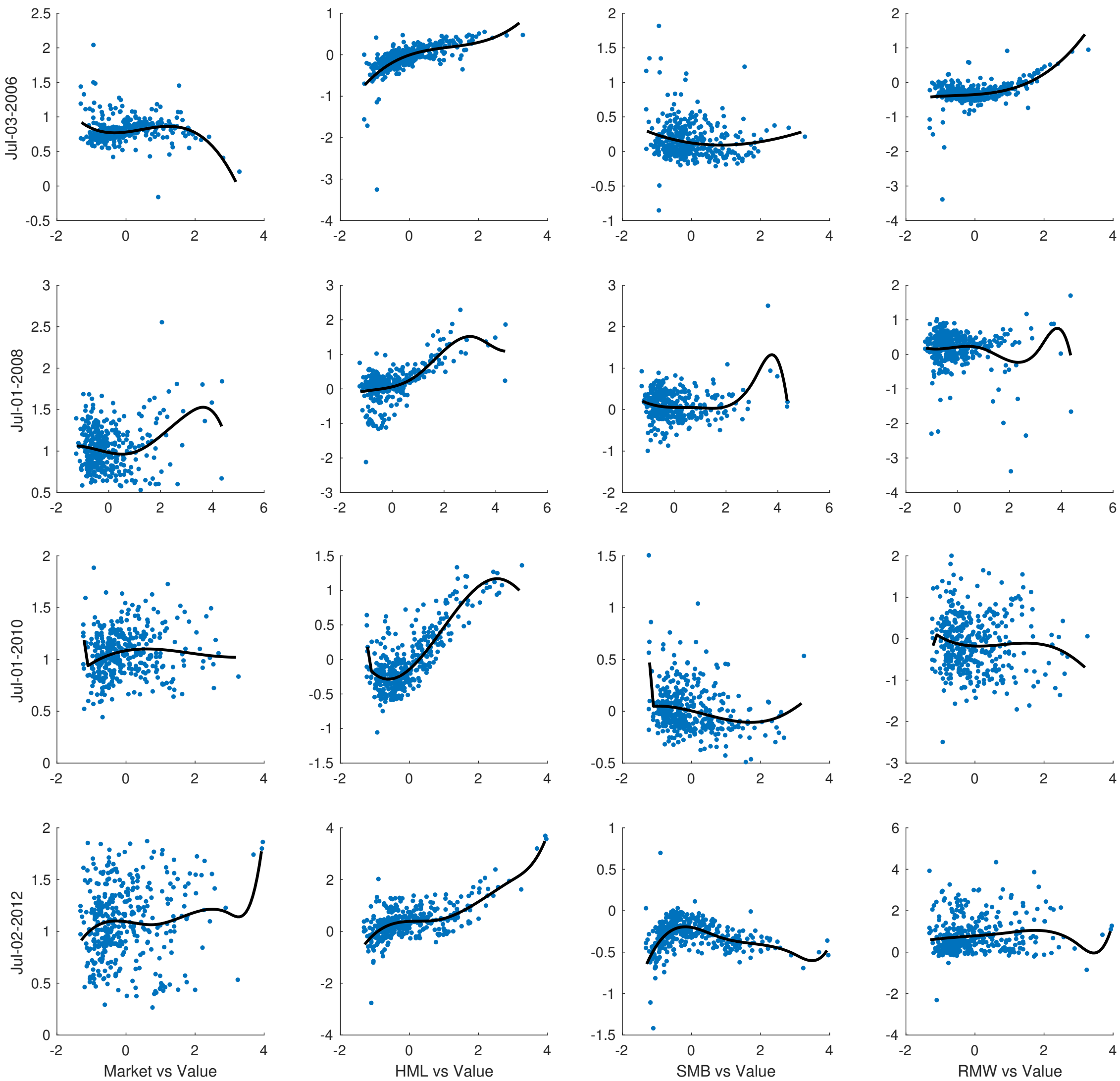

Figure 2: Different factors' G versus value at representative days

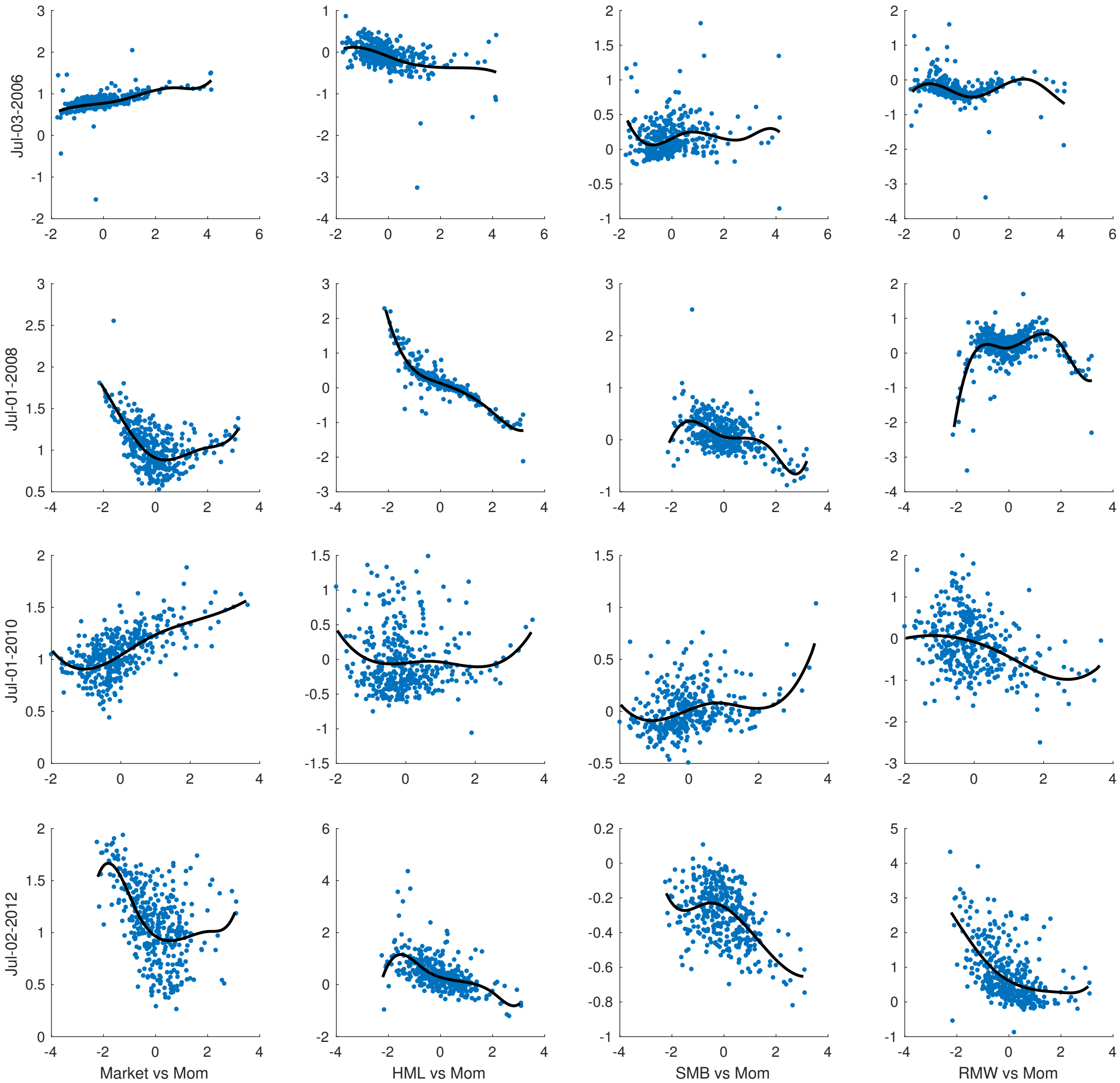

Figure 3: Different factors' G versus momemtum at representative days

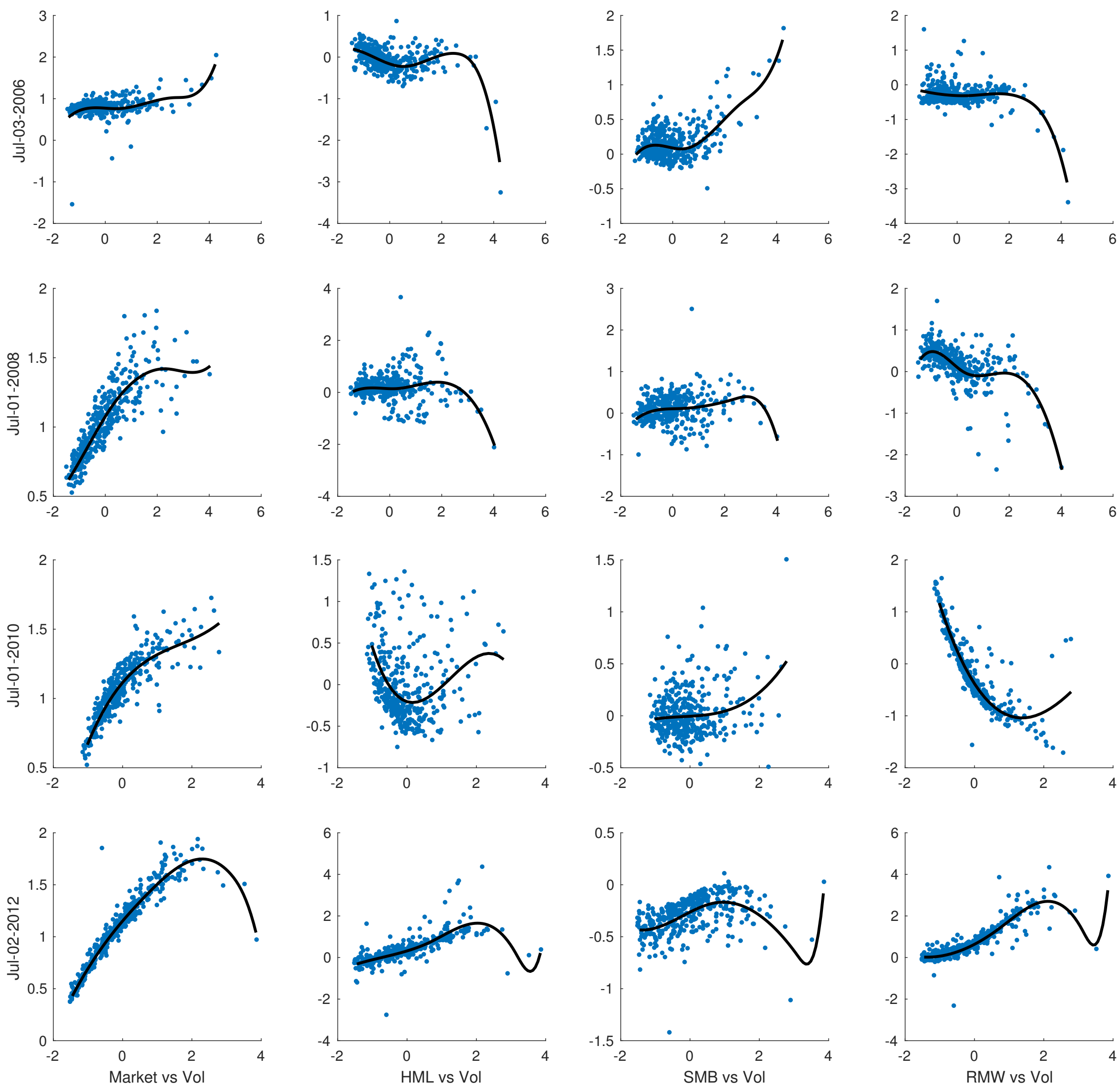

Figure 4: Different factors' G versus volatility at representative days

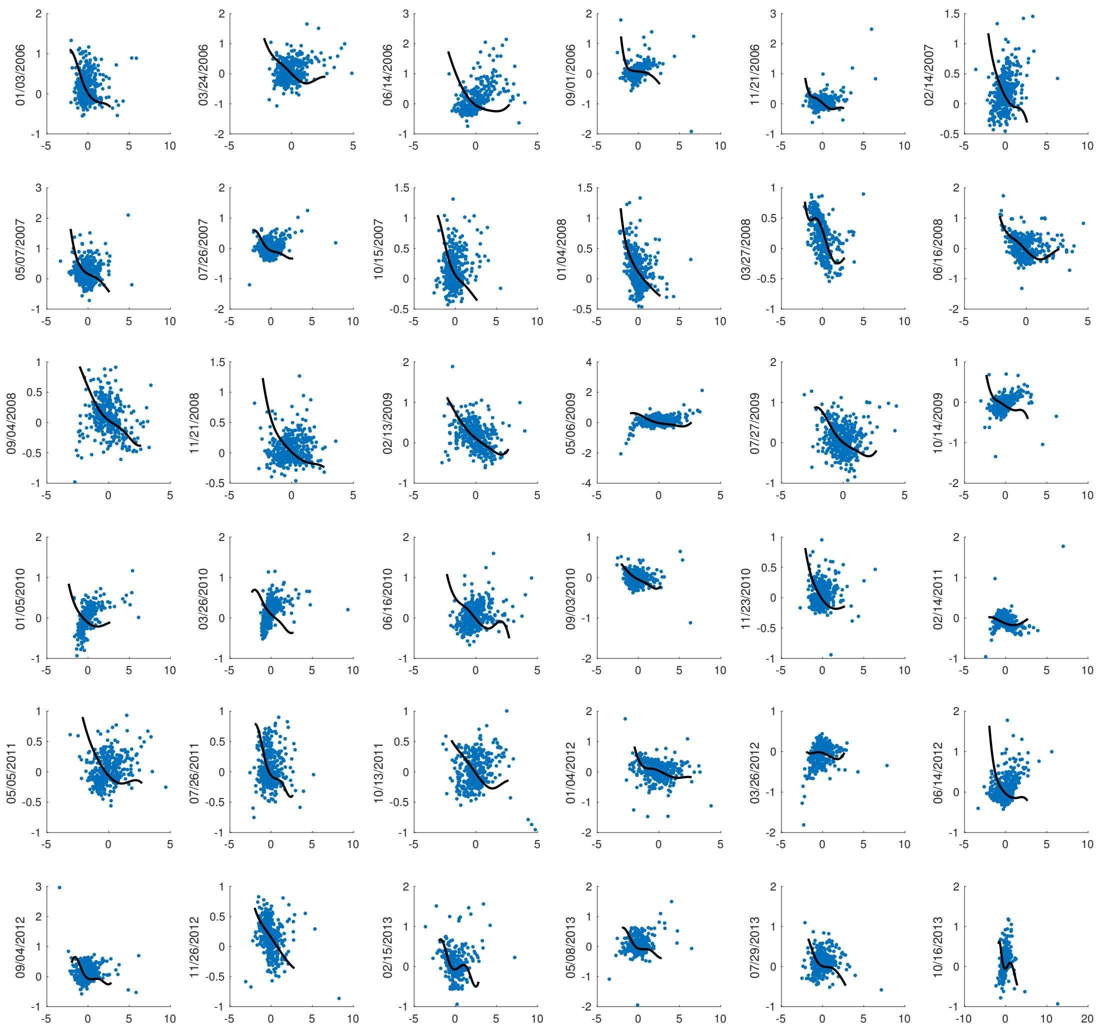

Figure 5: Estimated size effect on the SMB factor at representative days