

Statistical inferences using large estimated covariances for Panel Data and Factor Models

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Abstract

While most of the convergence results in the literature on high dimensional covariance matrix are concerned about the accuracy of estimating the covariance matrix (and precision matrix), relatively less is known about the effect of estimating large covariances on statistical inferences. We study two important models: factor analysis and panel data model with interactive effects, and focus on the statistical inference and estimation efficiency of structural parameters based on large covariance estimators. For efficient estimation, both models call for a weighted principle components (WPC), which relies on a high dimensional weight matrix. This paper derives an efficient and feasible WPC using the covariance matrix estimator of Fan et al. (2013). However, we demonstrate that existing results on large covariance estimation based on absolute convergence are not suitable for statistical inferences of the structural parameters. What is needed is some weighted consistency and the associated rate of convergence, which are obtained in this paper. Finally, the proposed method is applied to the US divorce rate data. We find that the efficient WPC identifies the significant effects of divorce-law reforms on the divorce rate, and it provides more accurate estimation and tighter confidence intervals than existing methods.

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1 Introduction

Estimating a high-dimensional covariance matrix has been an active research area in the recent literature. Many methods are proposed for estimating the covariance matrix and the precision (inverse covariance) matrix, e.g. El Karoui (2008), Bickel and Levina (2008), Rothman et al. (2009), Lam and Fan (2009), Cai and Liu (2011), Fan et al. (2013). Among many theoretical results, rates of convergence under various interesting matrix norms have been derived. In particular, if we write N to denote the dimension and T to denote the sample size, when the $N \times N$ covariance matrix Σ is sparse whose eigenvalues are bounded away from zero, we can obtain an estimator $\widehat{\Sigma}$ that achieves a *near*- \sqrt{T} -rate under the *operator norm*:

$$\|\widehat{\Sigma} - \Sigma\| = O_p(m_N(\frac{\log N}{T})^{\frac{1-q}{2}}) = \|\widehat{\Sigma}^{-1} - \Sigma^{-1}\| \quad (1.1)$$

where m_N and q are parameters that measure the level of sparsity. Cai and Zhou (2012) showed that the rate of convergence (1.1) is minimax optimal. However, there is relatively less knowledge about the effect of estimating a high-dimensional covariance matrix on statistical inferences, e.g., the estimation efficiency for a parametric model, and the effect of estimating large covariances on the limiting distributions for estimators of some structural parameters.

We find that when a high-dimensional covariance estimator is applied for statistical inferences (precisely, deriving limiting distributions of estimated structural parameters), most of the results in the literature based on *absolute convergence* like (1.1) are not suitable, even with the minimax optimal rate. Instead, a “*weighted convergence*” is needed, which takes the form $\|A_1(\widehat{\Sigma}^{-1} - \Sigma^{-1})A_2\|$, where both A_1, A_2 are stochastic matrices that weight the estimation error $\widehat{\Sigma}^{-1} - \Sigma^{-1}$. The weights A_1 and A_2 further “average down” the estimation errors, which significantly improve the rate of convergence to make valid statistical inferences. However, the weighted convergence cannot be implied by the usual results in the literature. One of our contributions is to tackle this problem.

This paper focuses on two models that are of increasing importance in many statistical applications: factor analysis and panel data model with interactive effects. In factor analysis, the notion of sparsity is a natural assumption based on the factor structure, which is proved to be a successful approach (e.g., Boivin and Ng 2006, Phan 2012, Andersen et al. 2011). This

paper gives a theoretical justification about how such a sparse structure can be used to improve the estimation efficiency in two general models. Both problems involve estimating a large weight matrix, where the problem of proving “weighted convergence” is present.

1.1 Approximate factor model

We consider a high-dimensional approximate factor model:

$$y_{it} = \lambda_i' f_t + u_{it}, \quad i \leq N, t \leq T. \quad (1.2)$$

where f_t is an $r \times 1$ vector of common factors, λ_i is a vector of factor loadings, and u_{it} represents the error term, often known as the *idiosyncratic component*. If we denote $Y_t = (y_{1t}, \dots, y_{Nt})'$, $\Lambda = (\lambda_1, \dots, \lambda_N)'$, and $u_t = (u_{1t}, \dots, u_{Nt})'$, model (1.2) can be written as

$$Y_t = \Lambda f_t + u_t, \quad t \leq T.$$

Only Y_t is observable in the model. In a data-rich environment, both N and T can be large and the dimension N might be even much larger than T . The goal is to make efficient inference about $\lambda_i' f_t$, λ_i , f_t or their rotations.

Approximate factor models often require the $N \times N$ covariance matrix $\Sigma_u = \text{cov}(u_t)$ be non-diagonal matrix and the diagonal entries may vary over a large range (Chamberlain and Rothschild 1983). The traditional method of principal components (PC) essentially treats u_{it} to be homoskedastic and uncorrelated over i . As a result, it is inefficient. In this paper, we consider a *weighted principal components* (WPC) method to efficiently estimate the heteroskedastic approximate factor models. The WPC solves a weighted least squares problem:

$$\min_{\Lambda, f_t} \sum_{t=1}^T (Y_t - \Lambda f_t)' W (Y_t - \Lambda f_t) \quad (1.3)$$

subject to certain normalization constraints. Here W is an $N \times N$ positive definite weight matrix. We propose a feasible efficient WPC that requires consistently estimating the high-dimensional Σ_u^{-1} (when $N > T$) as the weight matrix, and is shown to be optimal over a broad class of estimators.

1.2 Large panel data model with interactive effects

A closely related model is the panel data with a factor structure in the error term:

$$y_{it} = X'_{it}\beta + \varepsilon_{it}, \quad \varepsilon_{it} = \lambda'_i f_t + u_{it}, \quad i \leq N, t \leq T, \quad (1.4)$$

where X_{it} is a $d \times 1$ vector of regressors; β is a $d \times 1$ vector of unknown coefficients. The regression noise ε_{it} has a factor structure with unknown loadings and factors. In the model, the only observables are (y_{it}, X_{it}) . The goal is to estimate the structural parameter β , whose dimension is fixed. In this model, the factor component $\lambda'_i f_t$ is regarded as an *interactive effect* of the individual and time effects. Because the regressor and factor can be correlated, simply regressing y_{it} on X_{it} is not consistent.

Similarly, we propose to estimate β via:

$$\min_{\beta, \Lambda, f_t} \sum_{t=1}^T (Y_t - X'_{it}\beta - \Lambda f_t)' W (Y_t - X'_{it}\beta - \Lambda f_t), \quad (1.5)$$

with a high-dimensional weight matrix W . The method is also WPC because the estimated factors are shown to be principal components of the weighted sample covariance matrix. In particular, it allows a consistent estimator for Σ_u^{-1} as the optimal weight matrix even when Σ_u^{-1} is non-diagonal and $N/T \rightarrow \infty$. Except for sparsity, the off-diagonal structure of Σ_u is unknown. The WPC takes into account both cross-sectional correlation and heteroskedasticity of u_{it} over i , while the existing methods in the literature, e.g., Bai 2009, Moon and Weidner 2010, do not.

1.3 Summary of contributions

First of all, we develop the inferential theory using a general high-dimensional weight W . This admits many promising choices of the weight matrices that are suitable for specific applied problems for factor analysis. Especially, in cases where estimating Σ_u is difficult, our inferential theory is still useful when suitable weight matrices are chosen to improve the estimation efficiency. Secondly, we show that when $W = \Sigma_u^{-1}$ is used, the WPC yields an optimal estimator in the sense that the estimated common component $\lambda'_i f_t$ and structural parameter β have the minimum asymptotic variance over a broad class of estimators.

Third, we focus on the effect of estimating large covariance matrices on efficient statistical inferences. In both pure factor analysis and the large panel data with a factor structure, we employ a consistent estimator for Σ_u^{-1} recently proposed by Fan et al. (2013), as an operational weight matrix. Therefore, our optimal estimator is still feasible under $N/T \rightarrow \infty$. However, substituting a consistent estimator Σ_u^{-1} is highly non-trivial when $N > T$. An interesting phenomenon is observed: most existing results on estimating large covariances are not suitable for statistical inferences of the models being considered. We develop a new strategy that investigates the *weighted consistency* for the estimated optimal weight matrix to address this problem.

Fourth, we consistently estimate the asymptotic variances of the proposed estimators under both cross-sectional and serial correlations in u_{it} . Hence the new WPC estimator for the interactive effect model is readily used for statistical inferences in practice. In contrast, existing methods usually require additionally modeling the large error covariance (e.g., assuming diagonality, parametrizing the off-diagonal structure) in order for practical inferences.

Given the popularity of the PC method, why do we need a new estimator to incorporate the large covariance Σ_u ? Even though most of the existing methods for panel data models avoid estimating Σ_u , to demonstrate the potential efficiency loss for existing methods, we present a real-data application in Section 7, which studies the effect of divorce reform law on the change of divorce rates. The WPC is applied to the year-state divorce rate data of U.S. during 1956-1985. It illustrates that after incorporating Σ_u^{-1} in the estimation, WPC captures the significant (negative) effects from nine to twelve years after the law was reformed, consistent with the previous empirical findings in the social science literature. In contrast, the existing method (PC) without estimating Σ_u^{-1} would result in wide confidence intervals and potentially conservative conclusions. Numerically, we find an average of 46% efficiency gained using WPC, relative to the existing method. In addition, the proposed WPC also enjoys the computational convenience, as it also admits analytical solutions.

Realizing the limitation of the regular PC method, some important works have been developed to improve the estimation efficiency for factor analysis, e.g., Breitung and Tenhofen (2011), Bai and Li (2012) and Doz et al. (2012). They require the cross-sectional dependences' structure be specifically modeled. Recently, Choi (2012) specified $W = \Sigma_u^{-1}$, which essentially requires Σ_u be known. Recently, Fan et al. (2013) proposed a thresholding method to estimate Σ_u^{-1} . They focused on covariance matrix estimations and did

not address the efficient estimation for the factors, loadings and panel data models. As we discussed, replacing Σ_u^{-1} with its consistent estimator is technically challenging when $N/T \rightarrow \infty$. Additional literature on factor analysis and panel data with interactive effects includes, e.g., Pesaran (2006), Ahn et al. (2001), Su and Chen (2013), Su et al. (2012), Wang (2009), Forni et al. (2000), Hallin and Liška (2007), Lam and Yao (2012), Cheng and Hansen (2013), Caner and Han (2012), etc. None of these incorporated Σ_u^{-1} or studied efficient estimation for panel data models. We also remark that there is a rapidly growing literature on estimating high-dimensional (inverse) covariance matrices. Besides those mentioned, the list also includes, e.g., d'Aspremont et al. (2008), Bien and Tibshirani (2011), Luo (2011), Pati et al. (2012), Xue et al. (2012), among many others.

We assume the number of factors $r = \dim(f_t)$ to be known. When r is unknown, it can be consistently estimated by certain information criteria as in, e.g., Bai and Ng (2002), as we shall briefly discuss in Section 5.

The rest of the paper is organized as follows. Section 2 describes the general problem of statistical inference based on large covariance matrices. Section 3 formally proposes the WPC method. The large-sample inferential theory of WPC with a general weight matrix is presented. Section 4 introduces the efficient WPC. Section 5 applies the WPC method to the panel data model with interactive effects. Section 6 illustrates numerical comparisons of related methods. Section 7 applies WPC to a real data problem of divorce rate study. Finally, Section 8 concludes. All proofs are given in the supplementary material.

Throughout the paper, we use $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ to denote the minimum and maximum eigenvalues of matrix A . We also let $\|A\|$, $\|A\|_1$ and $\|A\|_F$ denote the operator norm, L_1 -norm and Frobenius norm of a matrix, defined as $\|A\| = \sqrt{\lambda_{\max}(A'A)}$, $\|A\|_1 = \max_i \sum_j |A_{ij}|$ and $\|A\|_F = \sqrt{\sum_{i,j} A_{ij}^2}$ respectively. Note that if A is a vector, $\|A\| = \|A\|_F$ is equal to the Euclidean norm. Finally, for two sequences, we write $a_T \ll b_T$ (and equivalently $b_T \gg a_T$) if $a_T = o(b_T)$ as $T \rightarrow \infty$.

2 Challenge of Inference based on Large Estimated Covariance

Consider estimating a low-dimensional structural parameter θ that arises from a model involving a high-dimensional covariance matrix Σ . It is often the case that when Σ were known, incorporating it in the estimator may achieve a better estimation accuracy, e.g., smaller standard errors and tighter confidence intervals. Taking into account Σ , the estimator can be written as a function of the data D_T and Σ as (T denotes the sample size):

$$\hat{\theta} = f(D_T, \Sigma),$$

and the limiting distribution may be derived. In practice, we replace Σ by a consistent estimator $\hat{\Sigma}$ and obtain a feasible efficient estimator $f(D_T, \hat{\Sigma})$.

To show that replacing Σ with its consistent estimator does not affect the limiting distribution of $\hat{\theta}$, one often needs $a_T(f(D_T, \Sigma) - f(D_T, \hat{\Sigma})) = o_p(1)$ where a_T^{-1} can be understood as the rate of convergence of $\hat{\theta}$. However, such a simple substitution is technically difficult if $N > T$. To see this, note that often $f(D_T, \Sigma)$ depends on the precision matrix Σ^{-1} , and the effect of estimating Σ^{-1} is approximately linearly dependent on $\hat{\Sigma}^{-1} - \Sigma^{-1}$. We can often write

$$a_T(f(D_T, \Sigma) - f(D_T, \hat{\Sigma})) = a_T A_1 (\hat{\Sigma}^{-1} - \Sigma^{-1}) A_2 + o_p(1)$$

where A_1, A_2 are typically non-sparse stochastic matrices of dimensions $\dim(\theta) \times N$ and $N \times 1$ respectively. Applying the Cauchy-Schwarz inequality,

$$a_T \|A_1 (\hat{\Sigma}^{-1} - \Sigma^{-1}) A_2\| \leq a_T \|A_1\| \|A_2\| \|\hat{\Sigma}^{-1} - \Sigma^{-1}\|.$$

As both A_1 and A_2 are high-dimensional matrices (vectors), the right hand side of the above inequality is typically not stochastically negligible even if the “absolute convergence” $\|\hat{\Sigma}^{-1} - \Sigma^{-1}\|$ achieves the optimal convergence rate.¹ The problem arises because $\|A_1\|$ and $\|A_2\|$ grow fast with the dimensionality, so they accumulate the estimation errors and lead to a crude bound.

We further illustrate this issue in two examples, which are to be studied in detail in the paper.

¹When Σ is sparse enough, one can obtain a near \sqrt{T} -rate of convergence for the L_1 -norm $\|\hat{\Sigma}^{-1} - \Sigma^{-1}\|_1$, but this still yields a crude bound for $a_T A_1 (\hat{\Sigma}^{-1} - \Sigma^{-1}) A_2$.

Example 2.1. Consider the high-dimensional factor model (1.2). The parameter of interest is the common component $\lambda_i' f_t$. The efficient estimation crucially depends on $\frac{1}{\sqrt{N}} \Lambda' (\widehat{\Sigma}_u^{-1} - \Sigma_u^{-1}) u_t$, for a sparse covariance estimator $\widehat{\Sigma}_u^{-1}$. However, the existing results on the optimal convergence of $\|\widehat{\Sigma}_u^{-1} - \Sigma_u^{-1}\|$ in the literature (e.g., Fan et al. 2013) are not applicable directly when $N > T$, because $\|\Lambda\| = O(\sqrt{N})$ and $\|u_t\| = O_p(\sqrt{N})$, but the minimax rate for $\|\widehat{\Sigma}_u^{-1} - \Sigma_u^{-1}\|$ is no faster than $O_p(T^{-1/2})$. Applying the absolute convergence for $\widehat{\Sigma}_u^{-1}$, $\frac{1}{\sqrt{N}} \|\Lambda\| \|\widehat{\Sigma}_u^{-1} - \Sigma_u^{-1}\| \|u_t\| = O_p(\sqrt{\frac{N}{T}}) \neq o_p(1)$ when $N > T$.

Example 2.2. Consider the high-dimensional panel data model (1.4). The efficient estimation of β requires estimating the inverse covariance Σ_u^{-1} . Suppose $\widetilde{\Sigma}_u^{-1}$ is a consistent estimator. We require

$$\frac{1}{\sqrt{NT}} Z' [(\widetilde{\Sigma}_u^{-1} - \Sigma_u^{-1}) \otimes I_T] U = o_p(1),$$

where I_T is a T -dimensional identity matrix and Z and U are stochastic matrices whose dimensions are $NT \times \dim(\beta)$ and $NT \times 1$ respectively. However, because $\|Z\| = O_p(\sqrt{NT}) = \|U\|$, it is difficult to apply the absolute convergence $\|\widetilde{\Sigma}_u^{-1} - \Sigma_u^{-1}\|$ (whose minimax rate is no faster than $O_p(T^{-1/2})$) to achieve the desired convergence when $N > T$. The crude bound gives $\frac{1}{\sqrt{NT}} \|Z\| \|\widetilde{\Sigma}_u^{-1} - \Sigma_u^{-1}\| \|U\| = O_p(\sqrt{N}) \neq o_p(1)$. \square

As one of the main contributions of this paper, a new strategy of “weighted convergence” is developed. When analyzing $a_T A_1 (\widehat{\Sigma}^{-1} - \Sigma^{-1}) A_2$, we should not separate the covariance estimation error from the weighting matrices A_1, A_2 . Intuitively, the weights further “average down” the estimation errors, to ensure the asymptotic negligibility of the weighted error. We demonstrate that the weighted convergence is useful for high-dimensional inferences in panel data and factor models, and cannot be simply implied by the usual results on “absolute convergence” in the literature.

3 Approximate Factor Models

3.1 Weighted principal components

In model (1.2), the only observables are $\{Y_t\}_{t=1}^T$, and both the factors $\{f_t\}_{t=1}^T$ and loadings $\Lambda = (\lambda_1, \dots, \lambda_N)'$ are parameters to estimate. We esti-

mate them via the following weighted least squares:

$$(\widehat{\Lambda}, \widehat{f}_t) = \min_{\Lambda, f_t} \sum_{t=1}^T (Y_t - \Lambda f_t)' W_T (Y_t - \Lambda f_t) \quad (3.1)$$

subject to:

$$\frac{1}{T} \sum_{t=1}^T \widehat{f}_t \widehat{f}_t' = I_r; \quad \widehat{\Lambda}' W_T \widehat{\Lambda} \text{ is diagonal.} \quad (3.2)$$

Here W_T is an $N \times N$ weight matrix, which can be either stochastic or deterministic. When W_T is stochastic, we mean W_T to be a consistent estimator of some positive definite W under the operator norm. We will show in Section 4 that the optimal weight is Σ_u^{-1} . On the other hand, keeping a general W_T admits other choices of the weight for specific applied problems, especially when it is difficult to estimate the optimal weight matrix.

Solving (3.1) subjected to the restriction (3.2) gives the WPC estimators: $\widehat{\lambda}_j$ and \widehat{f}_t are both $r \times 1$ vectors such that, the columns of the $T \times r$ matrix $\widehat{F}/\sqrt{T} = (\widehat{f}_1, \dots, \widehat{f}_T)'/\sqrt{T}$ are the eigenvectors corresponding to the largest r eigenvalues of $Y'W_T Y$, and $\widehat{\Lambda} = T^{-1}Y\widehat{F} = (\widehat{\lambda}_1, \dots, \widehat{\lambda}_N)'$. We call our method to be *weighted principal components* (WPC), to distinguish from the traditional principal components (PC) method that uses $W_T = I_N$. Note that PC does not encounter the problem of estimating large covariance matrices, and is not efficient when $\{u_{it}\}$'s are cross-sectionally correlated across i .

It has been well known that the factors and loadings are not separably identifiable without further restrictions. The WPC estimates rotated factors and loadings with rotation matrix H_W . Let \widehat{V} be the $r \times r$ diagonal matrix of the first r largest eigenvalues of $YW_T Y'/(TN)$. Let $F = (f_1, \dots, f_T)'$, then $H_W = \widehat{V}^{-1}\widehat{F}'F\Lambda'W_T\Lambda/(NT)$. We use the subscript W to emphasize the dependence of the rotation on W .

3.2 General conditions

We present general results for the proposed WPC with a general weight matrix, which hold for a broad class of estimators. For the general weight matrix W and its data-dependent version W_T , the following assumption is needed:

Assumption 3.1. (i) $\|W_T - W\| = o_p(\min\{T^{-1/4}, N^{-1/4}, \sqrt{\frac{N}{T}}, \sqrt{\frac{T}{N \log N}}\})$.
(ii) $\|\frac{1}{\sqrt{N}}\Lambda'(W_T - W)u_t\| = o_p(1)$.

Condition (i) is easy to satisfy by using many “good” covariance estimators given in the literature. However, the main challenge described in Section 2 arises from proving condition (ii) in the above assumption. When W_T is a consistent estimator for Σ_u^{-1} , we shall see in Section 4.2 that this requires a new “weighted convergence”, which is necessary but challenging to the high-dimensional inference problems being considered.

We allow the factors and idiosyncratic components to be weakly serially dependent via the strong mixing condition. Let $\mathcal{F}_{-\infty}^0$ and \mathcal{F}_T^∞ denote the σ -algebras generated by $\{(f_t, u_t) : -\infty \leq t \leq 0\}$ and $\{(f_t, u_t) : T \leq t \leq \infty\}$ respectively. In addition, define the mixing coefficient

$$\alpha(T) = \sup_{A \in \mathcal{F}_{-\infty}^0, B \in \mathcal{F}_T^\infty} |P(A)P(B) - P(AB)|. \quad (3.3)$$

Assumption 3.2. (i) $\{u_t, f_t\}_{t \geq 1}$ is strictly stationary. In addition, $Eu_{it} = Eu_{it}f_{jt} = 0$ for all $i \leq p, j \leq r$ and $t \leq T$.

(ii) There exist constants $c_1, c_2 > 0$ such that $c_2 < \lambda_{\min}(\Sigma_u) \leq \lambda_{\max}(\Sigma_u) < c_1$, $\max_{j \leq N} \|\lambda_j\| < c_1$, and $c_2 < \lambda_{\min}(\text{cov}(f_t)) \leq \lambda_{\max}(\text{cov}(f_t)) < c_1$.

(iii) Exponential tail: There exist $r_1, r_2 > 0$ and $b_1, b_2 > 0$, such that for any $s > 0$, $i \leq p$ and $j \leq r$, $P(|u_{it}| > s) \leq \exp(-(s/b_1)^{r_1})$, and $P(|f_{jt}| > s) \leq \exp(-(s/b_2)^{r_2})$.

(iv) Strong mixing: There exists $r_3 > 0$ and $C > 0$ satisfying: for all $T \in \mathbb{Z}^+$,

$$\alpha(T) \leq \exp(-CT^{r_3}).$$

We assume that W has bounded row sums, that is, $\|W\|_1 < M$ for some $M > 0$. Write $\Lambda'W = (d_1, \dots, d_N)$, with each d_i being an $r \times 1$ vector. Then $\max_{j \leq N} \|d_j\| < \infty$.

The following assumptions are standard in the literature. Assumption 3.3 requires the factors be *pervasive*, which holds when the factors impact a non-vanishing proportion of individual time series. Assumption 3.4 extends similar conditions in Stock and Watson (2002) and Bai (2003). When $W = I_N$ is used, they reduce to those in the literature of the regular PC. A simple sufficient condition for Assumption 3.4 is that u_{it} is i.i.d. in both i and t .

Assumption 3.3. All the eigenvalues of the $r \times r$ matrix $\Lambda'\Lambda/N$ are bounded away from both zero and infinity as $N \rightarrow \infty$.

Assumption 3.4. (i) $E\|\frac{1}{\sqrt{NT}} \sum_{s=1}^T f_s(u'_s W u_t - Eu'_s W u_t)\|^2 = O(1)$.

(ii) For each $i \leq N$, $E\|\frac{1}{\sqrt{NT}} \sum_{t=1}^T \sum_{j=1}^N d_j(u_{jt}u_{it} - Eu_{jt}u_{it})\| = O(1)$.

- (iii) For each $i \leq r$, $E\|\frac{1}{\sqrt{NT}}\sum_{t=1}^T\sum_{j=1}^N d_j u_{jt} f_{it}\| = O(1)$.
(iv) There is a constant $\delta \geq 4$ and $M > 0$ such that for all large N , $E|\frac{1}{\sqrt{N}}(u'_s W u_t - E u'_s W u_t)|^\delta < M$ and $E\|\frac{1}{\sqrt{N}}\Lambda' W u_t\|^\delta < M$.

3.3 Limiting distributions

The factors and loadings are two sets of parameters to estimate. The limiting distributions of their estimators depend on the following asymptotic expansions, to be shown in the appendix: for some positive definite matrix J_W , and the rotation matrix H_W ,

$$\begin{aligned}\sqrt{N}(\widehat{f}_t - H_W f_t) &= J_W \frac{\Lambda' W u_t}{\sqrt{N}} + O_p(a_T) \\ \sqrt{T}(\widehat{\lambda}_j - H_W^{-1} \lambda_j) &= H_W \frac{1}{\sqrt{T}} \sum_{t=1}^T f_t u_{jt} + O_p(b_T).\end{aligned}\quad (3.4)$$

where the asymptotic normality arises from the leading terms while a_T and b_T are some remaining stochastic sequences.

The limiting distribution of $\widehat{\lambda}_j$ requires H_W to have a limit. We thus need the following condition:

Assumption 3.5. (i) There is an $r \times r$ matrix Σ_Λ such that $\Lambda' W \Lambda / N \rightarrow \Sigma_\Lambda$ as $N \rightarrow \infty$. In addition, the eigenvalues of the $\Sigma_\Lambda \text{cov}(f_t)$ are distinct.

(ii) For each $t \leq T$, $(\Lambda' W \Sigma_u W \Lambda)^{-1/2} \Lambda' W u_t \rightarrow^d \mathcal{N}(0, I_r)$.

According to the expansions of (3.4), the above condition (ii) is almost a necessary condition for the asymptotic normality of \widehat{f}_t . Note that $\frac{1}{\sqrt{N}} \Lambda' W u_t = \frac{1}{\sqrt{N}} \sum_{i=1}^N d_i u_{it}$. Hence a cross-sectional central limit theorem can indeed apply. Condition (ii) is only for \widehat{f}_t , and the limiting distribution of the estimated loading $\widehat{\lambda}_j$ in Theorem 3.1 below does not depend on this condition.

We now introduce some notation that are needed to present the limiting distributions. Let V be an $r \times r$ diagonal matrix with element as the largest r eigenvalues of $\Sigma_\Lambda^{1/2} \text{cov}(f_t) \Sigma_\Lambda^{1/2}$, and Γ_W be the corresponding eigenvector matrix such that $\Gamma'_W \Gamma_W = I_r$. We use the subscript W to indicate that Γ_W depends on W via Σ_Λ . Recall that Σ_Λ is defined in Assumption 3.5. Let $Q_W = V^{1/2} \Gamma'_W \Sigma_\Lambda^{-1/2}$. In fact $H_W \rightarrow^p Q_W^{-1}$. In addition, to account for the serial correlation over t , let

$$\Phi_j = E(f_t f'_t u_{jt}^2) + \sum_{t=1}^{\infty} E[(f_1 f'_{1+t} + f_{1+t} f'_1) u_{j1} u_{j,1+t}]. \quad (3.5)$$

Theorem 3.1. Assume $(\log N)^2 = o(T)$ and $T = o(N^2)$. Under Assumptions 3.1-3.5(i), for each $j \leq N$,

$$\sqrt{T}(\widehat{\lambda}_j - H_W'^{-1}\lambda_j) \rightarrow^d \mathcal{N}(0, Q_W'^{-1}\Phi_j Q_W^{-1}).$$

If in addition, $N = o(T^2)$ and Assumption 3.5(ii) holds,

$$N(V^{-1}Q_W\Lambda'W\Sigma_u W\Lambda Q_W'V^{-1})^{-1/2}(\widehat{f}_t - H_W f_t) \rightarrow^d \mathcal{N}(0, I_r).$$

For the common component, we have

$$\frac{\widehat{\lambda}'_i \widehat{f}_t - \lambda'_i f_t}{(\lambda'_i \Xi_W \lambda_i / N + f'_t \Omega_i f_t / T)^{1/2}} \rightarrow^d \mathcal{N}(0, 1).$$

where $\Xi_W = \Sigma_\Lambda^{-1} \Lambda' W \Sigma_u W \Lambda \Sigma_\Lambda^{-1} / N$ and $\Omega_i = \text{cov}(f_t)^{-1} \Phi_i \text{cov}(f_t)^{-1}$.

Remark 3.1. The eigenvalues of $(V^{-1}Q_W\Lambda'W\Sigma_u W\Lambda Q_W'V^{-1})^{-1/2}$ are of order $O(N^{-1/2})$. Hence Theorem 3.1 implies the \sqrt{N} -consistency of the estimated factors. If we further assume that $\Lambda'W\Sigma_u W\Lambda/N$ has a limit, say G , then immediately we have

$$\sqrt{N}(\widehat{f}_t - H_W f_t) \rightarrow^d \mathcal{N}(0, V^{-1}Q_W G Q_W' V^{-1}),$$

where the \sqrt{N} -consistency is more clearly demonstrated.

The uniform convergence of \widehat{f}_t and $\widehat{\lambda}_j$ are given below.

Theorem 3.2. Let $\alpha = \max\{1/r_1, 1/r_2\}$ with r_1, r_2 defined in Assumption 3.2. Let $\delta \geq 4$ be as defined in Assumption 3.4. Under Assumptions 3.1-3.4, as $N, T \rightarrow \infty$,

$$\max_{t \leq T} \|\widehat{f}_t - H_W f_t\| = O_p \left((\log T)^\alpha \|W_T - W\| + \frac{T^{1/\delta}}{\sqrt{N}} + \frac{1}{\sqrt{T}} \right), \quad (3.6)$$

$$\max_{j \leq N} \|\widehat{\lambda}_j - H_W'^{-1}\lambda_j\| = O_p \left(\|W_T - W\| + \frac{1}{\sqrt{N}} + \sqrt{\frac{\log N}{T}} \right). \quad (3.7)$$

Remark 3.2. The uniform convergence in (3.6) and (3.7) is important under large N and T . For example, in estimating large covariance matrices, it is used to derive the proper levels of thresholding or shrinkage (e.g., Fan et al. 2013, Ledoit and Wolf 2012).

3.4 Heteroskedastic WPC

As a simple choice for W ,

$$W = (\text{diag}(\Sigma_u))^{-1}.$$

This choice improves the regular PC when cross-sectional heteroskedasticity is present. This weight can be easily estimated using the residuals. First apply the regular PC by taking $W_T = I_N$, and obtain a consistent estimator \hat{C}_{it} of the common component $\lambda_i' f_t$ for each $i \leq N, t \leq T$. Define

$$W_T^h = \text{diag}\{\hat{\sigma}_{u,11}^{-1}, \dots, \hat{\sigma}_{u,NN}^{-1}\}, \text{ where } \hat{\sigma}_{u,ii} = \frac{1}{T} \sum_{t=1}^T (y_{it} - \hat{C}_{it})^2.$$

Then in the second step, apply the WPC with weight matrix W_T^h .

The heteroskedastic WPC (which we call HWPC) method has been previously suggested by, e.g., Breitung and Tenhofen (2011). Investigations of its theoretical properties can be found in the appendix. Moreover, numerical studies in Section 6 show that this method improves the efficiency relative to the regular PC method.

4 Efficient Principal Components Under Conditional Sparsity

In the approximate factor models, u_{it} 's are correlated (over i). A more efficient estimator (which we call EWPC) should take $W = \Sigma_u^{-1}$ as the weight matrix. This estimator has been recently suggested by Choi (2012), but Σ_u^{-1} was assumed to be known.

There are two main challenges in practice: on one hand, when $N > T$, Σ_u^{-1} is hard to estimate as the sample covariance based on the residual \hat{u}_{it} is no longer invertible. On the other hand, as we illustrated in Section 2, even if a consistent estimator for Σ_u^{-1} is available, it is technically difficult to prove that the effect of covariance estimation is negligible when $N/T \rightarrow \infty$. We first apply Fan et al. (2013)'s method to estimate Σ_u^{-1} , and then address the second problem in Section 4.2.

4.1 Conditional Sparsity

We apply a thresholded covariance estimator to estimate Σ_u^{-1} , which is recently proposed by Fan et al. (2013) for factor analysis. Let $(\nu_j, \xi_j)_{j=1}^N$ be the eigenvalues-eigenvectors of the sample covariance S_y of Y_t , in a decreasing order such that $\nu_1 \geq \nu_2 \geq \dots \geq \nu_N$. Let

$$R = S_y - \sum_{i=1}^r \nu_i \xi_i \xi_i'$$

Define a general thresholding function $s_{ij}(z) : \mathbb{R} \rightarrow \mathbb{R}$ as in Rothman et al. (2009) and Cai and Liu (2011) with an entry-dependent threshold τ_{ij} such that:

- (i) $s_{ij}(z) = 0$ if $|z| < \tau_{ij}$;
- (ii) $|s_{ij}(z) - z| \leq \tau_{ij}$.
- (iii) There are constants $a > 0$ and $b > 1$ such that $|s_{ij}(z) - z| \leq a\tau_{ij}^2$ if $|z| > b\tau_{ij}$.

Examples of $s_{ij}(z)$ include the hard-thresholding: $s_{ij}(z) = zI_{(|z| > \tau_{ij})}$; SCAD (Fan and Li 2001), MPC (Zhang 2010) etc. As for the threshold value, we specify

$$\tau_{ij} = C \sqrt{R_{ii}R_{jj}}\omega_T, \text{ where } \omega_T = \sqrt{\frac{\log N}{T}} + \frac{1}{\sqrt{N}} \quad (4.1)$$

for some pre-determined universal $C > 0$, chosen from cross-validation as in Fan et al. (2013). Then estimate Σ_u by $\hat{\Sigma}_u = (\hat{\Sigma}_{u,ij})_{N \times N}$,

$$\hat{\Sigma}_{u,ij} = \begin{cases} R_{ii}, & i = j \\ s_{ij}(R_{ij}), & i \neq j \end{cases}, \text{ where } R = (R_{ij})_{N \times N}.$$

Intuitively, $\hat{\Sigma}_u$ thresholds off the small entries of the residual covariance $\frac{1}{T} \sum_{t=1}^T \hat{u}_t \hat{u}_t'$ obtained from the regular PC estimate.

To apply such a weight estimator, we assume Σ_u to be a sparse matrix. In an approximate factor model, such a special structure is known to be *conditionally sparse* (given the common factors). Consider the notion of generalized sparsity: write $\Sigma_u = (\Sigma_{u,ij})_{N \times N}$. For some $q \in [0, 1/2)$, define

$$m_N = \max_{i \leq N} \sum_{j=1}^N |\Sigma_{u,ij}|^q. \quad (4.2)$$

In particular, when $q = 0$, define $m_N = \max_{i \leq N} \sum_{j=1}^N I_{(\Sigma_{u,ij} \neq 0)}$. Mathematically, the conditional sparse structure on Σ_u assumes, there is $q \in [0, 1/2)$, such that

$$m_N = o \left(\min \left\{ \frac{1}{N^{1/4}} \left(\frac{T}{\log N} \right)^{(1-q)/2}, N^{1/4-q/2} \right\} \right). \quad (4.3)$$

In the sparse covariance estimation literature, Condition (4.3) itself is enough to achieve a covariance estimator such that $\|\Sigma_u^{-1} - \widehat{\Sigma}_u^{-1}\| = o_p(1)$, whose rate of convergence is nearly \sqrt{T} (e.g., Cai and Zhou 2012, Fan et al. 2013, etc.). But for the “weighted convergence” needed for efficient estimations in factor analysis and large panel data models, this condition is not sufficient. Therefore, we introduce a more refined description of the sparse structure of Σ_u (condition (ii) in Assumption 4.1 below), which is similar to those in Rothman et al. (2008).

Let S_L and S_U denote two disjoint sets and respectively include the indices of small and large elements of Σ_u in absolute value, and

$$\{(i, j) : i \leq N, j \leq N\} = S_L \cup S_U. \quad (4.4)$$

We assume $(i, i) \in S_U$ for all $i \leq N$. The sparsity condition assumes that most of the indices (i, j) belong to S_L when $i \neq j$. A special case arises when Σ_u is strictly sparse, in the sense that its elements with small magnitudes (S_L) are exactly zero. For the banded matrix as an example, $\Sigma_{u,ij} = 0$ if $|i - j| > k$ for some fixed $k \geq 1$. Then $S_L = \{(i, j) : |i - j| > k\}$ and $S_U = \{(i, j) : |i - j| \leq k\}$. Another example is the block-diagonal matrix.

The following assumption mathematically defines the “conditional sparsity” for the approximate factor model.

Define

$$\omega_T = \sqrt{\frac{\log N}{T}} + \frac{1}{\sqrt{N}}.$$

Assumption 4.1. (i) There is $q \in [0, 1/2)$ such that (4.3) holds.
(ii) There is a partition $\{(i, j) : i \leq N, j \leq N\} = S_L \cup S_U$ such that $\sum_{i \neq j, (i,j) \in S_U} 1 = O(N)$ and $\sum_{(i,j) \in S_L} |\Sigma_{u,ij}| = O(1)$. In addition,

$$\max_{(i,j) \in S_L} |\Sigma_{u,ij}| = O(\omega_T), \quad \omega_T = O\left(\min_{(i,j) \in S_U} |\Sigma_{u,ij}|\right).$$

If for example, Σ_u is a block covariance matrix with finite block sizes, this assumption is naturally satisfied as long as the signal is not too-weak (that is, $\omega_T = o(\min_{(i,j) \in S_U} |\Sigma_{u,ij}|)$). Condition (ii) requires the elements in S_L and S_U be well-separable. The partition $\{(i, j) : i \leq N, j \leq N\} = S_L \cup S_U$ may not be unique. Most importantly, we do not need to know either S_L or S_U ; hence the block size, the banding length, or the locations of the zero entries can be completely unknown. Our analysis suffices as long as such a partition exists.

4.2 Weighted convergence using the optimal weight matrix

We now formally discuss the issue brought by Assumption 3.1. In order for the effect of estimating Σ_u^{-1} to be negligible, $\|\frac{1}{\sqrt{N}}\Lambda'(\widehat{\Sigma}_u^{-1} - \Sigma_u^{-1})u_t\| = o_p(1)$ is required, which is a tight condition. However, a direct application of the optimal rate of convergence (i.e., Fan et al. 2013, Cai and Zhou 2012) $\|\widehat{\Sigma}_u^{-1} - \Sigma_u^{-1}\| = O_p(m_N\omega_T^{1-q})$ implies

$$\left\| \frac{1}{\sqrt{N}} \Lambda'(\widehat{\Sigma}_u^{-1} - \Sigma_u^{-1})u_t \right\| \leq \frac{1}{\sqrt{N}} \|\Lambda\| \|\widehat{\Sigma}_u^{-1} - \Sigma_u^{-1}\| \|u_t\| = O_p(\sqrt{N}m_N\omega_T^{1-q}),$$

which is $O_p(1 + \sqrt{N(\log N)/T})$ even if m_N is bounded and $q = 0$. Hence this leads to a crude bound that does not converge. The problem is present even if Σ_u^{-1} is estimated with the optimal rate of convergence.

We realize that such a technical problem is common for statistical inferences that involve estimating a high-dimensional covariance. In fact, most of the existing approaches in the literature only produce “absolute convergence” $\|\widehat{\Sigma}_u^{-1} - \Sigma_u^{-1}\|$. For statistical inference purposes like the primary interest of this paper, however, the absolute convergence is not sufficient when $N/T \rightarrow \infty$.

We propose a new technical strategy to solve this problem, by directly investigating the “weighted convergence” of the weighted error:

$$\left\| \frac{1}{\sqrt{N}} \Lambda'(\widehat{\Sigma}_u^{-1} - \Sigma_u^{-1})u_t \right\|. \tag{4.5}$$

Intuitively, the weights Λ' and u_t “average down” the estimation errors, and improve the rate of convergence. Formal analysis requires us to re-investigate

the asymptotic behavior of the thresholded covariance estimator. We require the following technical assumption.

Let $\Lambda'\Sigma_u^{-1} = (\xi_1, \dots, \xi_N)$. Assuming $\|\Sigma_u^{-1}\|_1 = O(1)$, we then have $\max_{j \leq N} \|\xi_j\| < C$ for some $C > 0$. In addition, let $e_t = \Sigma_u^{-1}u_t$, then e_t has mean zero and covariance Σ_u^{-1} .

Assumption 4.2. For each $t \leq T$ and $k \leq r$,

- (i) $\frac{1}{T\sqrt{N}} \sum_{i=1}^N \sum_{s=1}^T (u_{is}^2 - Eu_{is}^2) \xi_i e_{it} = o_p(1)$
- (ii) $\frac{1}{NT\sqrt{N}} \sum_{i=1}^N \sum_{s=1}^T \sum_{j=1}^N (u_{js}u_{is} - Eu_{js}u_{is}) \lambda_j \lambda'_i e_{it} \xi_{ik} = o_p(1)$,
- (iii) $\frac{1}{T\sqrt{N}} \sum_{i \neq j, (i,j) \in S_U} \sum_{s=1}^T (u_{is}u_{js} - Eu_{is}u_{js}) \xi_i e_{jt} = o_p(1)$,
- (iv) $\frac{1}{NT\sqrt{N}} \sum_{i \neq j, (i,j) \in S_U} \sum_{v=1}^N \sum_{s=1}^T (u_{is}u_{vs} - Eu_{is}u_{vs}) \xi_{ik} e_{jt} \lambda_v \lambda'_j = o_p(1)$.

The above conditions are new in the literature and essential to establish the weighed convergence. The intuition of these conditions is that, the weighted average of the standardized sum $\frac{1}{\sqrt{T}} \sum_{t=1}^T (u_{it}u_{jt} - Eu_{it}u_{jt})$ is $o_p(1)$ once averaged across i and j . The extra term $\frac{1}{N}$ appeared in $\frac{1}{NT\sqrt{N}}$ of Conditions (ii) and (iv) is a scaling factor because under the sparsity condition, the number of summands of $\sum_{i=1}^N$ and $\sum_{i \neq j, (i,j) \in S_U}$ is at most $O(N)$ (e.g., in block diagonal and banded matrices).

We verify the key assumption 4.2 in the following lemma, when $\{u_t\}_{t \leq T}$ is serially independent. We require $N = o(T^2)$ but still allow $N/T \rightarrow \infty$.

Lemma 4.1. Suppose $\{u_{it}\}_{t \leq T}$ is independent across t (but can still be correlated across i), and the sparse condition Assumption 4.1 holds. Then when $N = o(T^2)$, Assumption 4.2 is satisfied.

We have the following *weighted consistency* for the estimated weight matrix, which as we have explained, cannot be implied directly by the absolute convergence $\|\widehat{\Sigma}_u^{-1} - \Sigma_u^{-1}\|$ even when Σ_u is diagonal. As one of the main contributions of this paper, result of this type is potentially widely useful for high-dimensional inferences when large covariance estimation is involved.

Proposition 4.1. Suppose $\sqrt{N}m_N^2\omega_T^{2-2q} = o(1)$, and Assumptions 3.2- 3.5, 4.1, 4.2 hold. For q , m_N and ω_T defined in (4.2) and (4.1), and for each $t \leq T$, we have

$$\left\| \frac{1}{\sqrt{N}} \Lambda' (\widehat{\Sigma}_u^{-1} - \Sigma_u^{-1}) u_t \right\| = o_p(1).$$

Therefore Assumption 3.1 is satisfied for $W = \Sigma_u^{-1}$.

Remark 4.1. Consider a strictly sparse case where $m_N = \max_{i \leq N} \sum_{j=1}^N I(\Sigma_{u,ij} \neq 0) = O(1)$. The condition in the theorem $\sqrt{N}m_N^2\omega_T^{2-2q} = o(1)$ then holds as long as $\sqrt{N} \log N = o(T)$. As always the case, requiring $N = o(T^2)$ is needed for the asymptotic normality of \hat{f}_t .

4.3 Efficient estimation

We use $W_T = \hat{\Sigma}_u^{-1}$ as the feasible weight matrix. Let the columns of the $T \times r$ matrix $\hat{F}^e/\sqrt{T} = (\hat{f}_1^e, \dots, \hat{f}_T^e)'/\sqrt{T}$ be the eigenvectors corresponding to the largest r eigenvalues of $Y'\hat{\Sigma}_u^{-1}Y$, and $\hat{\Lambda}^e = T^{-1}Y\hat{F}^e = (\hat{\lambda}_1^e, \dots, \hat{\lambda}_N^e)'$. Here the superscript e denotes ‘‘efficient’’ WPC.

We denote $\Sigma_{\Lambda,e}$ as the limit of $\Lambda'\Sigma_u^{-1}\Lambda/N$. Let V_e be an $r \times r$ diagonal matrix with elements as the largest r eigenvalues of $\Sigma_{\Lambda,e}^{1/2}\text{cov}(f_t)\Sigma_{\Lambda,e}^{1/2}$, and Γ_e be the corresponding eigenvector matrix such that $\Gamma_e'\Gamma_e = I_r$. In addition, let $Q_e = V_e^{1/2}\Gamma_e'\Sigma_{\Lambda,e}^{-1/2}$. We have the following limiting distributions for the estimated factors and loadings.

Theorem 4.1. *Under the assumptions of Proposition 4.1, for each $t \leq T$ and $j \leq N$,*

$$\begin{aligned} \sqrt{T}(\hat{\lambda}_j^e - H_e'^{-1}\lambda_j) &\rightarrow^d \mathcal{N}(0, Q_e'^{-1}\Phi_j Q_e^{-1}). \\ \sqrt{N}(\hat{f}_t^e - H_e f_t) &\rightarrow^d \mathcal{N}(0, V_e^{-1}). \end{aligned}$$

where Φ_j is as defined in (3.5). In addition, for the estimated common component,

$$\frac{\hat{\lambda}_i^e \hat{f}_t^e - \lambda_i f_t}{(\lambda_i' \Xi_e \lambda_i / N + f_t' \Omega_i f_t / T)^{1/2}} \rightarrow^d \mathcal{N}(0, 1).$$

where $\Xi_e = (\Lambda'\Sigma_u^{-1}\Lambda/N)^{-1}$ and Ω_i is defined as in Theorem 3.1.

For completeness, the following result gives the uniform rate of convergence.

Theorem 4.2. *Suppose $N^{1/(2-2q)} \log N = o(T)$ and $T = o(N^2)$. Under the assumptions of Theorem 4.1, there is an $r \times r$ matrix H_e such that*

$$\begin{aligned} \max_{t \leq T} \|\hat{f}_t^e - H_e f_t\| &= O_p \left(\frac{T^{1/\delta}}{\sqrt{N}} + (\log T)^\alpha m_N \omega_T^{1-q} \right), \\ \max_{j \leq N} \|\hat{\lambda}_j^e - H_e'^{-1}\lambda_j\| &= O_p(m_N \omega_T^{1-q}). \end{aligned}$$

Table 1: Three interesting choices of W

	Objective function	Eigenvectors of	W
regular PC	$\sum_{t=1}^T (Y_t - \Lambda f_t)'(Y_t - \Lambda f_t)$	$Y'Y$	I_r
HWPC	$\sum_{t=1}^T (Y_t - \Lambda f_t)' \text{diag}(\widehat{\Sigma}_u)^{-1} (Y_t - \Lambda f_t)$	$Y' \text{diag}(\widehat{\Sigma}_u)^{-1} Y$	$\text{diag}(\Sigma_u)^{-1}$
EWPC	$\sum_{t=1}^T (Y_t - \Lambda f_t)' \widehat{\Sigma}_u^{-1} (Y_t - \Lambda f_t)$	$Y' \widehat{\Sigma}_u^{-1} Y$	Σ_u^{-1}

The estimated \widehat{F}/\sqrt{T} is the eigenvectors of the largest r eigenvalues of $Y'W_T Y$, and $\widehat{\Lambda} = T^{-1}Y\widehat{F}$. HWPC represents the heteroskedastic WPC; EWPC represents the efficient WPC.

Remark 4.2. Typically in the strictly sparse case $m_N = O(1)$ and $q = 0$. When $N/T \rightarrow \infty$, the above rates become:

$$\max_{t \leq T} \|\widehat{f}_t^e - H_e f_t\| = O_p \left(\frac{T^{1/\delta}}{\sqrt{N}} + \frac{(\log T)^\alpha \sqrt{\log N}}{\sqrt{T}} \right),$$

$$\max_{j \leq N} \|\widehat{\lambda}_j^e - H_e^{-1} \lambda_j\| = O_p \left(\sqrt{\frac{\log N}{T}} \right).$$

4.4 Optimal weight matrix

Regular PC, heteroskedastic WPC and the efficient WPC minimize different objective functions, depending on the choices of the weight matrix. Thus the estimated \widehat{F}/\sqrt{T} are the eigenvectors from three different matrices. Table 1 summarizes the main differences of the estimators.

A natural question arises: is the consistent estimator for $W = \Sigma_u^{-1}$ indeed the optimal choice over a broad class of positive definite weight matrices? One can answer this question via looking at the asymptotic variance of the estimators, as choosing the optimal weight for GMM (Hansen 1982). However, because WPC estimators are estimating rotated factors and loadings, the rotation depends on the choice of W . But regardless of the choice W , the common component $\lambda_i' f_t$ is always directly estimated. The following result

demonstrates that $W_T = \widehat{\Sigma}_u^{-1}$ yields the minimum asymptotic variance of $\widehat{\lambda}'_i \widehat{f}_t$ among WPC estimators.

Theorem 4.3. *Let $(\lambda'_i \Xi_e \lambda_i / N + f'_t \Omega_i f_t / T)$ denote the asymptotic variance of $\widehat{\lambda}'_i \widehat{f}_t$ based on $\widehat{\Sigma}_u^{-1}$ as in Theorem 4.1. For any positive definite matrix W , let $(\lambda'_i \Xi_W \lambda_i / N + f'_t \Omega_i f_t / T)$ denote the asymptotic variance of $\widehat{\lambda}'_i \widehat{f}_t$ as in Theorem 3.1 based on W . Then for each $i \leq N$ and $t \leq T$,*

$$\lambda'_i \Xi_e \lambda_i / N + f'_t \Omega_i f_t / T \leq \lambda'_i \Xi_W \lambda_i / N + f'_t \Omega_i f_t / T.$$

In fact, for all large N , $\Xi_W - \Xi_e$ is semi-positive definite for each positive definite matrix W .

4.5 Estimating asymptotic covariances

We derive consistent estimators for the asymptotic variances that appeared in Theorem 4.1. Hence the derived optimal limiting distributions can be used for statistical inferences. These estimators account for the serial and cross-sectional correlations of the data in both i and t .

The factor estimator has an asymptotic expansion:

$$\sqrt{N}(\widehat{f}_t^e - H_e f_t) = \widehat{V}^{-1} \frac{\widehat{F}^{e'} F \Lambda' \Sigma_u^{-1} u_t}{T \sqrt{N}} + o_p(1)$$

where \widehat{V} is the $r \times r$ diagonal matrix of the first r largest eigenvalues of $\frac{1}{TN} Y \widehat{\Sigma}_u^{-1} Y'$. Theorem 4.1 shows that the asymptotic variance is V_e^{-1} . Hence,

$$\widehat{V}^{-1} \frac{\widehat{F}^{e'} F \Lambda' \Sigma_u^{-1} \Lambda F' \widehat{F}^e}{T N T} \widehat{V}^{-1} \rightarrow^p V_e^{-1} \quad (4.6)$$

The left hand side involves the product $F \Lambda'$, which can be estimated by $\widehat{F}^e \widehat{\Lambda}^e$. A consistent estimator of V_e^{-1} is then given by (note that $\frac{1}{T} \widehat{F}^{e'} \widehat{F}^e = I_r$)

$$\widehat{V}_e^{-1} = \widehat{V}^{-1} \frac{\widehat{F}^{e'} \widehat{F}^e \widehat{\Lambda}^e \widehat{\Sigma}_u^{-1} \widehat{\Lambda}^e \widehat{F}^{e'} \widehat{F}^e}{T N T} \widehat{V}^{-1} = \frac{1}{N} \widehat{V}^{-1} \widehat{\Lambda}^e \widehat{\Sigma}_u^{-1} \widehat{\Lambda}^e \widehat{V}^{-1}.$$

The loading estimator has an asymptotic expansion:

$$\sqrt{T}(\widehat{\lambda}_j - H_e^{-1} \lambda_j) = \frac{1}{\sqrt{T}} \sum_{t=1}^T H_e f_t u_{jt} + o_p(1).$$

Here $H_e f_t u_{jt}$ can be estimated by $\widehat{f}_t^e \widehat{u}_{jt}$, where \widehat{u}_{jt} is a WPC estimator of the error term (e.g., $\widehat{u}_{jt} = y_{it} - \widehat{\lambda}_j^e f_t^e$). We apply the HAC (heteroskedasticity and autocorrelation consistent) estimator of Newey and West (1987) to estimate $Q_e'^{-1} \Phi_j Q_e^{-1}$, the asymptotic variance of $\sqrt{T}(\widehat{\lambda}_j - H_e'^{-1} \lambda_j)$, based on the sequence $\{\widehat{f}_t^e \widehat{u}_{jt}\}$:

$$\widehat{\Psi}_j = \frac{1}{T} \sum_{t=1}^T \widehat{u}_{jt}^2 \widehat{f}_t^e \widehat{f}_t^{e'} + \sum_{l=1}^K \left(1 - \frac{l}{K+1}\right) \frac{1}{T} \sum_{t=l+1}^T \widehat{u}_{jt} \widehat{u}_{j,t-l} (\widehat{f}_t^e \widehat{f}_{t-l}^{e'} + \widehat{f}_{t-l}^e \widehat{f}_t^{e'}),$$

where $K = K_{T,N} \rightarrow \infty$ is an increasing sequence such that $K = o(\min\{T^{1/4}, N^{1/4}\})$. The advantages of using the HAC estimator are: it accounts for the serial correlations of $\{f_t u_t'\}_{t \geq 1}$, and it also guarantees the positive semi-definiteness for any given finite sample as shown by Newey and West (1987).

The asymptotic variance of the common component in Theorem 4.1 consists of $\lambda_i' \Xi_e \lambda_i$ and $f_t' \Omega_i f_t$, where $\Xi_e = (\frac{1}{N} \Lambda' \Sigma_u^{-1} \Lambda)^{-1}$ and $\Omega_i = \text{cov}(f_t)^{-1} \Phi_i \text{cov}(f_t)^{-1}$. We respectively estimate them by

$$\widehat{\Theta}_{1i} = \frac{1}{N} \widehat{\lambda}_i^{e'} \widehat{V}^{-1} \widehat{\Lambda}^{e'} \widehat{\Sigma}_u^{-1} \widehat{\Lambda}^e \widehat{V}^{-1} \widehat{\lambda}_i^e, \quad \widehat{\Theta}_{2,it} = \widehat{f}_t^{e'} \widehat{\Psi}_i \widehat{f}_t^e.$$

Theorem 4.4. *Under the assumptions of Theorem 4.2, as $T, N \rightarrow \infty$, and $K = K_{T,N} = o(\min\{T^{1/4}, N^{1/4}\})$,*

$$\begin{aligned} \widehat{V}_e^{-1} &\rightarrow^p V_e^{-1}, & \widehat{\Psi}_j &\rightarrow^p Q_e'^{-1} \Phi_j Q_e^{-1}, \\ \widehat{\Theta}_{1i} &\rightarrow^p \lambda_i' \Xi_e \lambda_i, & \widehat{\Theta}_{2,it} &\rightarrow^p f_t' \Omega_i f_t. \end{aligned}$$

These covariance estimators can be easily computed.

5 WPC for Panel data Models with Interactive Effects

The factor model we have considered so far is closely related to the following panel data model:

$$y_{it} = X_{it}' \beta + \varepsilon_{it}, \quad \varepsilon_{it} = \lambda_i' f_t + u_{it}, \quad i \leq N, t \leq T \quad (5.1)$$

The regression noise has a factor structure with unknown λ_i and f_t , and u_{it} still represents the idiosyncratic error component. It is assumed that u_{it} is

independent of (X_{it}, f_t) . In the model, the only observables are (y_{it}, X_{it}) . The goal is to estimate β , the structural parameter of the model.

Substituting the second equation to the first one in (5.1), we obtain

$$y_{it} = X_{it}'\beta + \lambda_i'f_t + u_{it}. \quad (5.2)$$

If we treat λ_i as the “individual effect” and f_t as the “time effect”, then the factor structure $\lambda_i'f_t$ represents the interaction between the individual and time effects, so called “interactive effect”. This model was previously studied by, e.g., Ahn et al. (2001), Pesaran (2006), Bai (2009), Moon and Weidner (2010).

The difficulty of estimating β is that, in many applied problems the regressor X_{it} is correlated with the time effect (common factor) f_t , individual effect λ_i , or both. As a result, X_{it} and ε_{it} are also correlated, so regressing y_{it} directly on X_{it} cannot produce a consistent estimator for β . In addition, existing methods ignore the heteroskedasticity and correlation in $\{u_{it}\}_{i \leq N}$. Hence efficiency is lost, for instance, when Σ_u is non-diagonal or its diagonal entries vary over a large range. We shall illustrate the consequence of efficiency loss using a real data application in Section 7.

5.1 WPC estimation of β

Let $X_t = (X_{1t}, \dots, X_{Nt})'$, $(N \times d)$. We estimate β via

$$\min_{\beta, f_t, \Lambda} \sum_{t=1}^T (Y_t - \Lambda f_t - X_t \beta)' W (Y_t - \Lambda f_t - X_t \beta), \quad (5.3)$$

for some positive definite $N \times N$ weight matrix. Similar to the generalized least squares estimator (GLS) for linear regressions, we choose the weight matrix to be

$$W = \Sigma_u^{-1}.$$

This choice produces similar estimators as the efficient WPC. The estimator is feasible once we consistently estimate Σ_u^{-1} , which can be done under the assumption that Σ_u is sparse. Suppose $\tilde{\Sigma}_u^{-1}$ is a consistent covariance estimator. The feasible WPC estimates β by:

$$\hat{\beta} = \arg \min_{\beta} \min_{f_t, \Lambda} \sum_{t=1}^T (Y_t - \Lambda f_t - X_t \beta)' \tilde{\Sigma}_u^{-1} (Y_t - \Lambda f_t - X_t \beta), \quad (5.4)$$

where the minimization is subjected to the constraint $\frac{1}{T} \sum_{t=1}^T f_t f_t' / T = I_r$ and $\Lambda' \tilde{\Sigma}_u^{-1} \Lambda$ being diagonal. The estimated β for each given (Λ, f_t) is simply

$$\beta(\Lambda, f_t) = \left(\sum_{t=1}^T X_t' \tilde{\Sigma}_u^{-1} X_t \right)^{-1} \sum_{t=1}^T X_t' \tilde{\Sigma}_u^{-1} (Y_t - \Lambda f_t).$$

On the other hand, given β , the variable $Y_t - X_t \beta$ has a factor structure. Hence the estimated (Λ, f_t) are the WPC estimators: let $X(\hat{\beta})$ be an $N \times T$ matrix $X(\hat{\beta}) = (X_1 \hat{\beta}, \dots, X_T \hat{\beta})$. The columns of the $T \times r$ matrix $\tilde{F} / \sqrt{T} = (\tilde{f}_1, \dots, \tilde{f}_T)' / \sqrt{T}$ are the eigenvectors corresponding to the largest r eigenvalues of $(Y - X(\hat{\beta}))' \tilde{\Sigma}_u^{-1} (Y - X(\hat{\beta}))$, and $\tilde{\Lambda} = T^{-1} (Y - X(\hat{\beta})) \tilde{F}$. Therefore, given (Λ, f_t) , we can estimate β , and given β , we can estimate (Λ, f_t) . So $\hat{\beta}$ can be simply obtained by iterations, with an initial value $\hat{\beta}_0$. This iteration scheme only requires two matrix inverses: $\tilde{\Sigma}_u^{-1}$ and $(\sum_{t=1}^T X_t' \tilde{\Sigma}_u^{-1} X_t)^{-1}$, which do not update during iterations. Based on our experience of numerical studies, the iterations converge fast.

Similar to Fan et al. (2013), the covariance estimator can be constructed based on thresholding. Let $\hat{\beta}_0$ be a ‘‘regular PC estimator’’ that takes $W = I_N$ in (5.3), which is known to be \sqrt{NT} -consistent (e.g., Bai 2009, Moon and Weidner 2010). Apply the singular value decomposition to

$$\frac{1}{T} \sum_{t=1}^T (Y_t - X_t \hat{\beta}_0)(Y_t - X_t \hat{\beta}_0)' = \sum_{i=1}^N \nu_i g_i g_i',$$

where $(\nu_j, g_j)_{j=1}^N$ are the eigenvalues-eigenvectors of $\frac{1}{T} \sum_{t=1}^T (Y_t - X_t \hat{\beta}_0)(Y_t - X_t \hat{\beta}_0)'$ in a decreasing order such that $\nu_1 \geq \nu_2 \geq \dots \geq \nu_N$. Then $\tilde{\Sigma}_u = (\tilde{\Sigma}_{u,ij})_{N \times N}$,

$$\tilde{\Sigma}_{u,ij} = \begin{cases} \tilde{R}_{ii}, & i = j \\ s_{ij}(\tilde{R}_{ij}), & i \neq j \end{cases}, \quad \tilde{R} = (\tilde{R}_{ij})_{N \times N} = \sum_{i=r+1}^N \nu_i g_i g_i',$$

where $s_{ij}(\cdot)$ is the same thresholding function as defined in Section 4.2 with the same threshold τ_{ij} .

5.2 Assumptions for asymptotic analysis

Rearrange the design matrix

$$Z = (X_{11}, \dots, X_{1T}, X_{21}, \dots, X_{2T}, \dots, X_{N1}, \dots, X_{NT})', \quad NT \times d.$$

For any $T \times r$ matrix F , let $M_F = I_T - F(F'F)^{-1}F'/T$. The following matrices play an important role in the identification and asymptotic analysis:

$$\begin{aligned} A_F &= \left[\Sigma_u^{-1} - \Sigma_u^{-1} \Lambda (\Lambda' \Sigma_u^{-1} \Lambda)^{-1} \Lambda' \Sigma_u^{-1} \right] \otimes M_F, \\ V(F) &= \frac{1}{NT} Z' A_F Z, \end{aligned} \quad (5.5)$$

where (Λ, Σ_u^{-1}) in the above represent the true loading matrix and inverse error covariance in the data generating process, and \otimes denotes the Kronecker product. Our first condition assumes that $V(F)$ is positive definite in the limit uniformly over a class of F .

Assumption 5.1. *With probability approaching one,*

$$\inf_{F: F'F/T=I_r} \lambda_{\min}(V(F)) > 0.$$

If we write $B_F = \left[\Sigma_u^{-1/2} - \Sigma_u^{-1} \Lambda (\Lambda' \Sigma_u^{-1} \Lambda)^{-1} \Lambda' \Sigma_u^{-1/2} \right] \otimes M_F$, then $A_F = B_F B_F'$. So $V(F)$ is at least semi-positive definite. Also, summing over NT rows of Z should lead to a strictly positive definite matrix $V(F)$. As a sufficient condition, if X_{it} depends on the factors and loadings through:

$$X_{it} = \tau_i + \theta_t + \sum_{k=1}^r a_k \lambda_{ik} + \sum_{k=1}^r b_k f_{kt} + \sum_{k=1}^r c_k \lambda_{ik} f_{kt} + \eta_{it}$$

where a_k, b_k, c_k are constants (can be zero) and η_{it} is i.i.d. over both i and t , then Assumption 5.1 is satisfied (see Bai 2009).

Let $U = (u_{11}, \dots, u_{1T}, u_{21}, \dots, u_{2T}, \dots, u_{N1}, \dots, u_{NT})'$, and F_0 be the $T \times r$ matrix of true factors.

Assumption 5.2. *There is a $\dim(\beta) \times \dim(\beta)$ positive definite matrix Γ such that*

$$V(F_0) \rightarrow^p \Gamma, \quad \frac{1}{\sqrt{NT}} Z' A_{F_0} U \rightarrow^d \mathcal{N}(0, \Gamma).$$

This assumption is required for the asymptotic normality of $\hat{\beta}$, because it can be shown that,

$$\sqrt{NT}(\hat{\beta} - \beta) = V(F_0)^{-1} \frac{1}{\sqrt{NT}} Z' A_{F_0} U + o_p(1).$$

Hence the asymptotic normality depends on that of $\frac{1}{\sqrt{NT}}Z'A_{F_0}U$. Assumption 5.2 is not stringent because if we write $B'_{F_0}U = (\tilde{u}_{11}, \dots, \tilde{u}_{1T}, \tilde{u}_{21}, \dots, \tilde{u}_{NT})'$, and $Z'B_{F_0} = (\tilde{Z}_{11}, \dots, \tilde{Z}_{1T}, \tilde{Z}_{21}, \dots, \tilde{Z}_{NT})$, then $\frac{1}{\sqrt{NT}}Z'A_{F_0}U = \frac{1}{\sqrt{NT}}\sum_{t=1}^T\sum_{i=1}^N\tilde{Z}_{it}\tilde{u}_{it}$ is a standardized summation. We can further write

$$\sqrt{NT}(\hat{\beta} - \beta) = \left(\frac{1}{NT} \sum_{t=1}^T \sum_{i=1}^N \tilde{Z}_{it} \tilde{Z}'_{it} \right)^{-1} \frac{1}{\sqrt{NT}} \sum_{t=1}^T \sum_{i=1}^N \tilde{Z}_{it} \tilde{u}_{it} + o_p(1).$$

Hence the second statement of Assumption 5.2 is a central limit theorem for $\frac{1}{\sqrt{NT}}\sum_{t=1}^T\sum_{i=1}^N\tilde{Z}_{it}\tilde{u}_{it}$ on both cross-sectional and time domains. In addition, in the absence of serial correlation, the conditional covariance of $\frac{1}{\sqrt{NT}}Z'A_{F_0}U$ given Z and F_0 equals $\frac{1}{NT}Z'A_{F_0}(\Sigma_u \otimes I_T)A_{F_0}Z = V(F_0)$. This implies that the asymptotic variance of $\sqrt{NT}(\hat{\beta} - \beta_0)$ is simply Γ^{-1} .

5.3 Weighted convergence for estimating the weight matrix

The issue described in Section 2 arises in establishing

$$\frac{1}{\sqrt{NT}}Z'[(\tilde{\Sigma}_u^{-1} - \Sigma_u^{-1}) \otimes I_T]U = o_p(1), \quad (5.6)$$

which is the effect of estimating the large covariance Σ_u^{-1} . In fact, the first order condition of $\hat{\beta}$ leads to

$$\sqrt{NT}(\hat{\beta} - \beta) = V(F_0)^{-1} \frac{1}{\sqrt{NT}}Z'\hat{A}U + o_p(1),$$

where \hat{A} is as A_{F_0} with Σ_u^{-1} replaced with $\tilde{\Sigma}_u^{-1}$ and F_0 replaced with \tilde{F} . Hence we need

$$\frac{1}{\sqrt{NT}}Z'(\hat{A} - A_{F_0})U = o_p(1). \quad (5.7)$$

This requires the weighted convergence (5.6). However, when $N/T \rightarrow \infty$, achieving (5.6) is technically difficult. Similar to the case described in the approximate factor model, the absolute convergence of $\|\tilde{\Sigma}_u^{-1} - \Sigma_u^{-1}\|$ is not suitable for inferences.

We consider the Gaussian case for simplicity, and the problem is still highly technically involved. Non-Gaussian case will be even more challenging, and we shall leave it for future research.

Assumption 5.3. (i) u_t is distributed as $\mathcal{N}(0, \Sigma_u)$.
(ii) $\{u_t\}_{t \geq 1}$ is independent of $\{f_t, X_t\}_{t \geq 1}$, and $\{u_t, f_t, X_t\}$ are serially independent across t .

It is possible to relax Condition (ii) to allow for serial correlations, but $\hat{\beta}$ will be asymptotically biased.

5.4 Limiting distribution

We require the same conditions on the data generating process for the factors, loadings and the sparsity of Σ_u as in Sections 2 and 4.

Proposition 5.1. Under Assumptions 3.2- 3.4, 4.1, 5.1-5.3, as $N/T \rightarrow \infty$, and $m_N = o(T^2)$, we have the weighted convergence:

$$\frac{1}{\sqrt{NT}} Z'(\hat{A} - A_{F_0})U = o_p(1).$$

We have the following limiting distribution.

Theorem 5.1. Under the assumptions of Proposition 5.1, the asymptotic limiting distribution of $\hat{\beta}$ is the same when either $W = \Sigma_u^{-1}$ or the feasible weight $W_T = \tilde{\Sigma}_u^{-1}$ is used as the weight matrix, and is given by

$$\sqrt{NT}(\hat{\beta} - \beta) \rightarrow^d \mathcal{N}(0, \Gamma^{-1}),$$

where Γ is as defined in Assumption 5.2.

The asymptotic variance Γ^{-1} is the limit of $V(F_0)^{-1}$. Note that under the same set of conditions, the regular PC method of Bai (2009) and Moon and Weidner (2010) gives an asymptotic conditional covariance (given Z, F_0) of the sandwich-formula:

$$V_2 \equiv \left(\frac{1}{NT} Z'GZ\right)^{-1} \frac{1}{NT} Z'G(\Sigma_u \otimes I_T)GZ \left(\frac{1}{NT} Z'GZ\right)^{-1},$$

where G is defined as A_{F_0} with Σ_u^{-1} replaced with I_N . It is not hard to show that $V_2 - V(F_0)^{-1}$ is semi-positive definite. So relative efficiency is gained when WPC is used. In fact, the choice $W = \tilde{\Sigma}_u^{-1}$ is also the optimal weight matrix for WPC in this case.

To estimate the asymptotic variance of $\hat{\beta}$, let \tilde{A} equal A_F with F, Λ and Σ_u^{-1} replaced with $\tilde{F}, \tilde{\Lambda}$ and $\tilde{\Sigma}_u^{-1}$. Define $\tilde{\Gamma} = \frac{1}{NT} Z' \tilde{A} Z$. The following result enables us to construct confidence intervals and conduct hypothesis tests for β under large samples.

Theorem 5.2. *Under the assumptions of Theorem 5.1,*

$$\tilde{\Gamma}^{-1} \rightarrow^p \Gamma^{-1}.$$

The methods of Section 4 also carry over to derive the limiting distributions of the estimated interactive effects $\lambda_i' f_t$. The procedure and corresponding results are very similar given the \sqrt{NT} -consistency of $\hat{\beta}$. Hence we omit repeated discussions.

5.5 Estimation with unknown number of factors

For simplicity of presentations, we have assumed the number of factors r to be known. As was shown by many authors, estimation results are often robust to over-estimating r . For instance, Moon and Weidner (2011) have shown that for inference on the regression coefficients one does not need to estimate r consistently, as long as the “working number” is not less than the true value. On the other hand, we can also start with a consistent estimator \hat{r} using a similar method of Bai and Ng (2002) and Bai (2009).

Specifically, suppose there is a known upper bound \bar{r} of the number of factors. For each $k \leq \bar{r}$, define

$$\hat{\sigma}^2(k) = \min_{\beta, \Lambda_k, f_{t,k}} \frac{1}{NT} \sum_{t=1}^T (Y_t - \Lambda_k' f_{t,k} - X_t \beta)' (Y_t - \Lambda_k' f_{t,k} - X_t \beta)$$

where each row of Λ_k is a k -dimensional loading vector, and $f_{t,k}$ is also k -dimensional. The above minimization is subject to the constraint that $\frac{1}{T} \sum_{t=1}^T f_{t,k} f_{t,k}' = I_k$ and $\Lambda_k' \Lambda_k$ is diagonal. The iterative algorithm based on principal components can calculate the above minimization fast. Under our conditions, Bai (2009) showed that r can be consistently estimated by either minimizing CP(k) or IC(k), where

$$\text{CP}(k) = \hat{\sigma}^2(k) + \hat{\sigma}^2(\bar{k}) [k(N+T) - k^2] \frac{\log(NT)}{NT},$$

and

$$\text{IC}(k) = \log \hat{\sigma}^2(k) + [k(N+T) - k^2] \frac{\log(NT)}{NT}.$$

We then can apply the estimator \hat{r} to construct the WPC estimator, and achieve the same limiting distributions. Estimation procedure and its theoretical properties can be proved to be the same as before, so details are not presented to avoid repetition.

6 Simulated Experiments

We conduct numerical experiments to compare the proposed WPC with the popular methods in the literature². The idiosyncratic error terms are generated as follows: let $\{\epsilon_{it}\}_{i \leq N, t \leq T}$ be i.i.d. $\mathcal{N}(0, 1)$ in both t, i . Let

$$u_{1t} = \epsilon_{1t}, \quad u_{2t} = \epsilon_{2t} + a_1\epsilon_{1t}, \quad u_{3t} = \epsilon_{3t} + a_2\epsilon_{2t} + b_1\epsilon_{1t},$$

$$u_{i+1,t} = \epsilon_{i+1,t} + a_i\epsilon_{it} + b_{i-1}\epsilon_{i-1,t} + c_{i-2}\epsilon_{i-2,t},$$

where $\{a_i, b_i, c_i\}_{i=1}^N$ are i.i.d. $\mathcal{N}(0, 1)$. Then Σ_u is a banded matrix, possessing both cross-sectional correlation and heteroskedasticity. Let the two factors $\{f_{1t}, f_{2t}\}$ be i.i.d. $\mathcal{N}(0, 1)$, and $\{\lambda_{i,1}, \lambda_{i,2}\}_{i \leq N}$ be uniform on $[0, 1]$. We estimate the optimal weight matrix by soft-thresholding the ‘‘correlation matrix’’ of R as suggested by Fan et al. (2013).

Design 1

Consider the pure factor model $y_{it} = \lambda_{i1}f_{1,t} + \lambda_{i2}f_{2,t} + u_{it}$, where we estimate the factor loadings $\{\lambda_{i,1}, \lambda_{i,2}\}_{i \leq N}$ and factors $\{f_{1t}, f_{2t}\}$. For each estimator, the smallest canonical correlation (the larger the better) between the estimators and parameters are calculated, as an assessment of the estimation accuracy. The simulation is replicated for one hundred times, and the average canonical correlations for several competing methods are reported in Table 2. The mean squared error of the estimated common components are also compared.

We see that the estimation becomes more accurate when we increase the dimensionality. HWPC improves the regular PC, while the EWPC gives the best estimation results.

Design 2

Adding a regression term to the model of Design 1, we consider the panel data model with interactive effect: $y_{it} = X'_{it}\beta + \lambda_{i1}f_{1,t} + \lambda_{i2}f_{2,t} + u_{it}$, where the true $\beta = (1, 3)'$. The regressors are generated to be dependent on (f_t, λ_i) :

$$X_{it,1} = 2.5\lambda_{i1}f_{1,t} - 0.2\lambda_{i2}f_{2,t} - 1 + \eta_{it,1}, \quad X_{it,2} = \lambda_{i1}f_{1,t} - 2\lambda_{i2}f_{2,t} + 1 + \eta_{it,2}$$

where $\eta_{it,1}$ and $\eta_{it,2}$ are independent i.i.d. standard normal.

² We have written a Matlab code to implement the proposed WPC for any user-specified weight matrix as well as the optimal WPC for both the factor model and panel data model with interactive effects, available upon request.

Table 2: Canonical correlations for simulation study

T	N	Loadings			Factors			$(\frac{1}{NT} \sum_{i,t} (\hat{\lambda}'_i \hat{f}_t - \lambda'_i f_t)^2)^{1/2}$		
		PC (the larger the better)	HWPC	EWPC	PC (the larger the better)	HWPC	EWPC	PC (the smaller the better)	HWPC	EWPC
50	75	0.346	0.429	0.487	0.403	0.508	0.566	0.621	0.583	0.545
50	100	0.411	0.508	0.553	0.476	0.602	0.666	0.546	0.524	0.498
50	150	0.522	0.561	0.602	0.611	0.679	0.746	0.467	0.444	0.427
100	80	0.433	0.545	0.631	0.427	0.551	0.652	0.570	0.540	0.496
100	150	0.613	0.761	0.807	0.661	0.835	0.902	0.385	0.346	0.307
100	200	0.751	0.797	0.822	0.827	0.882	0.924	0.333	0.312	0.284
150	100	0.380	0.558	0.738	0.371	0.557	0.749	0.443	0.394	0.334
150	200	0.836	0.865	0.885	0.853	0.897	0.942	0.313	0.276	0.240
150	300	0.882	0.892	0.901	0.927	0.946	0.973	0.257	0.243	0.222

The columns of loadings and factors report the canonical correlations. PC is the regular principal components method; HWPC represents the heteroskedastic WPC; EWPC uses $\hat{\Sigma}_u^{-1}$ as the weight matrix.

Both the methods PC (Bai 2009 and Moon and Weidner 2011) and the proposed WPC are carried out to estimate β for the comparison. Also compared is the mean squared error of the estimated common components. The simulation is replicated for one hundred times; results are summarized in Table 3. We see that both methods are almost unbiased, while the efficient WPC indeed has significantly smaller standard errors than the regular PC method in the panel model with interactive effects.

7 Empirical Study : Effects of Divorce Law Reforms

This section shows the advantages of our proposed WPC method in a real data application. It demonstrates the gain of incorporating the estimated

Table 3: Method comparison for the panel data with interactive effects, simulation

T	N	$\beta_1 = 1$				$\beta_2 = 3$			
		Mean		Normalized SE		Mean		Normalized SE	
		WPC	PC	WPC	PC	WPC	PC	WPC	PC
50	75	1.005	1.013	0.758	1.413	2.998	3.002	0.744	1.472
50	100	1.005	1.010	0.662	1.606	2.997	2.998	0.731	1.616
50	150	1.004	1.008	0.964	1.913	2.999	2.999	0.951	1.881
100	100	1.002	1.010	0.550	1.418	3.000	3.003	0.416	1.353
100	150	1.003	1.007	0.681	1.626	2.999	3.000	0.611	1.683
100	200	1.002	1.005	0.631	1.800	3.000	3.000	0.774	1.752
150	100	1.003	1.006	0.772	1.399	3.000	2.999	0.714	1.458
150	150	1.001	1.005	0.359	1.318	3.000	3.001	0.408	1.379
150	200	1.001	1.003	0.547	1.566	3.000	3.000	0.602	1.762

WPC (with weight $\tilde{\Sigma}_u^{-1}$) and PC (existing method) comparison. “Mean” is the average of the estimators; “Normalized SE” is the standard error of the estimators multiplied by \sqrt{NT} .

Σ_u in the panel data estimation and the efficiency gains compared to the traditional PC.

7.1 Real Data Application

An important question in sociology is the cause of the sharp increase in the U.S. divorce rate in the 1960s and 1970s. The association between divorce rates and divorce law reforms has been considered a potential key, and during 1970s, about three quarters of states in the U.S. liberalized their divorce system, so-called “no-fault revolution”. There is plenty empirical research regarding the effects of divorce law reforms on the divorce rates (e.g., Peters 1986, Allen 1992), and statistical significance of these effects has been found (e.g., Friedberg 1998). In other words, states’ law reforms are found

to have significantly contributed to the increase in state-level divorce rates within the first eight years following reforms.

On the other hand, there is a puzzle about longer effects. Empirical evidence also illustrates the subsequent decrease of the divorce rates starting from (around) 1975, which is between nine and fourteen years after the law reforms in most states. So whether law reforms continue to contribute to the rate decrease has been an interesting question. Wolfers (2006) studied a treatment effect panel data model, and identified negative effects for the subsequent years. This suggests that, the increase in divorce following reform and the subsequent decrease may be two sides of the same treatment: after earlier dissolution of bad matches after law reforms, marital relations were gradually affected and changed. However, it has been argued that Wolfers (2006)'s approach may not capture the complex unobserved heterogeneity. The heterogeneity may exist through an interactive effect, where unobserved common factors may change over time.

Kim and Oka (2013) pioneered using interactive effect model for the study:

$$y_{it} = \sum_{k=1}^K X_{it,k} \beta_k + \lambda'_i f_t + \mu_i + \alpha_t + f(\delta_i, t) + u_{it}, \quad (7.1)$$

where y_{it} is the divorce rate for state i in year t ; $X_{it,k}$ is a binary regressor, representing the treatment effect $2k$ years after the reform. Specifically, we observe the law reform year T_i for each state. Then $X_{it,k} = 1$ if $2k - 1 \leq t - T_i \leq 2k$, and zero otherwise. In addition to the interactive effect $\lambda'_i f_t$ as being discussed, the model also contains unobserved state and time effects (μ_i, α_t) and time trend $f(\delta_i, t)$. For instance, the linear trend defines $f(\delta_i, t) = \delta_i t$ with unknown coefficient δ_i . Using the regular PC method, Kim and Oka (2013) concluded insignificant $(\beta_5, \dots, \beta_8)$, that is, the divorce rates after eight years and beyond are not affected by the reforms. However, We argue that using the regular PC method to estimate the model may lose efficiency because it ignores the off-diagonal entries. As a result, this can result in wide confidence intervals and possibly conservative conclusions.

We re-estimate Kim and Oka (2013)'s model using the new WPC, and compare with the regular PC. As a first step, we rewrite the model to fit in the form being considered in this paper. Introduce the conventional notation:

$$\dot{y}_{it} = y_{it} - \frac{1}{T} \sum_{t=1}^T y_{it} - \frac{1}{N} \sum_{i=1}^N y_{it} + \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T y_{it}.$$

Let $\dot{X}_{it,k}, \dot{u}_{it}$ be defined similarly. If the time trend $f(\delta_i, t)$ is not present,³ under the conventional normalizations $\sum_{i=1}^N \lambda_i = \sum_{t=1}^T f_t = 0$, $\sum_{i=1}^N \mu_i = \sum_{t=1}^T \alpha_t$, we have $\dot{y}_{it} = \dot{X}'_{it}\beta + \lambda'_i f_t + \dot{u}_{it}$.

The same data as in Wolfers (2006) and Kim and Oka (2013) are used, which contain the divorce rates, state-level reform years and binary regressors from 1956 to 1988 ($T = 33$) over 48 states. We fit the models both with and without linear time trend, and apply regular PC and our proposed WPC in each model to estimate β with confidence intervals. The number of factors is selected in a data-driven way as in Bai (2009). His IC and CP both suggested ten factors.⁴ Moreover, for the WPC, the threshold value in the estimated covariance is obtained using the suggested cross-validation procedure in Fan et al. (2013). The estimated $(\beta_1, \dots, \beta_8)$ and their confidence intervals are summarized in Table 4.

Both models produce similar estimates. Interestingly, WPC confirms that the law reforms significantly contribute to the subsequent decrease of the divorce rates, more specifically, 9-14 years after the reform in the model with linear time trends, and 11-14 years after in the model without linear time trends. In contrast, the regular PC reaches a more conservative conclusion as it does not capture these significant negative effects. Moreover, both methods show that the effect on the increase of divorce rates for the first 6 years are significant, which is consistent with previous findings in this literature.

We also report the relative efficiency using WPC, relative to the regular PC. The reported numbers are $\text{var}(\text{WPC})/\text{var}(\text{PC})$, where $\text{var}(A)$ calculates the estimated variance of the estimator using method A. It is clear from the table that WPC achieves almost 50% of efficiency gain relative to the regular PC method.

³When the time trend is present, we can do a simple projection to eliminate the time trend, and still estimate the untransformed β from the familiar interactive effect model. For instance, suppose $f(\delta_i, t) = \delta_i t$. Let $M = (1, 2, \dots, T)'$ and $P_M = I_T - M(M'M)^{-1}M'$. We can define $\tilde{Y}_i = P_M(y_{i1}, \dots, y_{iT})'$, and $\tilde{X}_i = P_M(X_{i1}, \dots, X_{iT})'$, and define \tilde{y}_{it} and \tilde{X}_{it} accordingly from \tilde{y}_{it} and \tilde{X}_{it} .

⁴This is the same as in Kim and Oka (2013). We also tried a few larger values for r , and the estimates are similar, consistent with previous findings that the estimation is robust to over-estimating r .

Table 4: Method comparison in effects of divorce law reform: real data

	Interactive effect				Relative efficiency
	WPC		PC		
	estimate	confidence interval	estimate	confidence interval	
First 2 years	0.014	[0.007, 0.021]*	0.018	[0.0091, 0.028]*	0.59
3-4 years	0.034	[0.027, 0.041]*	0.042	[0.032, 0.053]*	0.59
5-6 years	0.025	[0.017, 0.032]*	0.032	[0.022, 0.042]*	0.58
7-8 years	0.015	[0.007, 0.023]*	0.030	[0.019, 0.04]*	0.56
9-10 years	-0.006	[-0.014, 0.001]	0.008	[-0.002, 0.018]	0.56
11-12 years	-0.008	[-0.015, -0.001]*	0.010	[-0.001, 0.02]	0.53
13-14 years	-0.009	[-0.017, -0.001]*	0.005	[-0.005, 0.016]	0.53
15 years+	0.009	[0.001, 0.017]*	0.031	[0.020, 0.042]*	0.55
	Interactive effect+linear trend				Relative efficiency
	WPC		PC		
	estimate	confidence interval	estimate	confidence interval	
First 2 years	0.014	[0.006, 0.021]*	0.016	[0.006, 0.026]*	0.55
3-4 years	0.032	[0.024, 0.039]*	0.037	[0.026, 0.047]*	0.54
5-6 years	0.018	[0.010, 0.026]*	0.024	[0.012, 0.035]*	0.54
7-8 years	0.006	[-0.002, 0.014]	0.017	[0.005, 0.028]*	0.52
9-10 years	-0.017	[-0.025, -0.008]*	-0.007	[-0.019, 0.005]	0.52
11-12 years	-0.019	[-0.028, -0.010]*	-0.006	[-0.018, 0.006]	0.51
13-14 years	-0.021	[-0.030, -0.012]*	-0.012	[-0.025, 0.001]	0.50
15 years+	-0.003	[-0.012, 0.006]	0.014	[0.000, 0.028]*	0.46

*95% confidence intervals are reported; intervals with * are significant. Relative efficiency is referred to WPC relative to PC, as estimated $\text{var}(WPC)/\text{var}(PC)$.*

7.2 Simulated data

Let us further demonstrate the relative efficiency WPC gains by incorporating the estimated Σ_u^{-1} through simulated data. The true parameters are estimated from the real data as described above. Specifically, we use the first column from Table 4 (no linear trend) as the true β , and the corresponding

estimated Λ as the true loading matrix. We fix $N = 48$ as before. To pertain the actual cross-sectional dependence, in the simulation, the true error terms, factors, and regressors are generated as simple random samples (with replacement) of size T from the estimated residuals, factors and regressors from the real data.

Simulations are conducted with one hundred replications. The averages and the standard deviations for each estimated component are reported in Table 5, representing the bias and standard error. Also reported is the relative efficiency, defined as $\text{var}(\text{WPC})/\text{var}(\text{PC})$. It is clearly shown in the table that the standard errors of WPC are uniformly smaller than those of PC. In addition, most of the time WPC also reduces the finite sample bias. The relative efficiency varies from 39% to 52%, which illustrates 48%-61% efficiency gain. Overall, after incorporating the error covariance, the performance of the estimator is significantly improved.

8 Conclusion

The literature on estimating high-dimensional sparse covariance matrices has targeted on the covariance and inverse covariance directly, and the theoretical results are mostly in an absolute convergence form. We see that the absolute convergence, even though achieving the minimax optimal rate, is often not suitable for statistical inference. Thus using an estimated high-dimensional covariance matrix as the optimal weight matrix is highly-nontrivial. We study a new notion of “weighted convergence” to show that the effect of estimating a high-dimensional covariance matrix is indeed asymptotically negligible.

This paper studies in detail two models that are of increasing importance in applied statistics: approximate factor model and panel data with unobservable interactive effects. We propose a method of weighted principal components, which uses a high-dimensional weight matrix. The efficient weight uses the inverse error covariance matrix. The EWPC considers both heteroskedasticity and cross-sectional dependence. It is shown that EWPC uses the optimal weight matrix over the class of WPC estimators thus it is the most efficient.

The EWPC is applied to the year-state divorce rate data. The new method captures the significant (negative) effects from nine to twelve years after the law was reformed, consistent with the previous empirical findings

Table 5: Method comparison in effects of divorce law reform: simulated data

	Bias		Normalized SE		Relative Efficiency
	WPC	PC	WPC	PC	
	$T = 50$				
First 2 years	-0.008	-0.013	1.077	1.714	0.393
3-4 years	-0.023	-0.033	1.911	2.694	0.494
5-6 years	-0.040	-0.058	2.743	3.821	0.525
7-8 years	-0.054	-0.080	3.429	4.899	0.501
9-10 years	-0.068	-0.103	4.017	5.633	0.501
11-12 years	-0.073	-0.107	4.262	6.221	0.475
13-14 years	-0.081	-0.124	4.703	6.907	0.462
15 years+	-0.090	-0.133	5.094	7.691	0.439
	$T = 70$				
First 2 years	-0.002	-0.000	0.927	1.449	0.408
3-4 years	-0.008	-0.008	1.623	2.434	0.438
5-6 years	-0.021	-0.028	2.434	3.420	0.505
7-8 years	-0.030	-0.039	3.246	4.579	0.507
9-10 years	-0.043	-0.060	4.115	5.738	0.513
11-12 years	-0.048	-0.061	4.579	6.492	0.501
13-14 years	-0.055	-0.076	5.101	7.245	0.495
15 years+	-0.062	-0.079	5.564	8.173	0.465

“Normalized SE” is the standard error of the estimator multiplied by \sqrt{NT} .
 The relative efficiency is calculated as the square of the ratio of the third
 and fourth columns, estimating $\text{var}(WPC)/\text{var}(PC)$

in the social science literature. The estimator is more accurate and produces tighter confidence intervals.

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