Additional Online Appendix for “Jackknife Estimation of a Cluster-Sample IV Regression Model with Many Weak Instruments"*

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Abstract

This online appendix is comprised of three sections. Section 1 contains proofs for the key lemmas used to show the main theorems of the paper “Jackknife Estimation of a Cluster-Sample IV Regression with Many Weak Instruments." More specifically, in section 1, we provide proofs for Lemmas S2-1 to S2-18 which are stated without proof in the Supplemental Appendix of the aforementioned paper. In section 2, we provide a proof of Lemma 1 of the main paper. Finally, in section 3, we prove some additional lemmas which provide further supporting arguments for the proofs given in section 1.

Section 1: Proof of Lemmas S2-1 to S2-18

In this section of the online appendix, we provide proof of Lemmas S2-1 to S2-18, which have been stated without proof in the Appendix S2 of the Supplemental Appendix to our paper.

Lemma S2-1: Let $A = P^{\perp} - M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}$. Then, under Assumptions 2-7, the following statements hold as $K_{2,n}$, $n \to \infty$.

(a) $\sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A_{(i,t),(j,s)}^2 = O_{a.s.}\left(K_{2,n}\right)$.

(b) $\sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A_{(i,t),(j,s)}^4 = O_{a.s.}\left(K_{2,n}^3/n^2\right)$.

(c) $\sum_{(j,s)=1}^{m_n} \sum_{(i,t),(k,v)=1,(i,t)\neq(j,s),(k,v)\neq(j,s)}^{m_n} A_{(i,t),(j,s)}^2 A_{(j,s),(k,v)}^2 = O_{a.s.}\left(K_{2,n}^2/n\right)$.

(d) $\max_{1\leq(i,t)\leq m_n}\left(\sum_{(j,s)=1}^{m_n} A_{(i,t),(j,s)}^2\right) = O_{a.s.}\left(K_{2,n}/n\right)$.

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Proof of Lemma S2-1:

To show part (a), note first that, by Lemma OA-1 given in section 3 of this online appendix, we have

$$tr\left\{D^2_{\widehat{\vartheta}}\right\} = O_{a.s.}\left(\frac{K^2_{2,n}}{n}\right),$$

where $D_{\widehat{\vartheta}} = diag\left(\widehat{\vartheta}_1, ..., \widehat{\vartheta}_{m_n}\right)$. Now, by straightforward calculations and by making use of the inequality $\left|\sum_{i=1}^{G} a_i\right|^r \leq G^{r-1}\sum_{i=1}^{G}|a_i|^r$, we get

$$\begin{aligned}
\sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A^2_{(i,t),(j,s)} &= \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} \left(P^{\perp}_{(i,t),(j,s)} - e'_{(i,t)} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} e_{(j,s)}\right)^2 \\
&\leq 2 \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} \left(P^{\perp}_{(i,t),(j,s)}\right)^2 + 2 \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} \left(e'_{(i,t)} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} e_{(j,s)}\right)^2 \\
&\leq 2\left[K_{2,n} + \sum_{(i,t)=1}^{m_n} e'_{(i,t)} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} e_{(i,t)}\right] \\
&\leq 2\left[K_{2,n} + tr\left\{M^{(Z,Q)} D^2_{\widehat{\vartheta}} M^{(Z,Q)}\right\}\right] = 2\left[K_{2,n} + tr\left\{D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}}\right\}\right] \\
&\leq 2\left[K_{2,n} + tr\left\{D^2_{\widehat{\vartheta}}\right\}\right] = O_{a.s.}\left(K_{2,n}\right) + O_{a.s.}\left(\frac{K^2_{2,n}}{n}\right) = O_{a.s.}\left(K_{2,n}\right).
\end{aligned}$$

Next, to show part (b), note that, by applying the inequality $\left|\sum_{i=1}^{G} a_i\right|^r \leq G^{r-1}\sum_{i=1}^{G}|a_i|^r$

and the CS inequality and using the fact that $\lambda_{\max}\left(M^{(Z,Q)}\right)=1$, we have

$$
\begin{aligned}
& \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^4 \\
= & \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} \left(P_{(i,t),(j,s)}^{\perp} - e_{(i,t)}' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} e_{(j,s)}\right)^4 \\
\leq & \; 2^3 \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} \left\{ \left(P_{(i,t),(j,s)}^{\perp}\right)^4 + \left(e_{(i,t)}' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} e_{(j,s)}\right)^4 \right\} \\
\leq & \; 8 \sum_{(i,t),(j,s)=1}^{m_n} \left(P_{(i,t),(j,s)}^{\perp}\right)^4 \\
& + 8 \sum_{(i,t),(j,s)=1}^{m_n} \left\{ \left(e_{(i,t)}' M^{(Z,Q)} D_{\widehat{\vartheta}}^2 M^{(Z,Q)} e_{(i,t)}\right) \left(e_{(j,s)}' M^{(Z,Q)} e_{(j,s)}\right) \right. \\
& \qquad \left. \times e_{(i,t)}' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} e_{(j,s)} e_{(j,s)}' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} e_{(i,t)} \right\} \\
\leq & \; 8 \sum_{(i,t),(j,s)=1}^{m_n} \left(P_{(i,t),(j,s)}^{\perp}\right)^4 + 8 \max_{1\leq(i,t)\leq m_n} \left|\widehat{\vartheta}_{(i,t)}\right|^2 \sum_{(i,t)=1}^{m_n} e_{(i,t)}' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} e_{(i,t)} \\
\leq & \; 8 \left(\max_{1\leq(i,t)\leq m_n} P_{(i,t),(i,t)}^{\perp}\right)^2 \sum_{(i,t)=1}^{m_n} \sum_{(j,s)=1}^{m_n} \left(P_{(i,t),(j,s)}^{\perp}\right)^2 \\
& +8 \left(\max_{1\leq(i,t)\leq m_n} \left|\widehat{\vartheta}_{(i,t)}\right|^2\right) tr \left\{ D_{\widehat{\vartheta}} M^{(Z,Q)} \sum_{(i,t)=1}^{m_n} e_{(i,t)} e_{(i,t)}' M^{(Z,Q)} D_{\widehat{\vartheta}} \right\} \\
\leq & \; 8 \left(\max_{1\leq(i,t)\leq m_n} P_{(i,t),(i,t)}^{\perp}\right)^2 K_{2,n} + 8 \left(\max_{1\leq(i,t)\leq m_n} \left|\widehat{\vartheta}_{(i,t)}\right|^2\right) tr\left\{ D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} \right\} \\
\leq & \; 8 \left(\max_{1\leq(i,t)\leq m_n} P_{(i,t),(i,t)}^{\perp}\right)^2 K_{2,n} + 8 \left(\max_{1\leq(i,t)\leq m_n} \left|\widehat{\vartheta}_{(i,t)}\right|^2\right) tr\left\{ D_{\widehat{\vartheta}}^2 \right\} \\
= & \; O_{a.s.}\left(\frac{K_{2,n}^3}{n^2}\right) + O_{a.s.}\left(\frac{K_{2,n}^4}{n^3}\right) \quad \text{(by parts (a) and (b) of Lemma OA-1)} \\
= & \; O_{a.s.}\left(\frac{K_{2,n}^3}{n^2}\right).
\end{aligned}
$$

To show part (c), note that

$$
\begin{aligned}
& \sum_{(j,s)=1}^{m_n} \sum_{\substack{(i,t),(k,\upsilon)=1 \\ (i,t)\neq(j,s),(k,\upsilon)\neq(j,s)}}^{m_n} A^2_{(i,t),(j,s)} A^2_{(j,s),(k,\upsilon)} \\
= & \sum_{(j,s)=1}^{m_n} \sum_{\substack{(i,t),(k,\upsilon)=1 \\ (i,t)\neq(j,s) \\ (k,\upsilon)\neq(j,s)}}^{m_n} \left(P^{\perp}_{(i,t),(j,s)} - e'_{(i,t)} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} e_{(j,s)} \right)^2 \left(P^{\perp}_{(j,s),(k,\upsilon)} - e'_{(j,s)} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} e_{(k,\upsilon)} \right)^2 \\
\leq & \sum_{(i,t),(j,s)=1}^{m_n} \left(P^{\perp}_{(i,t),(j,s)} - e'_{(i,t)} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} e_{(j,s)} \right)^2 \\
& \times \sum_{(k,\upsilon)=1}^{m_n} \left(P^{\perp}_{(j,s),(k,\upsilon)} - e'_{(j,s)} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} e_{(k,\upsilon)} \right)^2 \\
\leq & \ 4 \sum_{(i,t),(j,s)=1}^{m_n} \left[\left(P^{\perp}_{(i,t),(j,s)} \right)^2 + \left(e'_{(i,t)} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} e_{(j,s)} \right)^2 \right] \\
& \times \sum_{(k,\upsilon)=1}^{m_n} \left[\left(P^{\perp}_{(j,s),(k,\upsilon)} \right)^2 + \left(e'_{(j,s)} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} e_{(k,\upsilon)} \right)^2 \right] \\
& \left(\text{by the inequality } \left| \sum\nolimits_{i=1}^{G} a_i \right|^r \leq G^{r-1} \sum\nolimits_{i=1}^{G} |a_i|^r \right) \\
= & \ 4 \sum_{(i,t),(j,s)=1}^{m_n} \left\{ \left[\left(P^{\perp}_{(i,t),(j,s)} \right)^2 + \left(e'_{(i,t)} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} e_{(j,s)} \right)^2 \right] \right. \\
& \qquad \left. \times \left[P^{\perp}_{(j,s),(j,s)} + e'_{(j,s)} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} e_{(j,s)} \right] \right\} \\
\leq & \ 4 \left(\max_{1\leq(j,s)\leq m_n} P^{\perp}_{(j,s),(j,s)} \right) \sum_{(i,t),(j,s)=1}^{m_n} \left[\left(P^{\perp}_{(i,t),(j,s)} \right)^2 + \left(e'_{(i,t)} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} e_{(j,s)} \right)^2 \right] \\
& +4 \sum_{(i,t),(j,s)=1}^{m_n} \left[\left(P^{\perp}_{(i,t),(j,s)} \right)^2 + \left(e'_{(i,t)} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} e_{(j,s)} \right)^2 \right] e'_{(j,s)} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} e_{(j,s)} \\
\leq & \ 4 \left(\max_{1\leq(j,s)\leq m_n} P^{\perp}_{(j,s),(j,s)} \right) \sum_{(i,t)=1}^{m_n} \left[P^{\perp}_{(i,t),(i,t)} + e'_{(i,t)} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} e_{(i,t)} \right] \\
& +4 \left(\max_{1\leq(j,s)\leq m_n} \left| \widehat{\vartheta}_{(j,s)} \right|^2 \right) \sum_{(i,t)=1}^{m_n} \left[P^{\perp}_{(i,t),(i,t)} + e'_{(i,t)} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} e_{(i,t)} \right] \\
\leq & \ 4 \left[\left(\max_{(j,s)\in\Lambda_2} P^{\perp}_{(j,s),(j,s)} \right) + \left(\max_{1\leq(j,s)\leq m_n} \left| \widehat{\vartheta}_{(j,s)} \right|^2 \right) \right] \left[K_{2,n} + tr \left\{ D^2_{\widehat{\vartheta}} \right\} \right] \\
= & \left[O_{a.s.} \left(\frac{K_{2,n}}{n} \right) + O_{a.s.} \left(\frac{K^2_{2,n}}{n^2} \right) \right] \left[O_{a.s.} \left(K_{2,n} \right) + O_{a.s.} \left(\frac{K^2_{2,n}}{n} \right) \right] \text{ (using Lemma OA-1)} \\
= & \ O_{a.s.} \left(\frac{K^2_{2,n}}{n} \right).
\end{aligned}
$$

Finally, to show part (d), note that

$$
\begin{aligned}
\sum_{(j,s)=1}^{m_n} A_{(i,t),(j,s)}^2 &= \sum_{(j,s)=1}^{m_n} \left(P_{(i,t),(j,s)}^{\perp} - e_{(i,t)}' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} e_{(j,s)} \right)^2 \\
&= \sum_{(j,s)=1}^{m_n} \left(P_{(i,t),(j,s)}^{\perp} \right)^2 - 2 e_{(i,t)}' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} \sum_{(j,s)=1}^{m_n} e_{(j,s)} e_{(j,s)}' P^{\perp} e_{(i,t)} \\
&\quad + e_{(i,t)}' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} \sum_{(j,s)=1}^{m_n} e_{(j,s)} e_{(j,s)}' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} e_{(i,t)} \\
&= P_{(i,t),(i,t)}^{\perp} + e_{(i,t)}' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} e_{(i,t)} \\
&\leq P_{(i,t),(i,t)}^{\perp} + e_{(i,t)}' M^{(Z,Q)} D_{\widehat{\vartheta}}^2 M^{(Z,Q)} e_{(i,t)} \\
&\leq P_{(i,t),(i,t)}^{\perp} + e_{(i,t)}' M^{(Z,Q)} e_{(i,t)} \left(\max_{1 \leq (i,t) \leq m_n} \left| \widehat{\vartheta}_{(i,t)} \right|^2 \right) \\
&\leq P_{(i,t),(i,t)}^{\perp} + \left(\max_{1 \leq (i,t) \leq m_n} \left| \widehat{\vartheta}_{(i,t)} \right|^2 \right) \text{ (since } e_{(i,t)}' M^{(Z,Q)} e_{(i,t)} \leq 1)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\max_{1 \leq (i,t) \leq m_n} \left(\sum_{(j,s)=1}^{m_n} A_{(i,t),(j,s)}^2 \right) &\leq \left(\max_{1 \leq (i,t) \leq m_n} P_{(i,t),(i,t)}^{\perp} \right) + \left(\max_{1 \leq (i,t) \leq m_n} \left| \widehat{\vartheta}_{(i,t)} \right|^2 \right) \\
&= O_{a.s.} \left(\frac{K_{2,n}}{n} \right) + O_{a.s.} \left(\frac{K_{2,n}^2}{n^2} \right) \\
&= O_{a.s.} \left(\frac{K_{2,n}}{n} \right). \ \square
\end{aligned}
$$

Lemma S2-2: Suppose that Assumptions 1-7 are satisfied. Then, the following statements are true: (a) $D_\mu^{-1} X' M^{(Z_1,Q)} X D_\mu^{-1} = O_p \left(n \left(\mu_n^{\min} \right)^{-2} \right)$; (b) $D_\mu^{-1} X' A X D_\mu^{-1} = H_n + o_p(1)$, where $H_n = \Gamma' M^{(Z_1,Q)} \Gamma / n = O_p(1)$.

Proof of Lemma S2-2:

To show part (a), note first that $D_\mu^{-1} X' M^{(Z_1,Q)} X D_\mu^{-1}$ $\leq 3 \left[\Gamma' M^{(Z_1,Q)} \Gamma / n + D_\mu^{-1} D_\kappa F' M^{(Z_1,Q)} F D_\kappa D_\mu^{-1} / n + D_\mu^{-1} U' M^{(Z_1,Q)} U D_\mu^{-1} \right]$, where $F = \left(f\left(W_{1,(1,1)}\right), .., f\left(W_{1,(1,T_1)}\right), ..., f\left(W_{1,(n,1)}\right), .., f\left(W_{1,(n,T_n)}\right) \right)'$ and where we take A $\leq B$ for two square matrices A and B to mean that $A - B$ is negative semi-definite, or, alternatively, $B - A$ is positive semidefinite. Now, for $a, b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$, we can apply the CS inequality and Assumption 3(iii) to obtain $\left| a' \Gamma' M^{(Z_1,Q)} \Gamma b / n \right| \leq \sqrt{a' \Gamma' M^{(Z_1,Q)} \Gamma a / n} \sqrt{b' \Gamma' M^{(Z_1,Q)} \Gamma b / n} \leq \sqrt{a' \Gamma' \Gamma a / n} \sqrt{b' \Gamma' \Gamma b / n} = O_{a.s.}(1)$. Since the above argument holds for all a, $b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$, we further deduce that $\Gamma' M^{(Z_1,Q)} \Gamma / n = O_p(1)$. Next, note that, for $a, b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$, we can apply the CS inequality along with Assumptions 3(ii), 4(i), 5(i), and 7(ii) to obtain

$$\begin{aligned}
\left|\frac{a'D_\mu^{-1}D_\kappa F'M^{(Z_1,Q)}FD_\kappa D_\mu^{-1}b}{n}\right| &\leq \sqrt{\frac{a'D_\mu^{-1}D_\kappa\left(F-Z_1\Theta^{K_{1,n}}\right)'M^{(Z_1,Q)}\left(F-Z_1\Theta^{K_{1,n}}\right)D_\kappa D_\mu^{-1}a}{n}} \\
&\quad\times\sqrt{\frac{b'D_\mu^{-1}D_\kappa\left(F-Z_1\Theta^{K_{1,n}}\right)'M^{(Z_1,Q)}\left(F-Z_1\Theta^{K_{1,n}}\right)D_\kappa D_\mu^{-1}b}{n}} \\
&\leq \frac{m_n d}{n}\left\|f(\cdot)-\Theta^{K_{1,n}\prime}Z_1(\cdot)\right\|_{\infty,d}^2\frac{\left(\kappa_n^{\max}\right)^2}{\left(\mu_n^{\min}\right)^2}=O_{a.s.}\left(\frac{\left(\kappa_n^{\max}\right)^2}{\left(\mu_n^{\min}\right)^2K_{1,n}^{2\varrho_f}}\right)
\end{aligned}$$

Since the above argument holds for all a, $b\in\mathbb{R}^d$ such that $\|a\|_2=\|b\|_2=1$, we further deduce that $D_\mu^{-1}D_\kappa F'M^{(Z_1,Q)}FD_\kappa D_\mu^{-1}/n=O_{a.s.}\left(\left(\kappa_n^{\max}\right)^2\left(\mu_n^{\min}\right)^{-2}K_{1,n}^{-2\varrho_f}\right)$. Finally, note that, by the CS inequality, $\left|a'D_\mu^{-1}U'M^{(Z_1,Q)}UD_\mu^{-1}b\right|\leq\sqrt{a'D_\mu^{-1}U'M^{(Z_1,Q)}UD_\mu^{-1}a}\sqrt{b'D_\mu^{-1}U'M^{(Z_1,Q)}UD_\mu^{-1}b}$. Now, by Assumptions 2(i), 3(ii), and 6(ii),

$$\begin{aligned}
E\left[a'D_\mu^{-1}U'M^{(Z_1,Q)}UD_\mu^{-1}a|\mathcal{F}_n^W\right] &\leq a'D_\mu^{-1}E\left[U'U|\mathcal{F}_n^W\right]D_\mu^{-1}a \\
&\leq \frac{\overline{T}\left(\max_{1\leq(i,t)\leq m_n}E\left[\left\|U_{(i,t)}\right\|_2^2|\mathcal{F}_n^W\right]\right)n}{\left(\mu_n^{\min}\right)^2}=O_{a.s.}\left(\frac{n}{\left(\mu_n^{\min}\right)^2}\right),
\end{aligned}$$

where $\overline{T}=\max_{1\leq(i,t)\leq m_n}T_i$. Hence, by Theorem 16.1 of Billingsley (1995), there exists a constant $\overline{C}<\infty$ such that for all n sufficiently large $E\left[\left(\left(\mu_n^{\min}\right)^2/n\right)a'D_\mu^{-1}U'M^{(Z_1,Q)}UD_\mu^{-1}a\right]=E_{W_n}\left(\left(\left(\mu_n^{\min}\right)^2/n\right)E\left[\frac{a'U'M^{(Z_1,Q)}Ua}{n}|\mathcal{F}_n^W\right]\right)\leq\overline{C}$. It follows from the Markov's inequality that $a'D_\mu^{-1}U'M^{(Z_1,Q)}UD_\mu^{-1}a=O_p\left(n\left(\mu_n^{\min}\right)^{-2}\right)$. In the same way, we also have $b'D_\mu^{-1}U'M^{(Z_1,Q)}UD_\mu^{-1}b=O_p\left(n/\left(\mu_n^{\min}\right)^2\right)$, so that $\left|a'D_\mu^{-1}U'M^{(Z_1,Q)}UD_\mu^{-1}b\right|\leq\sqrt{a'D_\mu^{-1}U'M^{(Z_1,Q)}UD_\mu^{-1}a}\sqrt{b'D_\mu^{-1}U'M^{(Z_1,Q)}UD_\mu^{-1}b}=O_p\left(n\left(\mu_n^{\min}\right)^{-2}\right)$. Since this result holds for all a, $b\in\mathbb{R}^d$ such that $\|a\|_2=\|b\|_2=1$, we further deduce that $D_\mu^{-1}U'M^{(Z_1,Q)}UD_\mu^{-1}=O_p\left(n\left(\mu_n^{\min}\right)^{-2}\right)$. Putting these results together, it follows that $D_\mu^{-1}X'M^{(Z_1,Q)}XD_\mu^{-1}=O_p(1)+O_p\left(\left(\kappa_n^{\max}\right)^2\left(\mu_n^{\min}\right)^{-2}K_{1,n}^{-2\varrho_f}\right)+O_p\left(n\left(\mu_n^{\min}\right)^{-2}\right)=O_p\left(n\left(\mu_n^{\min}\right)^{-2}\right)$, as required to show part (a).

To show part (b), note that, by applying parts (a)-(f) of Lemma OA-13, we obtain

$$
\begin{aligned}
D_{\mu}^{-1}X'AXD_{\mu}^{-1} &= D_{\mu}^{-1}\left(\Upsilon+\Phi+Q\Xi+U\right)'A\left(\Upsilon+\Phi+Q\Xi+U\right)D_{\mu}^{-1} \\
&= D_{\mu}^{-1}\left(\Upsilon+\Phi+U\right)'A\left(\Upsilon+\Phi+U\right)D_{\mu}^{-1} \text{ (since } AQ=0) \\
&= D_{\mu}^{-1}\Upsilon'A\Upsilon D_{\mu}^{-1}+D_{\mu}^{-1}\Phi'A\Phi D_{\mu}^{-1}+D_{\mu}^{-1}U'AUD_{\mu}^{-1} \\
&\quad +D_{\mu}^{-1}\Phi'A\Upsilon D_{\mu}^{-1}+D_{\mu}^{-1}\Upsilon'A\Phi D_{\mu}^{-1}+D_{\mu}^{-1}\Upsilon'AUD_{\mu}^{-1} \\
&\quad +D_{\mu}^{-1}U'A\Upsilon D_{\mu}^{-1}+D_{\mu}^{-1}\Phi'AUD_{\mu}^{-1}+D_{\mu}^{-1}U'A\Phi D_{\mu}^{-1} \\
&= \frac{D_{\mu}^{-1}D_{\mu}\Gamma'A\Gamma D_{\mu}D_{\mu}^{-1}}{n}+\frac{D_{\mu}^{-1}D_{\kappa}F'AFD_{\kappa}D_{\mu}^{-1}}{n} \\
&\quad +D_{\mu}^{-1}U'AUD_{\mu}^{-1}+\frac{D_{\mu}^{-1}D_{\kappa}F'A\Gamma D_{\mu}D_{\mu}^{-1}}{n} \\
&\quad +\frac{D_{\mu}^{-1}D_{\mu}\Gamma'AFD_{\kappa}D_{\mu}^{-1}}{n}+\frac{D_{\mu}^{-1}D_{\mu}\Gamma'AUD_{\mu}^{-1}}{\sqrt{n}} \\
&\quad +\frac{D_{\mu}^{-1}U'A\Gamma D_{\mu}D_{\mu}^{-1}}{\sqrt{n}}+\frac{D_{\mu}^{-1}D_{\kappa}F'AUD_{\mu}^{-1}}{\sqrt{n}} \\
&\quad +\frac{D_{\mu}^{-1}U'AFD_{\kappa}D_{\mu}^{-1}}{\sqrt{n}} \\
&= \frac{\Gamma'A\Gamma}{n}+\frac{D_{\mu}^{-1}D_{\kappa}F'AFD_{\kappa}D_{\mu}^{-1}}{n}+D_{\mu}^{-1}U'AUD_{\mu}^{-1} \\
&\quad +\frac{D_{\mu}^{-1}D_{\kappa}F'A\Gamma}{n}+\frac{\Gamma'AFD_{\kappa}D_{\mu}^{-1}}{n}+\frac{\Gamma'AUD_{\mu}^{-1}}{\sqrt{n}} \\
&\quad +\frac{D_{\mu}^{-1}U'A\Gamma}{\sqrt{n}}+\frac{D_{\mu}^{-1}D_{\kappa}F'AUD_{\mu}^{-1}}{\sqrt{n}}+\frac{D_{\mu}^{-1}U'AFD_{\kappa}D_{\mu}^{-1}}{\sqrt{n}} \\
&= \frac{\Gamma'M^{(Z_1,Q)}\Gamma}{n}+o_p\left(1\right).
\end{aligned}
$$

Moreover, $\Gamma'M^{(Z_1,Q)}\Gamma/n = O_p\left(1\right)$, as shown in the proof of part (a) above. This shows part (b). □

Lemma S2-3: Let $\underline{U} = U - \varepsilon\rho'$ and $\underline{U}_{(i,t)} = U_{(i,t)} - \rho\varepsilon_{(i,t)}$ and let $VC\left(X|\mathcal{F}_n^W\right)$ denote the conditional covariance matrix of the random vector X given $\mathcal{F}_n^W$. Under Assumptions 1-2, 5-6, and 8; there exists positive constants $0 < \underline{C} \leq \overline{C} < \infty$ such that the following statements are true.

(a) $\lambda_{\max}\left[VC\left(\Gamma'M^{(Z_1,Q)}\varepsilon/\sqrt{n}|\mathcal{F}_n^W\right)\right] \leq \overline{C}$ *a.s.* and $\lambda_{\min}\left[VC\left(\Gamma'M^{(Z_1,Q)}\varepsilon/\sqrt{n}|\mathcal{F}_n^W\right)\right] \geq \underline{C}$ *a.s.* for all n sufficiently large.

(b) $VC\left(\underline{U}'A\varepsilon/\sqrt{K_{2,n}}|\mathcal{F}_n^W\right) \geq \underline{C}I_d > \underset{d\times d}{0}$ *a.s.*, for all n sufficiently large.

(c) $\lambda_{\max}\left(VC\left[\underline{U}'A\varepsilon/\sqrt{K_{2,n}}|\mathcal{F}_n^W\right]\right) \leq \overline{C}$ *a.s.*, $\lambda_{\max}\left(VC\left[\underline{U}'A\varepsilon/\sqrt{K_{2,n}}\right]\right) \leq \overline{C}$, $\lambda_{\max}\left(VC\left[U'A\varepsilon/\sqrt{K_{2,n}}|\mathcal{F}_n^W\right]\right) \leq \overline{C}$ *a.s.*, and $\lambda_{\max}\left(VC\left[U'A\varepsilon/\sqrt{K_{2,n}}\right]\right) \leq \overline{C}$, for all n sufficiently large.

(d) For any $a \in \mathbb{R}^d$ with $\|a\|_2 = 1$ and for all n sufficiently large, $\lambda_{\min}\left(\Sigma_n\right) \geq \underline{C} > 0$ *a.s.* and $a'\Sigma_n^{-1}a \leq \overline{C} < \infty$ *a.s.*, where $\Sigma_n = VC\left(\mathcal{Y}_n|\mathcal{F}_n^W\right) = \Sigma_{1,n} + \Sigma_{2,n}$, as defined in section 3 of the main paper, and where $\mathcal{Y}_n = \Gamma'M^{(Z_1,Q)}\varepsilon/\sqrt{n} + D_{\mu}^{-1}\underline{U}'A\varepsilon$.

Proof of Lemma S2-3:

For part (a), note that, by Assumptions 1, 2, and 3(iii); there exists a pair of constants

$0<\underline{C}\leq\overline{C}<\infty$ such that, for any $b\in\mathbb{R}^d$ such that $\|b\|=1$ and for all n sufficiently large, $b'VC\left(\Gamma'M^{(Z_1,Q)}\varepsilon/\sqrt{n}|\mathcal{F}_n^W\right)b=b'\Gamma'M^{(Z_1,Q)}E\left[\varepsilon\varepsilon'|\mathcal{F}_n^W\right]M^{(Z_1,Q)}\Gamma b/n$ $\leq\left(\max_{1\leq(i,t)\leq m_n}E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right]\right)\lambda_{\max}\left(\Gamma'\Gamma/n\right)\leq\overline{C}$ *a.s.* and $b'VC\left(\Gamma'M^{(Z_1,Q)}\varepsilon/\sqrt{n}|\mathcal{F}_n^W\right)b$ $\geq\left(\min_{1\leq(i,t)\leq m_n}E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right]\right)b'\Gamma'M^{(Z_1,Q)}\Gamma b/n\geq\underline{C}$ *a.s.* Since the above bounds hold for any $b\in\mathbb{R}^d$ such that $\|b\|_2=1$, it follows that, almost surely, $\lambda_{\max}\left[VC\left(\Gamma'M^{(Z_1,Q)}\varepsilon/\sqrt{n}|\mathcal{F}_n^W\right)\right]=\max_{\|b\|=1}b'VC\left(\Gamma'M^{(Z_1,Q)}\varepsilon/\sqrt{n}|\mathcal{F}_n^W\right)b\leq\overline{C}<\infty$ and $\lambda_{\min}\left[VC\left(\Gamma'M^{(Z_1,Q)}\varepsilon/\sqrt{n}|\mathcal{F}_n^W\right)\right]$ $=\min_{\|b\|=1}b'VC\left(\Gamma'M^{(Z_1,Q)}\varepsilon/\sqrt{n}|\mathcal{F}_n^W\right)b\geq\underline{C}>0$ for all n sufficiently large, which establishes the required result.

To show part (b), let $a\in\mathbb{R}^d$ such that $\|a\|_2=1$, and we define $\underline{U}_{(i,t)}=U_{(i,t)}-\rho\varepsilon_{(i,t)}$, $\underline{u}_{a,(i,t)}=a'\underline{U}_{(i,t)}$, $\sigma_{(i,t)}^2=E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right]$, $\widetilde{\omega}_{a,(i,t)}^2=E\left[\underline{u}_{a,(i,t)}^2|\mathcal{F}_n^W\right]$, $\widetilde{\psi}_{a,(i,t)}=E\left[\varepsilon_{(i,t)}\underline{u}_{a,(i,t)}|\mathcal{F}_n^W\right]$, and $\varrho_{a,(i,t)}=\widetilde{\psi}_{a,(i,t)}/\left(\sigma_{(i,t)}\widetilde{\omega}_{a,(i,t)}\right)$; for $(i,t)=1,..,m_n$, where we have suppressed the dependence of $\sigma_{(i,t)}^2$, $\widetilde{\omega}_{a,(i,t)}^2$, $\widetilde{\psi}_{a,(i,t)}$, and $\varrho_{a,(i,t)}$ on $\mathcal{F}_n^W=\sigma\left(W_n\right)$ for notational convenience. Note also that we can write $a'VC\left(\underline{U}'A\varepsilon/\sqrt{K_{2,n}}|\mathcal{F}_n^W\right)a=$ $K_{2,n}^{-1}\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}A_{(i,t),(j,s)}^2E\left[\left(\varepsilon_{(j,s)}\underline{u}_{a,(i,t)}+\varepsilon_{(i,t)}\underline{u}_{a,(j,s)}\right)^2|\mathcal{F}_n^W\right]$, since, by construction, $A_{(i,t),(i,t)}=0$ for $(i,t)=1,...,m_n$. Moreover, define $\delta_{a,(i,t),(j,s)}=\begin{pmatrix}\sigma_{(j,s)}\widetilde{\omega}_{a,(i,t)} & \sigma_{(i,t)}\widetilde{\omega}_{a,(j,s)}\end{pmatrix}'$ and

$$\Delta_{(i,t),(j,s)}^a=\begin{pmatrix}1 & \varrho_{a,(i,t)}\varrho_{a,(j,s)}\\ \varrho_{a,(i,t)}\varrho_{a,(j,s)} & 1\end{pmatrix}$$

and note that, given that $(j,s)<(i,t)$, we have $E\left[\left(\varepsilon_{(j,s)}\underline{u}_{a,(i,t)}+\varepsilon_{(i,t)}\underline{u}_{a,(j,s)}\right)^2|\mathcal{F}_n^W\right]$ $=\delta_{a,(i,t),(j,s)}'\Delta_{(i,t),(j,s)}^a\delta_{a,(i,t),(j,s)}$. Now, by the quadratic formula, the smallest eigenvalue of $\Delta_{(i,t),(j,s)}^a$ is given by $\lambda_{\min}\left(\Delta_{(i,t),(j,s)}^a\right)=1-\left|\varrho_{a,(i,t)}\right|\left|\varrho_{a,(j,s)}\right|$. In addition, write

$$\begin{aligned}\widetilde{\Omega}_{(i,t)} &= \begin{pmatrix}E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right] & E\left[\varepsilon_{(i,t)}\underline{U}_{(i,t)}'|\mathcal{F}_n^W\right]\\ E\left[\underline{U}_{(i,t)}\varepsilon_{(i,t)}|\mathcal{F}_n^W\right] & E\left[\underline{U}_{(i,t)}\underline{U}_{(i,t)}'|\mathcal{F}_n^W\right]\end{pmatrix}\\ &= \begin{pmatrix}1 & 0\\ -\rho & I_d\end{pmatrix}\begin{pmatrix}E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right] & E\left[\varepsilon_{(i,t)}U_{(i,t)}'|\mathcal{F}_n^W\right]\\ E\left[U_{(i,t)}\varepsilon_{(i,t)}|\mathcal{F}_n^W\right] & E\left[U_{(i,t)}U_{(i,t)}'|\mathcal{F}_n^W\right]\end{pmatrix}\begin{pmatrix}1 & -\rho'\\ 0 & I_d\end{pmatrix}\\ &= L_\rho\Omega_{(i,t)}L_\rho',\text{ where } L_\rho=\begin{pmatrix}1 & 0\\ -\rho & I_d\end{pmatrix}.\end{aligned}$$

Note that L_ρ is nonsingular, so that $L_\rho L_\rho'$ is positive definite. Hence, by Assumption 2 part (ii) and by the fact that $L_\rho L_\rho'$ is a fixed, finite-dimensional positive definite matrix, there exists some constant $C_1>1$ such that

$$\min_{1\leq(i,t)\leq m_n}\lambda_{\min}\left(\widetilde{\Omega}_{(i,t)}\right)\geq\min_{1\leq(i,t)\leq m_n}\lambda_{\min}\left(\Omega_{(i,t)}\right)\lambda_{\min}\left(L_\rho L_\rho'\right)\geq 1/C_1>0\ a.s.n. \tag{1}$$

Next, let

$$\underset{(d+1)\times 2}{D_a} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \; D_{SD,(i,t)} = \begin{pmatrix} \sigma_{(i,t)} & 0 \\ 0 & \widetilde{\omega}_{a,(i,t)} \end{pmatrix}, \text{ and } D_{\varrho,(i,t)} = \begin{pmatrix} 1 & \varrho_{a,(i,t)} \\ \varrho_{a,(i,t)} & 1 \end{pmatrix}$$

and note that

$$\begin{aligned} D_a'\widetilde{\Omega}_{(i,t)}D_a &= \begin{pmatrix} 1 & 0 \\ 0 & a' \end{pmatrix} \begin{pmatrix} E\left[\varepsilon^2_{(i,t)}|\mathcal{F}_n^W\right] & E\left[\varepsilon_{(i,t)}\underline{U}'_{(i,t)}|\mathcal{F}_n^W\right] \\ E\left[\underline{U}_{(i,t)}\varepsilon_{(i,t)}|\mathcal{F}_n^W\right] & E\left[\underline{U}_{(i,t)}\underline{U}'_{(i,t)}|\mathcal{F}_n^W\right] \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \\ &= \begin{pmatrix} \sigma_{(i,t)} & 0 \\ 0 & \widetilde{\omega}_{a,(i,t)} \end{pmatrix} \begin{pmatrix} 1 & \varrho_{a,(i,t)} \\ \varrho_{a,(i,t)} & 1 \end{pmatrix} \begin{pmatrix} \sigma_{(i,t)} & 0 \\ 0 & \widetilde{\omega}_{a,(i,t)} \end{pmatrix} = D_{SD,(i,t)}D_{\varrho,(i,t)}D_{SD,(i,t)}, \end{aligned}$$

Now, as can be seen from expression (1) above, an implication of Assumption 2(ii) is that

$$\min_{1\le(i,t)\le m_n} \sigma^2_{(i,t)} = e'_{1,d+1}\widetilde{\Omega}_{(i,t)}e_{1,d+1} \ge 1/C_1 > 0 \; a.s.n. \tag{2}$$

$$\min_{1\le(i,t)\le m_n} \widetilde{\omega}^2_{a,(i,t)} = \underline{a}'\widetilde{\Omega}_{(i,t)}\underline{a} \ge 1/C_1 > 0 \; a.s.n. \tag{3}$$

where $\underset{(d+1)\times 1}{e_{1,d+1}} = \begin{pmatrix} 1 & 0' \end{pmatrix}'$ and $\underset{(d+1)\times 1}{\underline{a}} = \begin{pmatrix} 0 & a' \end{pmatrix}'$, from which we deduce that $D_{SD,(i,t)}$ is invertible almost surely for each $(i,t) \in \{1,...,m_n\}$ and for all n sufficiently large. The invertibility of $D_{SD,(i,t)}$ then allows us to write $D_{\varrho,(i,t)} = D^{-1}_{SD,(i,t)}D_a'\widetilde{\Omega}_{(i,t)}D_aD^{-1}_{SD,(i,t)}$. On the other hand, Assumption 2(i) implies that there exists some constant $C_2 > 1$ such that

$$\min_{1\le(i,t)\le m_n} \lambda_{\min}\left(D^{-1}_{SD,(i,t)}\right) = \frac{1}{\max_{1\le(i,t)\le m_n}\lambda_{\max}\left(D_{SD,(i,t)}\right)} \ge \frac{1}{C_2} > 0 \;\; a.s. \tag{4}$$

It follows from the fact that $\lambda_{\min}\left(D_a'D_a\right) = \lambda_{\min}\left(I_2\right) = 1$ and from making use of Assumptions 2(i) and (ii) and the lower bounds given in (1) and (4) that

$$\begin{aligned} \min_{1\le(i,t)\le m_n} \lambda_{\min}\left(D_{\varrho,(i,t)}\right) &\ge \min_{1\le(i,t)\le m_n} \lambda_{\min}\left(\widetilde{\Omega}_{(i,t)}\right)\lambda_{\min}\left(D_a'D_a\right)\lambda_{\min}\left(D^{-2}_{SD,(i,t)}\right) \\ &\ge \frac{\min_{1\le(i,t)\le m_n}\lambda_{\min}\left(\widetilde{\Omega}_{(i,t)}\right)}{\max_{1\le(i,t)\le m_n}\left(\lambda_{\max}\left(D_{SD,(i,t)}\right)\right)^2} \ge \frac{1}{C^3} > 0 \;\; a.s.n., \end{aligned}$$

where $C = \max\{C_1, C_2\}$. Moreover, by solving the characteristic equation of $D_{\varrho,(i,t)}$, we see that the smallest eigenvalue of $D_{\varrho,(i,t)}$ is given by $\lambda_{\min}\left(D_{\varrho,(i,t)}\right) = 1 - \left|\varrho_{a,(i,t)}\right|$, so that $\min_{1\le(i,t)\le m_n}\lambda_{\min}\left(D_{\varrho,(i,t)}\right) = 1 - \max_{1\le(i,t)\le m_n}\left|\rho_{a,(i,t)}\right| \ge 1/C^3 > 0 \;\; a.s.n.$, from which we further deduce that $\max_{1\le(i,t)\le m_n}\left|\varrho_{a,(i,t)}\right| \le 1 - \left(1/C^3\right) < 1 \;\; a.s.n.$ Applying this upper bound along with the lower bounds given by (2) and

(3) as well as the fact that $\lambda_{\min}\left(\Delta^a_{(i,t),(j,s)}\right) = 1 - \left|\varrho_{a,(i,t)}\right|\left|\varrho_{a,(j,s)}\right|$, as derived earlier, we have

$$\begin{aligned} E\left[\left(\varepsilon_{(j,s)}\underline{u}_{a,(i,t)} + \varepsilon_{(i,t)}\underline{u}_{a,(j,s)}\right)^2 |\mathcal{F}_n^W\right] &= \delta'_{a,(i,t),(j,s)}\Delta^a_{(i,t),(j,s)}\delta_{a,(i,t),(j,s)} \\ &\geq \left[1 - \left|\varrho_{a,(i,t)}\right|\left|\varrho_{a,(j,s)}\right|\right]\left[\sigma^2_{(j,s)}\widetilde{\omega}^2_{a,(i,t)} + \sigma^2_{(i,t)}\widetilde{\omega}^2_{a,(j,s)}\right] \\ &\geq \left(\frac{2}{C^3} - \frac{1}{C^6}\right)\left[\frac{1}{C^2} + \frac{1}{C^2}\right] \geq \frac{2}{C^5} > 0 \quad a.s.n. \end{aligned}$$

Summing over $1 \leq (j,s) < (i,t) \leq m_n$, we obtain
$K_{2,n}^{-1}\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1} A^2_{(i,t),(j,s)}E\left[\left(\varepsilon_{(j,s)}\underline{u}_{a,(i,t)} + \varepsilon_{(i,t)}\underline{u}_{a,(j,s)}\right)^2 |\mathcal{F}_n^W\right]$
$\geq \left(2/C^5\right)K_{2,n}^{-1}\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1} A^2_{(i,t),(j,s)} = \left(1/C^5\right)K_{2,n}^{-1}\sum_{(i,t)=1}^{m_n}\sum_{(j,s)=1}^{m_n} A^2_{(i,t),(j,s)}$, where the last equality follows from the symmetry of A and by the fact that $A_{(i,t),(i,t)} = 0$ for $(i,t) = 1, ..., m_n$. Furthermore, by straightforward calculation, we obtain

$$\begin{aligned} \frac{1}{K_{2,n}}\sum_{(i,t)=1}^{m_n}\sum_{(j,s)=1}^{m_n} A^2_{(i,t),(j,s)} &= \frac{1}{K_{2,n}}\sum_{(i,t)=1}^{m_n}\sum_{(j,s)=1}^{m_n}\left[e'_{(i,t)}P^{\perp}e_{(j,s)} - e'_{(i,t)}M^{(Z,Q)}D_{\widehat{\vartheta}}M^{(Z,Q)}e_{(j,s)}\right]^2 \\ &= 1 + \frac{1}{K_{2,n}}\sum_{(i,t)=1}^{m_n} e'_{(i,t)}M^{(Z,Q)}D_{\widehat{\vartheta}}M^{(Z,Q)}D_{\widehat{\vartheta}}M^{(Z,Q)}e_{(i,t)} \geq 1 \end{aligned}$$

Putting everything together, we have
$K_{2,n}^{-1}\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1} A^2_{(i,t),(j,s)}E\left[\left(\varepsilon_{(j,s)}\underline{u}_{a,(i,t)} + \varepsilon_{(i,t)}\underline{u}_{a,(j,s)}\right)^2 |\mathcal{F}_n^W\right] \geq$
$\left(1/C^5\right)K_{2,n}^{-1}\sum_{(i,t)=1}^{m_n}\sum_{(j,s)=1}^{m_n} A^2_{(i,t),(j,s)} \geq 1/C^5 \geq \underline{C} > 0$, by choosing $\underline{C}$ such that $0 < \underline{C} \leq 1/C^5$. Since the above argument holds for any $a \in \mathbb{R}^d$ such that $\|a\| = 1$, it further follows that $VC\left(\underline{U}'A\varepsilon/\sqrt{K_{2,n}}|\mathcal{F}_n^W\right) \geq \underline{C}I_d > 0$ $a.s.$, as required.

To show part (c), note first that, given Assumption 2(i), there exists a positive constant C such that

$$\begin{aligned} &\max_{1\leq(j,s)\leq m_n}\lambda_{\max}\left(E\left[\underline{U}_{(j,s)}\underline{U}'_{(j,s)}|\mathcal{F}_n^W\right]\right) \\ \leq\ &\max_{1\leq(j,s)\leq m_n} tr\left\{E\left[\underline{U}_{(j,s)}\underline{U}'_{(j,s)}|\mathcal{F}_n^W\right]\right\} \\ \leq\ &\max_{1\leq(j,s)\leq m_n}\left\{E\left[\left\|U_{(j,s)}\right\|_2^2|\mathcal{F}_n^W\right] + 2E\left[\left|U'_{(j,s)}\rho\varepsilon_{(j,s)}\right||\mathcal{F}_n^W\right] + \rho'\rho E\left[\varepsilon^2_{(j,s)}|\mathcal{F}_n^W\right]\right\} \\ \leq\ &\max_{1\leq(j,s)\leq m_n}\Bigg\{E\left[\left\|U_{(j,s)}\right\|_2^2|\mathcal{F}_n^W\right] + 2\left\|\rho\right\|_2\sqrt{E\left[\varepsilon^2_{(j,s)}|\mathcal{F}_n^W\right]}\sqrt{E\left[\left\|U_{(j,s)}\right\|_2^2|\mathcal{F}_n^W\right]} \\ &\qquad + \left\|\rho\right\|_2^2 E\left[\varepsilon^2_{(j,s)}|\mathcal{F}_n^W\right]\Bigg\} \\ \leq\ &2\left\{\left(\max_{1\leq(j,s)\leq m_n} E\left[\left\|U_{(j,s)}\right\|_2^2|\mathcal{F}_n^W\right]\right) + \left\|\rho\right\|_2^2\left(\max_{1\leq(j,s)\leq m_n} E\left[\varepsilon^2_{(j,s)}|\mathcal{F}_n^W\right]\right)\right\} \\ \leq\ &C < \infty \quad a.s. \end{aligned} \tag{5}$$

where the third inequality above follows from applying the CS inequality while the fourth inequality stems in part from applying the inequality $|XY| \leq (1/2)\, X^2 + (1/2)\, Y^2$. Now, for any $a \in \mathbb{R}^d$ such that $\|a\|_2 = 1$; we obtain by applying the triangle and CS inequalities, expression (5), as well as part (a) of Lemma S2-1 and Assumptions 2(i) and 8

$$\begin{aligned}
& a'VC\left(\underline{U}'A\varepsilon/\sqrt{K_{2,n}}|\mathcal{F}_n^W\right)a \\
\leq\ & \frac{1}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A^2_{(i,t),(j,s)} \left(E\left[\varepsilon^2_{(i,t)}|\mathcal{F}_n^W\right] a'E\left[\underline{U}_{(j,s)}\underline{U}'_{(j,s)}|\mathcal{F}_n^W\right] a \right. \\
& \left. + \sqrt{a'E\left[\underline{U}_{(i,t)}\underline{U}'_{(i,t)}|\mathcal{F}_n^W\right]a}\sqrt{E\left[\varepsilon^2_{(i,t)}|\mathcal{F}_n^W\right]}\sqrt{a'E\left[\underline{U}_{(j,s)}\underline{U}'_{(j,s)}|\mathcal{F}_n^W\right]a}\sqrt{E\left[\varepsilon^2_{(j,s)}|\mathcal{F}_n^W\right]} \right) \\
\leq\ & \frac{1}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A^2_{(i,t),(j,s)} \left\{ E\left[\varepsilon^2_{(i,t)}|\mathcal{F}_n^W\right] \lambda_{\max}\left(E\left[\underline{U}_{(j,s)}\underline{U}'_{(j,s)}|\mathcal{F}_n^W\right]\right) \right. \\
& \left. + \sqrt{\lambda_{\max}\left(E\left[\underline{U}_{(j,s)}\underline{U}'_{(j,s)}|\mathcal{F}_n^W\right]\right)}\sqrt{E\left[\varepsilon^2_{(i,t)}|\mathcal{F}_n^W\right]}\sqrt{\lambda_{\max}\left(E\left[\underline{U}_{(j,s)}\underline{U}'_{(j,s)}|\mathcal{F}_n^W\right]\right)}\sqrt{E\left[\varepsilon^2_{(j,s)}|\mathcal{F}_n^W\right]} \right\} \\
=\ & O_{a.s.}(1).
\end{aligned}$$

From this, we deduce that $\lambda_{\max}\left[VC\left(\underline{U}'A\varepsilon/\sqrt{K_{2,n}}|\mathcal{F}_n^W\right)\right] = \max_{\|a\|=1} a'VC\left(\underline{U}'A\varepsilon/\sqrt{K_{2,n}}|\mathcal{F}_n^W\right)a$ $\leq \overline{C}$ *a.s.n.* Moreover, by applying the law of iterated expectations and part (i) of Theorem 16.1 of Billingsley (1995) that there exists a constant $\overline{C} > 0$ such that $a'VC\left(\underline{U}'A\varepsilon/\sqrt{K_{2,n}}\right)a = E_{W_n}\left\{E\left[\left(a'\underline{U}'A\varepsilon\right)^2/K_{2,n}|\mathcal{F}_n^W\right]\right\} \leq \overline{C}$, for all $a \in \mathbb{R}^d$ such that $\|a\| = 1$, from which we further deduce the unconditional version of this inequality, i.e., $\lambda_{\max}\left(VC\left[\underline{U}'A\varepsilon/\sqrt{K_{2,n}}\right]\right)$ $= \max_{\|a\|=1} a'VC\left(\underline{U}'A\varepsilon/\sqrt{K_{2,n}}\right)a \leq \overline{C} < \infty$, where $\underline{U} = U - \varepsilon\rho'$ and $\underline{U}_{(i,t)} = U_{(i,t)} - \rho\varepsilon_{(i,t)}$.

Furthermore, since $\underline{U} = U - \varepsilon\rho'$, we see that, by setting $\rho = 0$ in the argument given above, we can also show that there exists a constant $\overline{C}$ such that $\lambda_{\max}\left(VC\left[U'A\varepsilon/\sqrt{K_{2,n}}|\mathcal{F}_n^W\right]\right) \leq \overline{C} < \infty$ *a.s.n.* and $\lambda_{\max}\left(VC\left[U'A\varepsilon/\sqrt{K_{2,n}}\right]\right) \leq \overline{C} < \infty$ for all n sufficiently large.

Finally, to show part (d), note first that, by straightforward calculations, we get $\Sigma_n = VC\left(\mathcal{Y}_n|\mathcal{F}_n^W\right) = VC\left(\Gamma'M^{(Z_1,Q)}\varepsilon/\sqrt{n}|\mathcal{F}_n^W\right) + VC\left(D_\mu^{-1}\underline{U}'A\varepsilon|\mathcal{F}_n^W\right) = \Sigma_{1,n} + \Sigma_{2,n}$. It follows by part (a) of this lemma that there exists a positive constant $\underline{C}$ such that $\lambda_{\min}\left(\Sigma_n\right) \geq \lambda_{\min}\left[VC\left(\Gamma'M^{(Z_1,Q)}\varepsilon/\sqrt{n}|\mathcal{F}_n^W\right)\right] + \lambda_{\min}\left[VC\left(D_\mu^{-1}\underline{U}'A\varepsilon|\mathcal{F}_n^W\right)\right] \geq \lambda_{\min}\left[VC\left(\Gamma'M^{(Z_1,Q)}\varepsilon/\sqrt{n}|\mathcal{F}_n^W\right)\right]$ $\geq \underline{C} > 0$ *a.s.n.*, so that Σ_n is positive definite *a.s.n.* Moreover, again by part (a) of this lemma, for any $a \in \mathbb{R}^d$ such that $\|a\| = 1$,

$$\begin{aligned}
a'\Sigma_n^{-1}a &\leq \frac{1}{\lambda_{\min}\left\{VC\left(\Gamma'M^{(Z_1,Q)}\varepsilon/\sqrt{n}|\mathcal{F}_n^W\right) + VC\left(D_\mu^{-1}\underline{U}'A\varepsilon|\mathcal{F}_n^W\right)\right\}} \\
&\leq \frac{1}{\lambda_{\min}\left[VC\left(\Gamma'M^{(Z_1,Q)}\varepsilon/\sqrt{n}|\mathcal{F}_n^W\right)\right]} \leq \frac{1}{\underline{C}} \leq \overline{C} < \infty \ a.s.n.
\end{aligned}$$

where $\overline{C}$ can be taken to be any finite, positive constant such that $\overline{C} \geq 1/\underline{C}$. □

Lemma S2-4: Under Assumptions 1-7, the following results hold: (a) $D_{\mu}^{-1}X'A\varphi_n = O_p\left(\tau_n / K_{1,n}^{\varrho_g}\right)$; (b) $D_{\mu}^{-1}X'A\varepsilon = \frac{\Gamma' M^{(Z_1,Q)}\varepsilon}{\sqrt{n}} + D_{\mu}^{-1}U'A\varepsilon + O_p\left(K_{2,n}^{-\varrho_\gamma}\right) + O_p\left(K_{2,n}^{-\left(\varrho_\gamma - 1\right)} n^{-1}\right) + O_p\left(\kappa_n^{\max} / \left(\mu_n^{\min} K_{1,n}^{\varrho_f}\right)\right) = O_p\left(\max\left\{1, \sqrt{K_{2,n}} / \left(\mu_n^{\min}\right)\right\}\right)$

Proof of Lemma S2-4:

To show part (a), first write $D_{\mu}^{-1}X'A\varphi_n = D_{\mu}^{-1}\left(\Upsilon_n + \Phi_n + Q\Xi + U\right)' A\varphi_n = D_{\mu}^{-1}\Upsilon_n' A\varphi_n + D_{\mu}^{-1}\Phi_n' A\varphi_n + D_{\mu}^{-1}U'A\varphi_n$, where the second equality above follows from the fact that $Q'A\varphi_n = 0$. We will analyze each term on the right-hand side of the expression above in turn. Consider first the term $D_{\mu}^{-1}\Upsilon_n' A\varphi_n$. For $b \in \mathbb{R}^d$ such that $\|b\|_2 = 1$, note that, making use of the triangle and CS inequalities and Assumptions 3(iii), 5, 7(i), and 7(iii); we get

$$
\begin{aligned}
\left|b' D_{\mu}^{-1}\Upsilon_n' A\varphi_n\right| &\leq \left|b' D_{\mu}^{-1}\Upsilon_n' P^{\perp}\varphi_n\right| + \left|b' D_{\mu}^{-1}\Upsilon_n' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\varphi_n\right| \\
&= \frac{\tau_n}{n}\left|b'\Gamma' P^{\perp}\left(g - Z_1\theta_{K_{1,n}}\right)\right| \\
&\quad + \frac{\tau_n}{n}\left|b'\left(\Gamma - Z_2\Pi^{K_{2,n}}\right)' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\left(g - Z_1\theta^{K_{1,n}}\right)\right| \\
&\leq \tau_n\sqrt{\frac{b'\Gamma'\Gamma b}{n}}\sqrt{\frac{m_n}{n}}\left\|g(\cdot) - \theta^{K_{1,n}\prime} Z_1(\cdot)\right\|_{\infty} \\
&\quad + \tau_n\sqrt{\max_{1\leq(i,t)\leq m_n}\left|\widehat{\vartheta}_{(i,t)}\right|^2}\left(\frac{m_n\sqrt{d}}{n}\right)\left\|\gamma(\cdot) - \Pi^{K_{2,n}\prime} Z_2(\cdot)\right\|_{\infty,d}\left\|g(\cdot) - \theta^{K_{1,n}\prime} Z_1(\cdot)\right\|_{\infty} \\
&= O_{a.s.}\left(\frac{\tau_n}{K_{1,n}^{\varrho_g}}\right) + O_{a.s.}\left(\frac{\tau_n}{nK_{2,n}^{\left(\varrho_\gamma - 1\right)} K_{1,n}^{\varrho_g}}\right) = O_{a.s.}\left(\frac{\tau_n}{K_{1,n}^{\varrho_g}}\right)
\end{aligned}
$$

where $g = \left(g\left(W_{1,(1,1)}\right), .., g\left(W_{1,(1,T_1)}\right), ..., g\left(W_{1,(n,1)}\right), .., g\left(W_{1,(n,T_n)}\right)\right)'$. Since the above argument holds for all $b \in \mathbb{R}^d$ such that $\|b\|_2 = 1$, we further deduce that $D_{\mu}^{-1}\Upsilon_n' A\varphi_n = O_{a.s.}\left(\tau_n K_{1,n}^{-\varrho_g}\right)$. Next, consider $D_{\mu}^{-1}\Phi_n' A\varphi_n$. Again, let $b \in \mathbb{R}^d$ such that $\|b\|_2 = 1$, note that, by applying the triangle and CS inequalities along with Assumptions 3(ii), 4(i), 5, 7(i), and 7(ii); we obtain

$$
\begin{aligned}
\left|b' D_{\mu}^{-1}\Phi_n' A\varphi_n\right| &\leq \frac{\tau_n}{n}\left|b' D_{\mu}^{-1} D_{\kappa} F' P^{\perp} g\right| + \frac{\tau_n}{n}\left|b' D_{\mu}^{-1} D_{\kappa} F' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} g\right| \\
&\leq \frac{\tau_n\left(\kappa_n^{\max}\right)}{\left(\mu_n^{\min}\right)}\frac{m_n\sqrt{d}}{n}\left\|f(\cdot) - \Theta^{K_{1,n}\prime} Z_1(\cdot)\right\|_{\infty,d}\left\|g(\cdot) - \theta^{K_{1,n}\prime} Z_1(\cdot)\right\|_{\infty} \\
&\quad + \frac{\tau_n\left(\kappa_n^{\max}\right)}{\left(\mu_n^{\min}\right)}\sqrt{\max_{1\leq(i,t)\leq m_n}\left|\widehat{\vartheta}_{(i,t)}\right|^2}\frac{m_n\sqrt{d}}{n}\left\{\left\|f(\cdot) - \Theta^{K_{1,n}\prime} Z_1(\cdot)\right\|_{\infty,d}\right. \\
&\qquad \left.\times\left\|g(\cdot) - \theta^{K_{1,n}\prime} Z_1(\cdot)\right\|_{\infty}\right\} \\
&= O_{a.s.}\left(\frac{\tau_n\left(\kappa_n^{\max}\right)}{\left(\mu_n^{\min}\right) K_{1,n}^{\varrho_f + \varrho_g}}\right) = o_{a.s.}(1),
\end{aligned}
$$

where $F = \left(f\left(W_{1,(1,1)}\right), .., f\left(W_{1,(1,T_1)}\right), ..., f\left(W_{1,(n,1)}\right), .., f\left(W_{1,(n,T_n)}\right)\right)'$ and where g is as defined above. Since the above argument holds for all $b \in \mathbb{R}^d$ such that $\|b\|_2 = 1$, we further deduce that

$D_{\mu}^{-1}\Phi_n' A\varphi_n = O_{a.s.}\left(\tau_n \kappa_n^{\max}\left(\mu_n^{\min}\right)^{-1} K_{1,n}^{-\varrho_f-\varrho_g}\right) = o_{a.s.}(1)$. Now, consider the term $D_{\mu}^{-1}U'A\varphi_n$. Let $u_b = UD_{\mu}^{-1}b$, for $b \in \mathbb{R}^d$ such that $\|b\|_2 = 1$. Making use of the conditional serial independence assumption given in Assumption 1, we deduce that $E\left[u_b u_b' | \mathcal{F}_n^W\right] = E\left[UD_{\mu}^{-1}bb'D_{\mu}^{-1}U'|\mathcal{F}_n^W\right] \leq \left(\mu_n^{\min}\right)^{-2} \max_{1\leq(i,t)\leq m_n} E\left[\left\|U_{(i,t)}\right\|_2^2 |\mathcal{F}_n^W\right] I_{m_n}$, where we take $A \leq B$ for two square matrices A and B to mean that $A - B$ is negative semi-definite, or, alternatively, $B - A$ is positive semidefinite. Applying the above inequality and Assumptions 2(i), 3(ii), 4(ii), 5, and 7(i); we obtain

$$
\begin{aligned}
E\left(\left[b'D_{\mu}^{-1}U'A\varphi_n\right]^2 |\mathcal{F}_n^W\right) &= \frac{\tau_n^2}{n}\left(g - Z_1\theta^{K_{1,n}}\right)' AE\left[u_b u_b' |\mathcal{F}_n^W\right] A\left(g - Z_1\theta^{K_{1,n}}\right) \\
&\leq \frac{\tau_n^2}{n}\frac{\max_{1\leq(i,t)\leq m_n} E\left[\left\|U_{(i,t)}\right\|_2^2 |\mathcal{F}_n^W\right]\left(g - Z_1\theta^{K_{1,n}}\right)' A^2\left(g - Z_1\theta^{K_{1,n}}\right)}{\left(\mu_n^{\min}\right)^2} \\
&\leq \frac{m_n C\tau_n^2}{n\left(\mu_n^{\min}\right)^2}\left[1 + \max_{1\leq(i,t)\leq m_n}\left|\widehat{\vartheta}_{(i,t)}\right|^2\right]\left\|g(\cdot) - \theta^{K_{1,n}\prime} Z_1(\cdot)\right\|_\infty^2 \\
&= O_{a.s.}\left(\frac{\tau_n^2}{\left(\mu_n^{\min}\right)^2 K_{1,n}^{2\varrho_g}}\right)
\end{aligned}
$$

Hence, there exists a positive constant $\overline{C} < \infty$ such that for all n sufficiently large $E\left(\left(\mu_n^{\min}\right)^2 K_{1,n}^{2\varrho_g}\tau_n^{-2}\left[b'D_{\mu}^{-1}U'A\varphi_n\right]^2\right) = E_{W_n}\left\{\left(\mu_n^{\min}\right)^2 K_{1,n}^{2\varrho_g}\tau_n^{-2}E\left(\left[b'D_{\mu}^{-1}U'A\varphi_n\right]^2 |\mathcal{F}_n^W\right)\right\}$ $\leq \overline{C}$. It follows by applying Markov's inequality that $b'D_{\mu}^{-1}U'A\varphi_n = O_p\left(\tau_n\left(\mu_n^{\min}\right)^{-1} K_{1,n}^{-\varrho_g}\right)$. Finally, it follows from these intermediate results that $D_{\mu}^{-1}X'A\varphi_n = D_{\mu}^{-1}\Upsilon_n' A\varphi_n + D_{\mu}^{-1}\Phi_n' A\varphi_n + D_{\mu}^{-1}U'A\varphi_n = O_p\left(\tau_n K_{1,n}^{-\varrho_g}\right)$, as required to show part (a).

For part (b), write

$$
\begin{aligned}
D_{\mu}^{-1}X'A\varepsilon &= D_{\mu}^{-1}\Upsilon' A\varepsilon + D_{\mu}^{-1}\Phi' A\varepsilon + D_{\mu}^{-1}\Xi' Q' A\varepsilon + D_{\mu}^{-1}U'A\varepsilon \\
&= \frac{\Gamma' A\varepsilon}{\sqrt{n}} + \frac{D_{\mu}^{-1}D_{\kappa}F'A\varepsilon}{\sqrt{n}} + D_{\mu}^{-1}U'A\varepsilon
\end{aligned}
$$

Note that by straightforward calculation, we have

$$
\frac{D_{\mu}^{-1}D_{\kappa}F'A\varepsilon}{\sqrt{n}} = O_p\left(\frac{\kappa_n^{\max}}{\mu_n^{\min}}\frac{1}{K_{1,n}^{\varrho_f}}\right) = o_p(1) \tag{6}
$$

Moreover, since

$$
\begin{aligned}
A &= P^{\perp} - M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} \\
&= P^{(Z,Q)} - P^{(Z_1,Q)} - M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} \\
&= M^{(Z_1,Q)} - \left(M^{(Z,Q)} + M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\right),
\end{aligned}
$$

by straightforward calculations, we can show that

$$
\begin{aligned}
\frac{\Gamma' A\varepsilon}{\sqrt{n}} &= \frac{\Gamma' M^{(Z_1,Q)}\varepsilon}{\sqrt{n}} - \frac{\Gamma' M^{(Z,Q)}\varepsilon}{\sqrt{n}} - \frac{\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\varepsilon}{\sqrt{n}} \\
&= \frac{\Gamma' M^{(Z_1,Q)}\varepsilon}{\sqrt{n}} + O_p\left(\frac{1}{K_{2,n}^{\varrho_\gamma}}\right) + O_p\left(\frac{1}{K_{2,n}^{\left(\varrho_\gamma - 1\right)} n}\right) \\
&= O_p(1) \qquad (7)
\end{aligned}
$$

Finally, let $b \in \mathbb{R}^d$ such that $\|b\|_2 = 1$, and, by straightforward calculations, we obtain

$$
\begin{aligned}
E\left(\left[b' D_\mu^{-1} U' A\varepsilon\right]^2 | \mathcal{F}_n^W\right) &= \frac{K_{2,n}}{\left(\mu_n^{\min}\right)^2} b' D_\mu^{-1} E\left(\frac{U' A\varepsilon\varepsilon' A U}{K_{2,n}} | \mathcal{F}_n^W\right) D_\mu^{-1} b \\
&= \frac{K_{2,n}}{\left(\mu_n^{\min}\right)^2} b' D_\mu^{-1} VC\left(U' A\varepsilon / \sqrt{K_{2,n}} | \mathcal{F}_n^W\right) D_\mu^{-1} b \\
&\leq \lambda_{\max}\left[VC\left(U' A\varepsilon / \sqrt{K_{2,n}} | \mathcal{F}_n^W\right)\right] \frac{K_{2,n}}{\left(\mu_n^{\min}\right)^2} \\
&\leq \overline{C} \frac{K_{2,n}}{\left(\mu_n^{\min}\right)^2} = O_{a.s.}\left(\frac{K_{2,n}}{\left(\mu_n^{\min}\right)^2}\right)
\end{aligned}
$$

for some positive constant $\overline{C} < \infty$. Hence, for all n sufficiently large

$$
E\left(\frac{\left(\mu_n^{\min}\right)^2}{K_{2,n}}\left[b' D_\mu^{-1} U' A\varepsilon\right]^2\right) = E_{W_n}\left\{\frac{\left(\mu_n^{\min}\right)^2}{K_{2,n}} E\left(\left[b' D_\mu^{-1} U' A\varepsilon\right]^2 | \mathcal{F}_n^W\right)\right\} \leq \overline{C}.
$$

It follows from Markov's inequality, for any $\epsilon > 0$, we can set $C_\epsilon = \sqrt{\overline{C}/\epsilon}$ so that for all n sufficiently large

$$
\begin{aligned}
\Pr\left(\left|\frac{\mu_n^{\min}}{\sqrt{K_{2,n}}} b' D_\mu^{-1} U' A\varepsilon\right| \geq C_\epsilon\right) &= \Pr\left(\left|\frac{\mu_n^{\min}}{\sqrt{K_{2,n}}} b' D_\mu^{-1} U' A\varepsilon\right|^2 \geq C_\epsilon^2\right) \\
&\leq \frac{1}{C_\epsilon^2} E\left(\frac{\left(\mu_n^{\min}\right)^2}{K_{2,n}}\left[b' D_\mu^{-1} U' A\varepsilon\right]^2\right) \\
&\leq \frac{\overline{C}}{\overline{C}/\epsilon} = \epsilon
\end{aligned}
$$

which shows that

$$
b' D_\mu^{-1} U' A\varepsilon = O_p\left(\frac{\sqrt{K_{2,n}}}{\left(\mu_n^{\min}\right)}\right).
$$

Since the above result holds for all $b \in \mathbb{R}^d$ such that $\|b\|_2 = 1$, we further deduce that

$$D_\mu^{-1} U' A \varepsilon = O_p\left(\frac{\sqrt{K_{2,n}}}{(\mu_n^{\min})}\right). \tag{8}$$

Expressions (6)-(8) together imply that

$$\begin{aligned}
\frac{D_\mu^{-1} X' A \varepsilon}{\mu_n^{\min}} &= \frac{\Gamma' A \varepsilon}{\sqrt{n}} + \frac{D_\mu^{-1} D_\kappa F' A \varepsilon}{\sqrt{n}} + D_\mu^{-1} U' A \varepsilon \\
&= \frac{\Gamma' M^{(Z_1,Q)} \varepsilon}{\sqrt{n}} + D_\mu^{-1} U' A \varepsilon + O_p\left(\frac{1}{K_{2,n}^{\varrho_\gamma}}\right) + O_p\left(\frac{\upsilon_n}{K_{2,n}^{\varrho_\gamma}}\right) + O_p\left(\frac{\kappa_n^{\max}}{\mu_n^{\min}} \frac{1}{K_{1,n}^{\varrho_f}}\right) \\
&= O_p\left(\max\left\{1, \frac{\sqrt{K_{2,n}}}{(\mu_n^{\min})}\right\}\right). \ \square
\end{aligned}$$

Lemma S2-5: Under Assumptions 1-7, the following results hold: (a) $D_\mu^{-1} X' M^{(Z_1,Q)} \varphi_n = O_p\left(\tau_n / K_{1,n}^{\varrho_g}\right)$; (b) $D_\mu^{-1} X' M^{(Z_1,Q)} \varepsilon = O_p\left(n / \mu_n^{\min}\right)$.

Proof of Lemma S2-5:

To show part (a), first let $b \in \mathbb{R}^d$ such that $\|b\| = 1$ and write

$$\begin{aligned}
b' D_\mu^{-1} X' M^{(Z_1,Q)} \varphi &= b' D_\mu^{-1} \left(\Upsilon + \Phi + Q\Xi + U\right)' M^{(Z_1,Q)} \varphi \\
&= D_\mu^{-1} \left(\Upsilon + \Phi + U\right)' M^{(Z_1,Q)} \varphi \\
&= b' D_\mu^{-1} \Upsilon' M^{(Z_1,Q)} \varphi + b' D_\mu^{-1} \Phi' M^{(Z_1,Q)} \varphi \\
&\quad + b' D_\mu^{-1} U' M^{(Z_1,Q)} \varphi
\end{aligned}$$

where the second equality above follows from the fact that $Q' M^{(Z_1,Q)} \varphi = 0$. We will analyze each term on the right-hand side of the expression above in turn. Consider first the term $D_\mu^{-1} \Upsilon' M^{(Z_1,Q)} \varphi$. Note that, by applying the CS inequality and Assumptions 3(iii), 4(ii), 5(i), and 7(i); we have

$$\begin{aligned}
& \left| b' D_\mu^{-1} \Upsilon' M^{(Z_1,Q)} \varphi \right| \\
= \ & \frac{\tau_n}{n} \left| b' D_\mu^{-1} D_\mu \Gamma' M^{(Z_1,Q)} g \right| \\
= \ & \frac{\tau_n}{n} \left| b' \Gamma' M^{(Z_1,Q)} \left(g - Z_1 \theta_{K_{1,n}}\right) \right| \\
\leq \ & \tau_n \sqrt{\frac{b' \Gamma' \Gamma b}{n}} \sqrt{\frac{\left(g - Z_1 \theta_{K_{1,n}}\right)' M^{(Z_1,Q)} \left(g - Z_1 \theta_{K_{1,n}}\right)}{n}} \\
\leq \ & \tau_n \sqrt{\frac{b' \Gamma' \Gamma b}{n}} \sqrt{\frac{m_n}{n}} \left\| g(\cdot) - \theta_{K_{1,n}}' Z_1(\cdot) \right\|_\infty \\
= \ & O_{a.s.}\left(\frac{\tau_n}{K_{1,n}^{\varrho_g}}\right)
\end{aligned}$$

Next, consider $b'D_\mu^{-1}\Phi' M^{(Z_1,Q)}\varphi$. Here, note that, by the CS inequality and by Assumptions 3(ii), 4, 5(i), 7(i), and 7(ii);

$$\begin{aligned}
& \left| b'D_\mu^{-1}\Phi' M^{(Z_1,Q)}\varphi \right| \\
= \;& \frac{\tau_n}{n}\left| b'D_\mu^{-1}D_\kappa \left(F - Z_1\Theta_{K_{1,n}}\right)' M^{(Z_1,Q)}\left(g - Z_1\theta_{K_{1,n}}\right)\right| \\
\leq \;& \tau_n \sqrt{\frac{b'D_\mu^{-1}D_\kappa \left(F - Z_1\Theta_{K_{1,n}}\right)' M^{(Z_1,Q)}\left(F - Z_1\Theta_{K_{1,n}}\right) D_\kappa D_\mu^{-1} b}{n}} \\
& \times \sqrt{\frac{\left(g - Z_1\theta_{K_{1,n}}\right)' M^{(Z_1,Q)}\left(g - Z_1\theta_{K_{1,n}}\right)}{n}} \\
\leq \;& \frac{\tau_n\left(\kappa_n^{\max}\right)}{\left(\mu_n^{\min}\right)}\frac{m_n\sqrt{d}}{n}\left\| f(\cdot) - \Theta'_{K_{1,n}} Z_1(\cdot)\right\|_{\infty,d}\left\| g(\cdot) - \theta'_{K_{1,n}} Z_1(\cdot)\right\|_{\infty} \\
= \;& O_{a.s.}\left(\frac{\tau_n\left(\kappa_n^{\max}\right)}{\left(\mu_n^{\min}\right) K_{1,n}^{\varrho_f+\varrho_g}}\right) = o_{a.s.}(1)
\end{aligned}$$

Now, consider the term $b'D_\mu^{-1}U' M^{(Z_1,Q)}\varphi$. Let $u_b = UD_\mu^{-1}b$, where $b \in \mathbb{R}^d$ such that $\|b\| = 1$, and note that

$$\begin{aligned}
E\left[u_b u_b' | \mathcal{F}_n^W\right] &= E\left[UD_\mu^{-1}bb'D_\mu^{-1}U' | \mathcal{F}_n^W\right] \leq \frac{1}{\left(\mu_n^{\min}\right)^2} E\left[UU' | \mathcal{F}_n^W\right] \\
&\leq \frac{\max_{1\leq(i,t)\leq m_n} tr\left\{E\left[U_{(i,t)}U'_{(i,t)} | \mathcal{F}_n^W\right]\right\}}{\left(\mu_n^{\min}\right)^2} I_{m_n}
\end{aligned}$$

where we take $A \leq B$ for two square matrices A and B to mean that $A - B$ is negative semi-definite, or, alternatively, $B - A$ is positive semidefinite. Note further that, by applying Assumptions 1, 2(i), 3(ii), 4(ii), and 7(i);

$$
\begin{aligned}
& E\left(\left[b' D_\mu^{-1} U' M^{(Z_1,Q)} \varphi\right]^2 | \mathcal{F}_n^W\right) \\
= \; & \frac{\tau_n^2}{n} E\left[g' M^{(Z_1,Q)} U D_\mu^{-1} b b' D_\mu^{-1} U' M^{(Z_1,Q)} g | \mathcal{F}_n^W\right] \\
= \; & \frac{\tau_n^2}{n} g' M^{(Z_1,Q)} E\left[u_b u_b' | \mathcal{F}_n^W\right] M^{(Z_1,Q)} g \\
\leq \; & \frac{\tau_n^2}{n} \frac{\max_{1 \leq (i,t) \leq m_n} tr\left\{E\left[U_{(i,t)} U_{(i,t)}' | \mathcal{F}_n^W\right]\right\}}{\left(\mu_n^{\min}\right)^2} g' M^{(Z_1,Q)} g \\
= \; & \frac{\tau_n^2}{\left(\mu_n^{\min}\right)^2} \left(\max_{1 \leq (i,t) \leq m_n} E\left[\left\|U_{(i,t)}\right\|_2^2 | \mathcal{F}_n^W\right]\right)^2 \\
& \times \frac{\left(g - Z_1 \theta_{K_{1,n}}\right)' M^{(Z_1,Q)} \left(g - Z_1 \theta_{K_{1,n}}\right)}{n} \\
\leq \; & C \frac{\tau_n^2}{\left(\mu_n^{\min}\right)^2} \frac{m_n}{n} \left\|g(\cdot) - \theta_{K_{1,n}}' Z_1(\cdot)\right\|_\infty^2 \\
= \; & O_{a.s.}\left(\frac{\tau_n^2}{\left(\mu_n^{\min}\right)^2 K_{1,n}^{2\varrho_g}}\right)
\end{aligned}
$$

Hence, there exists a positive constant $\overline{C} < \infty$ such that for all n sufficiently large

$$
E\left(\frac{\left(\mu_n^{\min}\right)^2 K_{1,n}^{2\varrho_g}}{\tau_n^2}\left[b' D_\mu^{-1} U' M^{(Z_1,Q)} \varphi\right]^2\right) = E_W\left\{\frac{\left(\mu_n^{\min}\right)^2 K_{1,n}^{2\varrho_g}}{\tau_n^2} E\left(\left[b' D_\mu^{-1} U' M^{(Z_1,Q)} \varphi\right]^2 | \mathcal{F}_n^W\right)\right\} \leq \overline{C}.
$$

It follows from Markov's inequality, for any $\epsilon > 0$, we can set $C_\epsilon = \sqrt{\overline{C}/\epsilon}$ so that for all n sufficiently large

$$
\begin{aligned}
\Pr\left(\left|\left(\frac{\left(\mu_n^{\min}\right) K_{1,n}^{\varrho_g}}{\tau_n}\right) b' D_\mu^{-1} U' M^{(Z_1,Q)} \varphi\right| \geq C_\epsilon\right) \; & = \; \Pr\left(\left|\left(\frac{\left(\mu_n^{\min}\right) K_{1,n}^{\varrho_g}}{\tau_n}\right) b' D_\mu^{-1} U' M^{(Z_1,Q)} \varphi\right|^2 \geq C_\epsilon^2\right) \\
& \leq \; \frac{1}{C_\epsilon^2} E\left(\frac{\left(\mu_n^{\min}\right)^2 K_{1,n}^{2\varrho_g}}{\tau_n^2}\left[b' D_\mu^{-1} U' M^{(Z_1,Q)} \varphi\right]^2\right) \\
& \leq \; \frac{\overline{C}}{\overline{C}/\epsilon} = \epsilon
\end{aligned}
$$

which shows that $b' D_\mu^{-1} U' M^{(Z_1,Q)} \varphi = O_p\left(\tau_n / \left[\left(\mu_n^{\min}\right) K_{1,n}^{\varrho_g}\right]\right)$. Putting the above results together, we get

$$
\begin{aligned}
& b' D_\mu^{-1} X' M^{(Z_1,Q)} \varphi \\
= \;& b' D_\mu^{-1} \Upsilon' M^{(Z_1,Q)} \varphi + b' D_\mu^{-1} \Phi' M^{(Z_1,Q)} \varphi + b' D_\mu^{-1} U' M^{(Z_1,Q)} \varphi \\
= \;& O_{a.s.}\left(\frac{\tau_n}{K_{1,n}^{\varrho_g}} \right) + O_{a.s.}\left(\frac{\tau_n \left(\kappa_n^{\max}\right)}{\left(\mu_n^{\min}\right) K_{1,n}^{\varrho_f + \varrho_g}} \right) + O_p\left(\frac{\tau_n}{\left(\mu_n^{\min}\right) K_{1,n}^{\varrho_g}} \right) \\
= \;& O_{a.s.}\left(\frac{\tau_n}{K_{1,n}^{\varrho_g}} \right).
\end{aligned}
$$

Since the above argument holds for all $b \in \mathbb{R}^d$ such that $\|b\|_2 = 1$, we further deduce that

$$
D_\mu^{-1} X' M^{(Z_1,Q)} \varphi = O_{a.s.}\left(\frac{\tau_n}{K_{1,n}^{\varrho_g}} \right).
$$

Next, to show part (b), let $b \in \mathbb{R}^d$ such that $\|b\|_2 = 1$ and let $u_b = U D_\mu^{-1} b$, $u_{b,(i,t)} = U'_{(i,t)} D_\mu^{-1} b$, and $F = \left(f\left(W_{1,(1,1)}\right), .., f\left(W_{1,(1,T_1)}\right), ..., f\left(W_{1,(n,1)}\right), .., f\left(W_{1,(n,T_n)}\right) \right)'$. Now, write $b' D_\mu^{-1} X' M^{(Z_1,Q)} \varepsilon = b' \Gamma' M^{(Z_1,Q)} \varepsilon / \sqrt{n} + b' D_\mu^{-1} D_\kappa F' M^{(Z_1,Q)} \varepsilon / \sqrt{n} + u_b' M^{(Z_1,Q)} \varepsilon$. By straightforward calculations and by applying Assumptions 1, 2(i), 3, 4, 5(i), 5(iii), and 7(ii) as well as the CS and Markov's inequalities, we can show that $b' \Gamma' M^{(Z_1,Q)} \varepsilon / \sqrt{n} = O_p(1)$ and $b' D_\mu^{-1} D_\kappa F' M^{(Z_1,Q)} \varepsilon / \sqrt{n} = O_p\left(\kappa_n^{\max} \left(\mu_n^{\min}\right)^{-1} K_{1,n}^{-\varrho_f} \right)$. From the CS inequality, we also obtain $E\left[\left| u_b' M^{(Z_1,Q)} \varepsilon \right| | \mathcal{F}_n^W \right]$ $\leq \sqrt{E\left[u_b' M^{(Z_1,Q)} u_b | \mathcal{F}_n^W \right]} \sqrt{E\left[\varepsilon' M^{(Z_1,Q)} \varepsilon | \mathcal{F}_n^W \right]}$. Next, note that, by applying Assumptions 2(i) and 6(ii), we have $E\left[\varepsilon' M^{(Z_1,Q)} \varepsilon | \mathcal{F}_n^W \right] \leq E\left[\varepsilon' \varepsilon | \mathcal{F}_n^W \right] \leq n \overline{T} \left(\max_{1 \leq (i,t) \leq m_n} E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W \right] \right) = O_{a.s.}(n)$. Similarly, by applying Assumptions 2(i), 3(ii), and 6(ii); we obtain $E\left[u_b' M^{(Z_1,Q)} u_b | \mathcal{F}_n^W \right]$ $= E\left[b' D_\mu^{-1} U' M^{(Z_1,Q)} U D_\mu^{-1} b | \mathcal{F}_n^W \right] \leq \overline{T} \left(\max_{1 \leq (i,t) \leq m_n} E\left[\left\| U_{(i,t)} \right\|_2^2 | \mathcal{F}_n^W \right] \right) \left[n / \left(\mu_n^{\min}\right)^2 \right]$ $= O_{a.s.}\left(n \left(\mu_n^{\min}\right)^{-2} \right)$. It follows from these results that $E\left[\left| u_b' M^{(Z_1,Q)} \varepsilon \right| | \mathcal{F}_n^W \right] = O_{a.s.}\left(n \left(\mu_n^{\min}\right)^{-1} \right)$. Hence, by the law of iterated expectations and Theorem 16.1 of Billingsley (1995), there exists a constant $\overline{C} < \infty$ such that for all n sufficiently large $E\left[\left(\left(\mu_n^{\min}\right) / n \right) \left| u_b' M^{(Z_1,Q)} \varepsilon \right| \right]$ $= E_W\left(\left(\left(\mu_n^{\min}\right) / n \right) E\left[\left| u_b' M^{(Z_1,Q)} \varepsilon \right| | \mathcal{F}_n^W \right] \right) \leq \overline{C}$. It follows by applying Markov's inequality that

$$
b' D_\mu^{-1} U' M^{(Z_1,Q)} \varepsilon = O_p\left(n \left(\mu_n^{\min}\right)^{-1} \right). \tag{9}
$$

Putting everything together, we have

$$
\begin{aligned}
b' D_\mu^{-1} X' M^{(Z_1,Q)} \varepsilon &= \frac{b' \Gamma' M^{(Z_1,Q)} \varepsilon}{\sqrt{n}} + \frac{b' D_\mu^{-1} D_\kappa F' M^{(Z_1,Q)} \varepsilon}{\sqrt{n}} + b' D_\mu^{-1} U' M^{(Z_1,Q)} \varepsilon \\
&= O_p(1) + O_p\left(\frac{\kappa_n^{\max}}{\left(\mu_n^{\min}\right) K_{1,n}^{\varrho_f}} \right) + O_p\left(\frac{n}{\left(\mu_n^{\min}\right)} \right) = O_p\left(\frac{n}{\left(\mu_n^{\min}\right)} \right)
\end{aligned}
$$

Since the above argument holds for all $b \in \mathbb{R}^d$ such that $\|b\|_2 = 1$, we further deduce that

$D_\mu^{-1}X'M^{(Z_1,Q)}\varepsilon = O_p\left(n\left(\mu_n^{\min}\right)^{-1}\right)$. □

Lemma S2-6: Suppose that Assumptions 2 and 8 hold. For $1 \leq p \leq 8$ and for all n, there exists a positive constant C such that $\max_{1\leq(i,t)\leq m_n} E\left[\left\|\underline{U}_{(i,t)}\right\|_2^p |\mathcal{F}_n^W\right] \leq C < \infty$ *a.s.*, where $\underline{U}_{(i,t)} = U_{(i,t)} - \rho\varepsilon_{(i,t)}$.

Proof of Lemma S2-6: Note that, for $1 \leq p \leq 8$ and for any $(i,t) \in \{1, ..., m_n\}$, there exists a positive constant C such that

$$\begin{aligned} E\left[\left\|\underline{U}_{(i,t)}\right\|_2^p |\mathcal{F}_n^W\right] &= E\left[\left(\left\|U_{(i,t)} - \rho\varepsilon_{(i,t)}\right\|_2\right)^p |\mathcal{F}_n^W\right] \\ &\leq 2^{p-1}\left\{\left(E\left[\left\|U_{(i,t)}\right\|_2^8 |\mathcal{F}_n^W\right]\right)^{p/8} + \|\rho\|_2^p\left(E\left[\left|\varepsilon_{(i,t)}\right|^8 |\mathcal{F}_n^W\right]\right)^{p/8}\right\} \\ &\leq C < \infty \ \ a.s., \end{aligned}$$

where the first inequality follows by applying the triangle inequality, Loève's c_r inequality, and Liapunov's inequality in sequence and where the second inequality follows from applying Assumption 2(i) and and from the fact that $\rho \in \mathcal{S}_\rho$, some compact subset of $\mathbb{R}^d$ as stated in Assumption 8. Since the upper bound above holds for all $(i,t) \in \{1, ..., m_n\}$, for all $\rho \in \mathcal{S}_\rho$, and for all n, it further follows that $\max_{1\leq(i,t)\leq m_n} E\left[\left\|\underline{U}_{(i,t)}\right\|_2^p |\mathcal{F}_n^W\right] \leq C < \infty$ *a.s.*, as required. □

Lemma S2-7: Under Assumptions 1-7, the following results hold: (a) $\widehat{\ell}_{L,n} = o_p\left(\left[\mu_n^{\min}\right]^2/n\right)$; (b) $\widehat{\ell}_{F,n} = o_p\left(\left[\mu_n^{\min}\right]^2/n\right)$.

Proof of Lemma S2-7: To proceed, first define

$$\overline{D}_n = \begin{pmatrix} \mu_n^{\min} & 0 \\ 0 & D_\mu \end{pmatrix} \text{ and } L_\delta = \begin{pmatrix} 1 & 0 \\ \delta_0 & I_d \end{pmatrix},$$

and note that

$$L_\delta^{-1} = \begin{pmatrix} 1 & 0 \\ -\delta_0 & I_d \end{pmatrix}$$

Now, for any $\beta \in \mathbb{R}^{d+1}$ such that $\|\beta\|_2 = 1$ and for $\overline{X} = \left[\begin{array}{cc} y & X \end{array}\right]$, we can write $\beta'\overline{X}'A\overline{X}\beta = \beta' L_\delta' \overline{D}_n\left(\overline{D}_n^{-1}L_\delta'^{-1}\overline{X}'A\overline{X}L_\delta^{-1}\overline{D}_n^{-1}\right)\overline{D}_n L_\delta\beta$. Moreover, by direct multiplication,

$$\overline{D}_n^{-1}L_\delta'^{-1}\overline{X}'A\overline{X}L_\delta^{-1}\overline{D}_n^{-1} = \begin{pmatrix} (y - X\delta_0)' A (y - X\delta_0) / \left(\mu_n^{\min}\right)^2 & (y - X\delta_0)' AXD_\mu^{-1} / \left(\mu_n^{\min}\right) \\ D_\mu^{-1}X'A(y - X\delta_0) / \left(\mu_n^{\min}\right) & D_\mu^{-1}X'AXD_\mu^{-1} \end{pmatrix},$$

Now, by straightforward but tedious calculations and by applying Assumptions 1, 2(i), 3(ii)-(iii), 4, 5, 6(i), and 7(i)-(iii) as well as Lemmas S2-3(c) and S2-1(a); we can show that, under the rate condition $\sqrt{K_2}/\left(\mu_n^{\min}\right)^2 \to 0$,

$$\overline{D}_n^{-1}L_\delta'^{-1}\overline{X}'A\overline{X}L_\delta^{-1}\overline{D}_n^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & \Gamma' M^{(Z_1,Q)}\Gamma/n \end{pmatrix} + o_p(1)^{1},$$

where, in light of Assumption 3(iii), $\Gamma' M^{(Z_1,Q)}\Gamma/n = O_p(1)$ and $\Gamma' M^{(Z_1,Q)}\Gamma/n$ is positive definite for all n large sufficiently large. It follows that $\overline{D}_n^{-1} L_\delta'^{-1} \overline{X}' A \overline{X} L_\delta^{-1} \overline{D}_n^{-1}$ is positive semidefinite w.p.a.1, so that $\beta' \overline{X}' A \overline{X} \beta = \beta' L_\delta' \overline{D}_n \left(\overline{D}_n^{-1} L_\delta'^{-1} \overline{X}' A \overline{X} L_\delta^{-1} \overline{D}_n^{-1} \right) \overline{D}_n L_\delta \beta \geq 0$ w.p.a.1 for all $\beta \in \mathbb{R}^{d+1}$ such that $\|\beta\|_2 = 1$. Moreover, by straightforward calculations, we can show that

$$\frac{\beta' \overline{X}' M^{(Z_1,Q)} \overline{X} \beta}{n} = \frac{\beta' L_{\delta,2}' D_\mu \Gamma' M^{(Z_1,Q)} \Gamma D_\mu L_{\delta,2} \beta}{n^2} + \beta' L_\delta' E \left[\frac{V' M^Q V}{n} \right] L_\delta \beta + o_p(1),$$

where $V = \begin{bmatrix} \varepsilon & U \end{bmatrix}$ and $L_{\delta,2} = \begin{bmatrix} \delta_0 & I_d \end{bmatrix}$ and where $E\left[V' M^Q V/n\right]$ is positive definite for all n sufficiently large in light of Assumptions 2(ii) and 6(i). Since $L_\delta \beta \neq 0$ for all $\beta \in \mathbb{R}^{d+1}$ such that $\|\beta\|_2 = 1$, it follows that $\beta' \overline{X}' M^{(Z_1,Q)} \overline{X} \beta / n > 0$ w.p.a.1 for all $\beta \in \mathbb{R}^{d+1}$ such that $\|\beta\|_2 = 1$. Hence, with probability approaching one as $n \to \infty$,

$$R(\beta) = \frac{\beta' \overline{X}' A \overline{X} \beta}{\beta' \overline{X}' M^{(Z_1,Q)} \overline{X} \beta}$$

is a continuous function of β for all values of β such that $\|\beta\|_2 = 1$. The Weierstrass extreme value theorem then implies that there exists some $\widetilde{\beta}$ such that $\widetilde{\beta} = \arg\min_{\|\beta\|_2=1} R(\beta)$ w.p.a.1. Next, note that $\widehat{\ell}_{L,n}$ is the smallest root of the determinantal equation
$\det\left\{ \overline{X}' A \overline{X} - \ell \overline{X}' M^{(Z_1,Q)} \overline{X} \right\} = 0$; and, thus, $\widehat{\ell}_{L,n}$ has the representation

$$\widehat{\ell}_{L,n} = R\left(\widetilde{\beta}\right) = \frac{\widetilde{\beta}' \overline{X}' A \overline{X} \widetilde{\beta}}{\widetilde{\beta}' \overline{X}' M^{(Z_1,Q)} \overline{X} \widetilde{\beta}} = \min_{\|\beta\|_2=1} \left(\frac{\beta' \overline{X}' A \overline{X} \beta}{\beta' \overline{X}' M^{(Z_1,Q)} \overline{X} \beta} \right).$$

Now, let $\delta_* = \begin{pmatrix} 1 & -\delta_0' \end{pmatrix}' / \left\| \begin{pmatrix} 1 & -\delta_0' \end{pmatrix}' \right\|_2$; and we have, with probability approaching one as $n \to \infty$,

$$\begin{aligned}
0 \quad &\leq \quad \widehat{\ell}_{L,n} = \min_{\|\beta\|_2=1} \left(\frac{\beta' \overline{X}' A \overline{X} \beta}{\beta' \overline{X}' M^{(Z_1,Q)} \overline{X} \beta} \right) \\
&\leq \quad \frac{\delta_*' \overline{X}' A \overline{X} \delta_*}{\delta_*' \overline{X}' M^{(Z_1,Q)} \overline{X} \delta_*} \\
&= \quad \frac{\left(\mu_n^{\min}\right)^2}{n} \left\{ \frac{(y - X\delta_0)' A (y - X\delta_0) / \sqrt{K_{2,n}}}{(y - X\delta_0) M^{(Z_1,Q)} (y - X\delta_0) / n} \right\} \frac{\sqrt{K_{2,n}}}{\left(\mu_n^{\min}\right)^2} \\
&= \quad O\left(\frac{\left[\mu_n^{\min}\right]^2}{n} \right) O_p(1)\, O_p\left(\frac{\sqrt{K_{2,n}}}{\left(\mu_n^{\min}\right)^2} \right) = o_p\left(\frac{\left[\mu_n^{\min}\right]^2}{n} \right),
\end{aligned}$$

given the rate condition $\sqrt{K_2} / \left(\mu_n^{\min}\right)^2 \to 0$. This shows part (a).

For part (b), we use the result in part (a) above and the fact that $m_n/n \sim 1$ by Assumption

5(i) to obtain

$$\widehat{\ell}_{F,n} = \frac{\widehat{\ell}_{L,n} - \left(1 - \widehat{\ell}_{L,n}\right) C/m_n}{1 - \left(1 - \widehat{\ell}_{L,n}\right) C/m_n} = \left[\widehat{\ell}_{L,n} + O_p\left(\frac{1}{n}\right)\right]\left[1 + O_p\left(\frac{1}{n}\right)\right] = o_p\left(\frac{\left[\mu_n^{\min}\right]^2}{n}\right). \square \tag{10}$$

Lemma S2-8: Let A be as defined above. Suppose that i) $\left(u_{(1,1),n}, \varepsilon_{(1,1)}\right), ..., \left(u_{(1,T_1),n}, \varepsilon_{(1,T_1)}\right)$, $\left(u_{(2,1),n}, \varepsilon_{(2,1),n}\right), ..., \left(u_{(2,T_2),n}, \varepsilon_{(2,T_2),n}\right), ..., \left(u_{(n,1),n}, \varepsilon_{(n,1),n}\right)$ ...., $\left(u_{(n,T_n),n}, \varepsilon_{(n,T_n),n}\right)$ are independent conditional on $\mathcal{F}_n^W = \sigma(W_n)$; ii) there exists a constant C such that, almost surely for all n sufficiently large, $\max_{1\leq(i,t)\leq m_n} E\left(u_{(i,t),n}^4 | \mathcal{F}_n^W\right) \leq C$, $\max_{1\leq(i,t)\leq m_n} E\left(\varepsilon_{(i,t),n}^4 | \mathcal{F}_n^W\right) \leq C$, and $\max_{1\leq(i,t)\leq m_n} \left|\phi_{(i,t),n}\right| \leq C$. In addition, define $\overline{\psi}_{(j,s),n} = E\left[u_{(j,s),n}\varepsilon_{(j,s),n} | \mathcal{F}_n^W\right]$ for $(j,s) = 1, ..., m_n$. Then, under Assumptions 5 and 6, the following statements are true:

(a) $K_{2,n}^{-1} \sum_{1\leq(j,s)<(i,t)\leq m_n} A_{(i,t),(j,s)}^2 \phi_{(i,t),n} \left\{u_{(j,s),n}\varepsilon_{(j,s),n} - \overline{\psi}_{(j,s),n}\right\} \xrightarrow{p} 0$;

(b) $K_{2,n}^{-1} \sum_{1\leq\ (k,v)<(j,s)<(i,t)\leq\ m_n} A_{(i,t),(j,s)} A_{(i,t),(k,v)} \phi_{(i,t),n} \left\{u_{(j,s),n}\varepsilon_{(k,v),n} + \varepsilon_{(j,s),n} u_{(k,v),n}\right\} \xrightarrow{p} 0$;

(c) $K_{2,n}^{-1} \sum_{1\leq\ (k,v)<(j,s)<(i,t)\leq\ m_n} A_{(i,t),(j,s)} A_{(i,t),(k,v)} \phi_{(i,t),n} \varepsilon_{(j,s),n} \varepsilon_{(k,v),n} \xrightarrow{p} 0$;

(d) $K_{2,n}^{-1} \sum_{1\leq\ (k,v)<(j,s)<(i,t)\leq\ m_n} A_{(i,t),(j,s)} A_{(i,t),(k,v)} \phi_{(i,t),n} u_{(j,s),n} u_{(k,v),n} \xrightarrow{p} 0$.

Proof of Lemma S2-8: To show part (a), note that

$$
\begin{aligned}
& E\left[\left(\frac{1}{K_{2,n}}\sum\nolimits_{1\leq(j,s)<(i,t)\leq m_n} A^2_{(i,t),(j,s)}\phi_{(i,t),n}\left\{u_{(j,s),n}\varepsilon_{(j,s),n}-\overline{\psi}_{(j,s),n}\right\}\right)^2 \mid \mathcal{F}^W_n\right] \\
= \; & \frac{1}{K^2_{2,n}}\sum\nolimits_{1\leq(j,s)<(i,t)\leq m_n} A^4_{(i,t),(j,s)}\phi^2_{(i,t),n}\left\{E\left(\varepsilon^2_{(j,s),n}u^2_{(j,s),n}|\mathcal{F}^W_n\right)-\overline{\psi}^2_{(j,s),n}\right\} \\
& +\frac{2}{K^2_{2,n}}\sum\nolimits_{1\leq(j,s)<(k,v)<(i,t)\leq m_n} A^2_{(k,v),(j,s)}A^2_{(i,t),(j,s)}\phi_{(k,v),n}\phi_{(i,t),n}\left\{E\left(u^2_{(j,s),n}\varepsilon^2_{(j,s),n}|\mathcal{F}^W_n\right)-\overline{\psi}^2_{(j,s),n}\right\} \\
\leq \; & \frac{1}{K^2_{2,n}}\sum\nolimits_{1\leq(j,s)<(i,t)\leq m_n}\left\{A^4_{(i,t),(j,s)}\phi^2_{(i,t),n}\right. \\
& \left.\times\left[\sqrt{E\left(u^4_{(j,s),n}|\mathcal{F}^W_n\right)E\left(\varepsilon^4_{(j,s),n}|\mathcal{F}^W_n\right)}+E\left(u^2_{(j,s),n}|\mathcal{F}^W_n\right)E\left(\varepsilon^2_{(j,s),n}|\mathcal{F}^W_n\right)\right]\right\} \\
& +\frac{2}{K^2_{2,n}}\sum\nolimits_{1\leq(j,s)<(k,v)<(i,t)\leq m_n} A^2_{(k,v),(j,s)}A^2_{(i,t),(j,s)}\left|\phi_{(k,v),n}\right|\left|\phi_{(i,t),n}\right| \\
& \qquad\qquad \times\left\{\sqrt{E\left(u^4_{(j,s),n}|\mathcal{F}^W_n\right)E\left(\varepsilon^4_{(j,s),n}|\mathcal{F}^W_n\right)}+E\left(u^2_{(j,s),n}|\mathcal{F}^W_n\right)E\left(\varepsilon^2_{(j,s),n}|\mathcal{F}^W_n\right)\right\} \\
\leq \; & C\left\{\frac{1}{K^2_{2,n}}\sum\nolimits_{1\leq(j,s)<(i,t)\leq m_n} A^4_{(i,t),(j,s)}+\frac{2}{K^2_{2,n}}\sum\nolimits_{1\leq(j,s)<(k,v)<(i,t)\leq m_n} A^2_{(k,v),(j,s)}A^2_{(i,t),(j,s)}\right\} \\
\leq \; & C\left\{\frac{1}{K^2_{2,n}}\sum_{\substack{(i,t),(j,s)=1\\(i,t)\neq(j,s)}}^{m_n} A^4_{(i,t),(j,s)}+\frac{2}{K^2_{2,n}}\sum_{(j,s)=1}^{m_n}\sum_{\substack{(i,t),(k,v)=1\\(i,t)\neq(j,s),(k,v)\neq(j,s)}}^{m_n} A^2_{(k,v),(j,s)}A^2_{(i,t),(j,s)}\right\} \\
= \; & o_{a.s.}(1)
\end{aligned}
$$

where the first inequality is the result of applying T and a conditional version of CS, the second inequality follows by hypothesis, and the convergence to zero almost surely follows from parts (b) and (c) of Lemma S2-1 and the symmetry of A. It follows by the conditional version of the Markov's inequality that for any $\epsilon > 0$

$$
\Pr\left(\left|\frac{1}{K_{2,n}}\sum\nolimits_{1\leq(j,s)<(i,t)\leq m_n} A^2_{(i,t),(j,s)}\phi_{(i,t),n}\left\{\varepsilon_{(j,s)}u_{(j,s),n}-\overline{\psi}_{(j,s),n}\right\}\right|\geq\epsilon \mid \mathcal{F}^W_n\right)\to 0 \;\; a.s.
$$

Note further that

$$
\sup_n E\left[\left|\Pr\left(\left|\sum_{1\leq(j,s)<(i,t)\leq m_n}\frac{A^2_{(i,t),(j,s)}\phi_{(i,t),n}\left\{\varepsilon_{(j,s)}u_{(j,s),n}-\overline{\psi}_{(j,s),n}\right\}}{K_{2,n}}\right|\geq\epsilon \mid \mathcal{F}^W_n\right)\right|^2\right]<\infty
$$

Hence, by a version of the dominated convergence theorem, as given by Theorem 25.12 of Billingsley

(1986), we have, as $n \to \infty$,

$$\begin{aligned}
&\Pr\left(\left|\frac{1}{K_{2,n}}\sum\nolimits_{1\leq(j,s)<(i,t)\leq m_n} A^2_{(i,t),(j,s)}\phi_{(i,t),n}\left\{\varepsilon_{(j,s),n}u_{(j,s),n}-\overline{\psi}_{(j,s),n}\right\}\right|\geq\epsilon\right)\\
=\;& E\left[\Pr\left(\left|\frac{1}{K_{2,n}}\sum_{1\leq(j,s)<(i,t)\leq m_n} A^2_{(i,t),(j,s)}\phi_{(i,t),n}\left\{\varepsilon_{(j,s),n}u_{(j,s),n}-\overline{\psi}_{(j,s),n}\right\}\right|\geq\epsilon \mid \mathcal{F}^W_n\right)\right]\to 0,
\end{aligned}$$

as required for part (a).

To show part (b), first let L be the lower triangular matrix such that $L_{(i,t),(j,s)} = A_{(i,t),(j,s)}\mathbb{I}\left\{(i,t)>(j,s)\right\}$, and define $D_{\overline{\psi}} = diag\left(\overline{\psi}_{(1,1),n},...,\overline{\psi}_{(n,T_n),n}\right) = diag\left(\overline{\psi}_1,...,\overline{\psi}_{m_n}\right)$, $D_\phi = diag\left(\phi_{(1,1)},...,\phi_{(n,T_n)}\right) = diag\left(\phi_1,...,\phi_{m_n}\right)$, $u = \left(u_{(1,1)},...,u_{(n,T_n)}\right)' = \left(u_1,...,u_{m_n}\right)'$, and $\varepsilon = \left(\varepsilon_{(1,1)},...,\varepsilon_{(n,T_n)}\right)' = \left(\varepsilon_1,...,\varepsilon_{m_n}\right)'$. It then follows by direct multiplication that

$$\begin{aligned}
&\varepsilon' L' D_\phi L u - tr\left\{L' D_\phi L D_{\overline{\psi}}\right\}\\
=\;& \sum_{1\leq(j,s)<(i,t)\leq m_n} A^2_{(i,t),(j,s)}\phi_{(i,t),n}\left\{u_{(j,s),n}\varepsilon_{(j,s),n}-\overline{\psi}_{(j,s),n}\right\}\\
&+\sum_{1\leq\ (k,v)<(j,s)<(i,t)\leq\ m_n} A_{(i,t),(j,s)}A_{(i,t),(k,v)}\phi_{(i,t),n}\left\{u_{(j,s),n}\varepsilon_{(k,v),n}+\varepsilon_{(j,s),n}u_{(k,v),n}\right\}
\end{aligned}$$

so that by making use of Loève's c_r inequality, we have that

$$\begin{aligned}
&E\left[\left(\sum_{\substack{1\leq\ (k,v)<(j,s)\\<(i,t)\leq\ m_n}}\frac{A_{(i,t),(j,s)}A_{(i,t),(k,v)}\phi_{(i,t),n}\left\{u_{(j,s),n}\varepsilon_{(k,v),n}+\varepsilon_{(j,s),n}u_{(k,v),n}\right\}}{K^2_{2,n}}\right)^2 \mid \mathcal{F}^W_n\right]\\
\leq\;& 2\frac{1}{K^2_{2,n}}E\left[\left(u'L'D_\phi L\varepsilon - tr\left\{L'D_\phi L D_{\overline{\psi}}\right\}\right)^2|\mathcal{F}^W_n\right]\\
&+2\frac{1}{K^2_{2,n}}E\left[\left(\sum_{1\leq(j,s)<(i,t)\leq m_n}A^2_{(i,t),(j,s)}\phi_{(i,t),n}\left\{u_{(j,s),n}\varepsilon_{(j,s),n}-\overline{\psi}_{(j,s),n}\right\}\right)^2|\mathcal{F}^W_n\right] \qquad (11)
\end{aligned}$$

From the proof of part (a), we already have

$$\frac{1}{K^2_{2,n}}E\left[\left(\sum\nolimits_{1\leq(j,s)<(i,t)\leq m_n}A^2_{(i,t),(j,s)}\phi_{(i,t),n}\left\{u_{(j,s),n}\varepsilon_{(j,s),n}-\overline{\psi}_{(j,s),n}\right\}\right)^2|\mathcal{F}^W_n\right] = o_{a.s.}(1).$$

To show that

$$\frac{1}{K^2_{2,n}}E\left[\left(u'L'D_\phi L\varepsilon - tr\left\{L'D_\phi L D_{\overline{\psi}}\right\}\right)^2|\mathcal{F}^W_n\right] = o_{a.s.}(1),$$

note first that

$$
\begin{aligned}
& \frac{1}{K_{2,n}^2} E\left[\left(u'L'D_\phi L\varepsilon - tr\left\{L'D_\phi LDD_{\overline{\psi}}\right\}\right)^2 \mid \mathcal{F}_n^W\right] \\
= & \frac{1}{K_{2,n}^2} E\left[\left(u'L'D_\phi L\varepsilon\right)^2 \mid \mathcal{F}_n^W\right] - \frac{1}{K_{2,n}^2}\left[tr\left\{L'D_\phi LD_{\overline{\psi}}\right\}\right]^2 \\
= & \frac{1}{K_{2,n}^2} E\left[u'L'D_\phi L\varepsilon \otimes u'L'D_\phi L\varepsilon \mid \mathcal{F}_n^W\right] - \frac{1}{K_{2,n}^2}\left[tr\left\{L'D_\phi LD_{\overline{\psi}}\right\}\right]^2 \\
= & \frac{1}{K_{2,n}^2} E\left[tr\left\{\left(u' \otimes u'\right)\left(L'D_\phi L \otimes L'D_\phi L\right)(\varepsilon \otimes \varepsilon)\right\} \mid \mathcal{F}_n^W\right] \\
& -\frac{1}{K_{2,n}^2}\left[tr\left\{L'D_\phi LD_{\overline{\psi}}\right\}\right]^2 \\
= & \frac{1}{K_{2,n}^2} tr\left\{\left(L'D_\phi L \otimes L'D_\phi L\right) E\left[\varepsilon u' \otimes \varepsilon u' | \mathcal{F}_n^W\right]\right\} - \frac{1}{K_{2,n}^2}\left[tr\left\{L'D_\phi LD_{\overline{\psi}}\right\}\right]^2 .
\end{aligned} \tag{12}
$$

Next, by straightforward calculations, we obtain

$$
\begin{aligned}
& E\left[\varepsilon u' \otimes \varepsilon u' | \mathcal{F}_n^W\right] \\
= & \begin{pmatrix}
\sigma^2_{(1,1),n}\overline{\omega}^2_{(1,1),n} e_{(1,1)} e'_{(1,1)} & \cdots & \sigma^2_{(1,1),n}\overline{\omega}^2_{(n,T_n),n} e_{(1,1)} e'_{(n,T_n)} \\
\vdots & & \vdots \\
\sigma^2_{(1,T_1),n}\overline{\omega}^2_{(1,1),n} e_{(1,T_1)} e'_{(1,1)} & \cdots & \sigma^2_{(1,T_1),n}\overline{\omega}^2_{(n,T_n),n} e_{(1,T_1)} e'_{(n,T_n)} \\
\vdots & & \vdots \\
\sigma^2_{(n,1),n}\overline{\omega}^2_{(1,1),n} e_{(n,1)} e'_{(1,1)} & \cdots & \sigma^2_{(n,1),n}\overline{\omega}^2_{(n,T_n),n} e_{(n,1)} e'_{(n,T_n)} \\
\vdots & & \vdots \\
\sigma^2_{(n,T_n),n}\overline{\omega}^2_{(1,1),n} e_{(n,T_n)} e'_{(1,1)} & \cdots & \sigma^2_{(n,T_n),n}\overline{\omega}^2_{(n,T_n),n} e_{(n,T_n)} e'_{(n,T_n)}
\end{pmatrix} \\
& + \begin{pmatrix}
\overline{\psi}^2_{(1,1),n} e_{(1,1)} e'_{(1,1)} & \cdots & \overline{\psi}_{(1,1),n}\overline{\psi}_{(n,T_n),n} e_{(n,T_n)} e'_{(1,1)} \\
\vdots & & \vdots \\
\overline{\psi}_{(1,T_1),n}\overline{\psi}_{(1,1),n} e_{(1,1)} e'_{(1,T_1)} & \cdots & \overline{\psi}_{(1,T_1),n}\overline{\psi}_{(n,T_n),n} e_{(n,T_n)} e'_{(1,T_1)} \\
\vdots & & \vdots \\
\overline{\psi}_{(n,1),n}\overline{\psi}_{(1,1),n} e_{(1,1)} e'_{(n,1)} & \cdots & \overline{\psi}_{(n,1),n}\overline{\psi}_{(n,T_n),n} e_{(n,T_n)} e'_{(n,1)} \\
\vdots & & \vdots \\
\overline{\psi}_{(n,T_n),n}\overline{\psi}_{(1,1),n} e_{(1,1)} e'_{(n,T_n)} & \cdots & \overline{\psi}_{(n,T_n),n}\overline{\psi}_{(n,T_n),n} e_{(n,T_n)} e'_{(n,T_n)}
\end{pmatrix} \\
& + \begin{pmatrix}
\overline{\varkappa}_{(1,1),n} e_{(1,1)} e'_{(1,1)} & \underset{m_n\times m_n}{0} & \cdots & \cdots & \underset{m_n\times m_n}{0} \\
\underset{m_n\times m_n}{0} & \ddots & \ddots & & \vdots \\
\vdots & \ddots & \overline{\varkappa}_{(1,T_1),n} e_{(1,T_1)} e'_{(1,T_1)} & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \underset{m_n\times m_n}{0} \\
\underset{m_n\times m_n}{0} & \underset{m_n\times m_n}{0} & \cdots & \underset{m_n\times m_n}{0} & \overline{\varkappa}_{(n,T_n),n} e_{(n,T_n)} e'_{(n,T_n)}
\end{pmatrix} \\
& + \begin{pmatrix}
\overline{\psi}_{(1,1),n} \otimes D_{\overline{\psi}} & \underset{m_n\times m_n}{0} & \cdots & \cdots & \underset{m_n\times m_n}{0} \\
\underset{m_n\times m_n}{0} & \ddots & \ddots & & \vdots \\
\vdots & \ddots & \overline{\psi}_{(1,T_1),n} \otimes D_{\overline{\psi}} & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \underset{m_n\times m_n}{0} \\
\underset{m_n\times m_n}{0} & \underset{m_n\times m_n}{0} & \cdots & \underset{m_n\times m_n}{0} & \overline{\psi}_{(n,T_n),n} \otimes D_{\overline{\psi}}
\end{pmatrix} \\
= & \left(D_\sigma \otimes I_{m_n}\right) vec\left(I_{m_n}\right) vec\left(I_{m_n}\right) \left(D_{\overline{\omega}} \otimes I_{m_n}\right) \\
& + \left(D_\psi \otimes I_{m_n}\right) \underline{K}_{m_n m_n} \left(D_\psi \otimes I_{m_n}\right) + \underline{E}' D_{\overline{\varkappa}} \underline{E} + \left(D_{\overline{\psi}} \otimes D_{\overline{\psi}}\right), \qquad (13)
\end{aligned}
$$

where $\underline{K}_{m_n m_n}$ is an $m_n^2 \times m_n^2$ commutation matrix such that for any $m_n \times m_n$ matrix A, $\underline{K}_{m_n m_n} vec\,(A) = vec\,(A')$. Also, here, $D_{\overline{\psi}} = diag\left(\overline{\psi}_1, ..., \overline{\psi}_{m_n}\right)$, $D_\sigma = diag\left(\sigma_1^2, ..., \sigma_{m_n}^2\right)$, $D_{\overline{\omega}} = diag\left(\overline{\omega}_1^2, ..., \overline{\omega}_{m_n}^2\right)$, $D_{\overline{\varkappa}} = diag\left(\overline{\varkappa}_1, ..., \overline{\varkappa}_{m_n}\right)$ with $\overline{\varkappa}_{(i,t),n} = E\left[\varepsilon^2_{(i,t),n} u^2_{(i,t),n} | \mathcal{F}_n^W\right] - \sigma^2_{(i,t),n}\overline{\omega}^2_{(i,t),n} - 2\overline{\psi}^2_{(i,t),n}$ for $(i,t) =$

$1, ..., m_n$. Also, let
$\underline{E} = (e_{1,m_n} \otimes e_{1,m_n}, \cdots, e_{m_n,m_n} \otimes e_{m_n,m_n})'$; and e_{i,m_n} is the i^{th} column of an $m_n \times m_n$ identity matrix . It follows from (12) and (13) that

$$
\begin{aligned}
& \frac{1}{K_{2,n}^2} E\left[\left(u'L'D_\phi L\varepsilon - tr\left\{L'D_\phi LD_{\overline{\psi}}\right\}\right)^2 \mid \mathcal{F}_n^W\right] \\
= \; & \frac{1}{K_{2,n}^2} tr\left\{\left(L'D_\phi L \otimes L'D_\phi L\right) E\left[\varepsilon u' \otimes \varepsilon u' | \mathcal{F}_n^W\right]\right\} - \frac{1}{K_{2,n}^2}\left[tr\left\{L'D_\phi LD_{\overline{\psi}}\right\}\right]^2 \\
= \; & \frac{1}{K_{2,n}^2} tr\left\{\left(L'D_\phi L \otimes L'D_\phi L\right)(D_\sigma \otimes I_{m_n}) vec(I_{m_n}) vec(I_{m_n})'(D_{\overline{\varpi}} \otimes I_{m_n})\right\} \\
& + \frac{1}{K_{2,n}^2} tr\left\{\left(L'D_\phi L \otimes L'D_\phi L\right)(D_\psi \otimes I_{m_n}) \underline{K}_{m_n m_n}\left(D_{\overline{\psi}} \otimes I_{m_n}\right)\right\} \\
& + \frac{1}{K_{2,n}^2} tr\left\{\left(L'D_\phi L \otimes L'D_\phi L\right) \underline{E}' D_{\overline{\varkappa}} \underline{E}\right\} + \frac{1}{K_{2,n}^2} tr\left\{\left(L'D_\phi L \otimes L'D_\phi L\right)\left(D_{\overline{\psi}} \otimes D_{\overline{\psi}}\right)\right\} \\
& - \frac{1}{K_{2,n}^2}\left[tr\left\{L'D_\phi LD_{\overline{\psi}}\right\}\right]^2 \\
= \; & \frac{1}{K_{2,n}^2} vec(I_{m_n})'\left(D_{\overline{\varpi}} L'D_\phi LD_\sigma \otimes L'D_\phi L\right) vec(I_{m_n}) \\
& + \frac{1}{K_{2,n}^2} tr\left\{\left(D_\psi L'D_\phi LD_{\overline{\psi}} \otimes L'D_\phi L\right) \underline{K}_{m_n m_n}\right\} + \frac{1}{K_{2,n}^2} tr\left\{\left(L'D_\phi L \otimes L'D_\phi L\right) \underline{E}' D_{\overline{\varkappa}} \underline{E}\right\} \\
& + \frac{1}{K_{2,n}^2} tr\left\{\left(L'D_\phi LD_{\overline{\psi}} \otimes L'D_\phi LD_{\overline{\psi}}\right)\right\} - \frac{1}{K_{2,n}^2}\left[tr\left\{L'D_\phi LD_{\overline{\psi}}\right\}\right]^2 \\
= \; & \frac{1}{K_{2,n}^2} tr\left\{L'D_\phi LD_{\overline{\varpi}} L'D_\phi LD_\sigma\right\} + \frac{1}{K_{2,n}^2} tr\left\{\left(D_\psi L'D_\phi LD_{\overline{\psi}} \otimes L'D_\phi L\right) \underline{K}_{m_n m_n}\right\} \\
& + \frac{1}{K_{2,n}^2} tr\left\{\left(L'D_\phi L \otimes L'D_\phi L\right) \underline{E}' D_{\overline{\varkappa}} \underline{E}\right\} + \frac{1}{K_{2,n}^2}\left[tr\left\{L'D_\phi LD_{\overline{\psi}}\right\}\right]^2 - \frac{1}{K_{2,n}^2}\left[tr\left\{L'D_\phi LD_{\overline{\psi}}\right\}\right]^2 \\
= \; & \frac{1}{K_{2,n}^2} tr\left\{L'D_\phi LD_{\overline{\varpi}} L'D_\phi LD_\sigma\right\} + \frac{1}{K_{2,n}^2} tr\left\{\left(D_{\overline{\psi}} L'D_\phi LD_{\overline{\psi}} \otimes L'D_\phi L\right) \underline{K}_{m_n m_n}\right\} \\
& + \frac{1}{K_{2,n}^2} tr\left\{\left(L'D_\phi L \otimes L'D_\phi L\right) \underline{E}' D_{\overline{\varkappa}} \underline{E}\right\} \qquad (14)
\end{aligned}
$$

Focusing first on the first term of (14), we get

$$\begin{aligned}
& \frac{1}{K_{2,n}^2} tr\left\{L' D_\phi L D_{\overline{\varpi}} L' D_\phi L D_\sigma\right\} \\
\leq\ & \sqrt{\frac{1}{K_{2,n}^2} tr\left\{L' D_\phi L D_{\overline{\varpi}}^2 L' D_\phi L\right\}} \sqrt{\frac{1}{K_{2,n}^2} tr\left\{L' D_\phi L D_\sigma^2 L' D_\phi L\right\}} \\
\leq\ & \sqrt{\max_{1 \leq (i,t) \leq m_n} \overline{\varpi}_{(i,t),n}^4} \sqrt{\max_{1 \leq (i,t) \leq m_n} \sigma_{(i,t),n}^4} \frac{1}{K_{2,n}^2} tr\left\{L' D_\phi L L' D_\phi L\right\} \\
\leq\ & \sqrt{\max_{1 \leq (i,t) \leq m_n} \overline{\varpi}_{(i,t),n}^4} \sqrt{\max_{1 \leq (i,t) \leq m_n} \sigma_{(i,t),n}^4} \sqrt{\frac{1}{K_{2,n}^2} tr\left\{D_\phi L L' D_\phi^2 L L' D_\phi\right\}} \sqrt{\frac{1}{K_{2,n}^2} tr\left\{L L' L L'\right\}} \\
\leq\ & \sqrt{\max_{1 \leq (i,t) \leq m_n} \overline{\varpi}_{(i,t),n}^4} \sqrt{\max_{1 \leq (i,t) \leq m_n} \sigma_{(i,t),n}^4} \sqrt{\max_{1 \leq (i,t) \leq m_n} \phi_{(i,t),n}^2} \\
& \times \sqrt{\frac{1}{K_{2,n}^2} tr\left\{L L' D_\phi^2 L L'\right\}} \sqrt{\frac{1}{K_{2,n}^2} tr\left\{L L' L L'\right\}} \\
\leq\ & \sqrt{\max_{1 \leq (i,t) \leq m_n} \overline{\varpi}_{(i,t),n}^4} \sqrt{\max_{1 \leq (i,t) \leq m_n} \sigma_{(i,t),n}^4} \left(\max_{1 \leq (i,t) \leq m_n} \phi_{(i,t),n}^2\right) \frac{1}{K_{2,n}^2} tr\left\{L' L L' L\right\} \\
\leq\ & \sqrt{\max_{1 \leq (i,t) \leq m_n} E\left(u_{(i,t),n}^4 | \mathcal{F}_n^W\right)} \sqrt{\max_{1 \leq (i,t) \leq m_n} E\left(\varepsilon_{(i,t),n}^4 | \mathcal{F}_n^W\right)} \left(\max_{1 \leq (i,t) \leq m_n} \phi_{(i,t),n}^2\right) \frac{1}{K_{2,n}^2} tr\left\{L' L L' L\right\} \\
\leq\ & C \frac{1}{K_{2,n}^2} tr\left\{L' L L' L\right\} = \frac{C}{K_{2,n}^2} \left\|L L'\right\|_F^2 \quad a.s., \qquad (15)
\end{aligned}$$

where the first and third inequalities follow from CS and where the sixth inequality follows from the conditional version of the Jensen's inequality and the last inequality follows in light of the assumptions of this lemma. In addition, let G be an $m_n \times m_n$ matrix and $D = diag\left(d_1, ..., d_{m_n}\right)$ such that $d_{(i,t)} \geq 0$ for all $(i,t) \in \{1, ..., m_n\}$, and note that the second and fourth inequalities in (15) above follows from the inequality

$$tr\left\{G' D G\right\} \leq \left\{\max_{(i,t)} \left(d_{(i,t)}\right)\right\} tr\left(G' G\right). \qquad (16)$$

Turning our attention now to the second term of (14), we see that, using the identity $tr\left\{(A \otimes B) \underline{K}_{m_n m_n}\right\} = tr\left\{AB\right\}$ for $m_n \times m_n$ matrices A and B,

$$\begin{aligned}
& \frac{1}{K_{2,n}^2} tr\left\{\left(D_{\overline{\psi}} L' D_\phi L D_{\overline{\psi}} \otimes L' D_\phi L\right) \underline{K}_{m_n m_n}\right\} \\
=\ & \frac{1}{K_{2,n}^2} tr\left\{D_{\overline{\psi}} L' D_\phi L D_{\overline{\psi}} L' D_\phi L\right\} = \frac{1}{K_{2,n}^2} tr\left\{L' D_\phi L D_{\overline{\psi}} L' D_\phi L D_{\overline{\psi}}\right\}.
\end{aligned}$$

It follows by calculations similar to that used to obtain (15), we obtain

$$
\begin{aligned}
& \frac{1}{K_{2,n}^2} tr \left\{ \left(D_{\overline{\psi}} L' D_\phi L D_{\overline{\psi}} \otimes L' D_\phi L \right) \underline{K}_{m_n m_n} \right\} \\
\leq \ & \left(\max_{1 \leq (i,t) \leq m_n} \overline{\psi}_{(i,t),n}^2 \right) \left(\max_{1 \leq (i,t) \leq m_n} \phi_{(i,t),n}^2 \right) \frac{1}{K_{2,n}^2} tr \left\{ L' L L' L \right\} \\
\leq \ & C \frac{1}{K_{2,n}^2} tr \left\{ L' L L' L \right\} = \frac{C}{K_{2,n}^2} \left\| L L' \right\|_F^2 \quad a.s.
\end{aligned}
\tag{17}
$$

Finally, to analyze the third term of (14), we note that

$$
\begin{aligned}
& \frac{1}{K_{2,n}^2} \left| tr \left\{ \left(L' D_\phi L \otimes L' D_\phi L \right) \underline{E}' D_{\overline{\varkappa}} \underline{E} \right\} \right| \\
\leq \ & \frac{1}{K_{2,n}^2} \sum_{(i,t)=1}^{m_n} \left| \overline{\varkappa}_{(i,t),n} \right| tr \left\{ e_{(i,t)}' L' D_\phi L e_{(i,t)} L' D_\phi L e_{(i,t)} e_{(i,t)}' \right\} \\
= \ & \frac{1}{K_{2,n}^2} \sum_{(i,t)=1}^{m_n} \left| \overline{\varkappa}_{(i,t),n} \right| \left(e_{(i,t)}' L' D_\phi L e_{(i,t)} \right)^2 \\
\leq \ & \frac{1}{K_{2,n}^2} \sum_{(i,t)=1}^{m_n} \left| \overline{\varkappa}_{(i,t),n} \right| \left(e_{(i,t)}' L' D_\phi^2 L e_{(i,t)} \right) \left(e_{(i,t)}' L' L e_{(i,t)} \right) \\
\leq \ & \left(\max_{1 \leq (i,t) \leq m_n} \phi_{(i,t),n}^2 \right) \frac{1}{K_{2,n}^2} \sum_{(i,t)=1}^{m_n} \left| \overline{\varkappa}_{(i,t),n} \right| \left(e_{(i,t)}' L' L e_{(i,t)} \right)^2 \\
\leq \ & \left\{ \sqrt{E \left[\varepsilon_{(i,t),n}^4 | \mathcal{F}_n^W \right]} \sqrt{E \left[u_{(i,t),n}^4 | \mathcal{F}_n^W \right]} + \left(\max_{1 \leq (i,t) \leq m_n} \sigma_{(i,t),n}^2 \right) \left(\max_{1 \leq (i,t) \leq m_n} \overline{\omega}_{(i,t),n}^2 \right) \right. \\
& \left. + 2 \left(\max_{1 \leq (i,t) \leq m_n} \overline{\psi}_{(i,t),n}^2 \right) \right\} \left(\max_{1 \leq (i,t) \leq m_n} \phi_{(i,t),n}^2 \right) \frac{1}{K_{2,n}^2} \sum_{(i,t)=1}^{m_n} \left(e_{(i,t)}' L' L e_{(i,t)} \right)^2 \\
\leq \ & C \frac{1}{K_{2,n}^2} \sum_{(i,t)=1}^{m_n} \left(e_{(i,t)}' L' L e_{(i,t)} \right)^2 \leq C \frac{1}{K_{2,n}^2} \sum_{(i,t)=1}^{m_n} \left(e_{(i,t)}' A' A e_{(i,t)} \right)^2 \\
= \ & C \frac{1}{K_{2,n}^2} \sum_{(i,t)=1}^{m_n} \left(e_{(i,t)}' \left[P^\perp - M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} \right]^2 e_{(i,t)} \right)^2 \\
= \ & C \frac{1}{K_{2,n}^2} \sum_{(i,t)=1}^{m_n} \left(e_{(i,t)}' P^\perp e_{(i,t)} + e_{(i,t)}' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} e_{(i,t)} \right)^2
\end{aligned}
$$

where the first inequality above follows from T, the second inequality follows from CS, the third inequality makes use of (16), the fourth inequality uses CS and T, and the fifth inequality stems from our assumption about the (almost sure) boundedness of the conditional moments. Applying the inequality $\left|\sum_{i=1}^{m} a_i\right|^r \leq m^{r-1}\sum_{i=1}^{m}|a_i|^r$ for $r \geq 1$, we further obtain

$$\begin{aligned}
&\frac{1}{K_{2,n}^2}\left|tr\left\{\left(L'D_\phi L \otimes L'D_\phi L\right)\underline{E}'D_{\overline{\varkappa}}\underline{E}\right\}\right| \\
\leq\ & 2C\left[\frac{1}{K_{2,n}^2}\sum_{(i,t)=1}^{m_n}\left(P^{\perp}_{(i,t),(i,t)}\right)^2 + \frac{1}{K_{2,n}^2}\sum_{(i,t)=1}^{m_n}\left(e'_{(i,t)}M^{(Z,Q)}D_{\widehat{\vartheta}}M^{(Z,Q)}D_{\widehat{\vartheta}}M^{(Z,Q)}e_{(i,t)}\right)^2\right] \\
\leq\ & 2C\left[\frac{1}{K_{2,n}^2}\sum_{(i,t)=1}^{m_n}\left(P^{\perp}_{(i,t),(i,t)}\right)^2 + \left(\max_{1\leq(i,t)\leq m_n}\left|\widehat{\vartheta}_{(i,t)}\right|^2\right)\frac{1}{K_{2,n}^2}tr\left\{D_{\widehat{\vartheta}}M^{(Z,Q)}D_{\widehat{\vartheta}}\right\}\right] \\
\leq\ & 2C\left[1+\left(\frac{1}{C_*}\right)^2\right]\left(\max_{1\leq(i,t)\leq m_n}P^{\perp}_{(i,t),(i,t)}\right)\left[\frac{1}{K_{2,n}^2}\sum_{(i,t)=1}^{m_n}P^{\perp}_{(i,t),(i,t)} + \frac{1}{K_{2,n}^2}tr\left\{D^2_{\widehat{\vartheta}}\right\}\right] \\
=\ & O_{a.s.}\left(\frac{1}{n}\right) \qquad (18)
\end{aligned}$$

where we have used Assumption 5(iv) and part (a) of Lemma OA-1 in arriving at the last line above.

In light of (14), it follows from (15), (17), (18), and Lemma OA-12 (given in section 3 below) that

$$\frac{1}{K_{2,n}^2}E\left[\left(u'L'D_\phi L\varepsilon - tr\left\{L'D_\phi LD_{\overline{\psi}}\right\}\right)^2 \mid \mathcal{F}_n^W\right] \leq 2C\left(1/K_{2,n}^2\right)\left\|LL'\right\|_F^2 + C\left(1/K_{2,n}\right) \leq C/K_{2,n} \quad a.s.$$

It follows from (11) that

$$\begin{aligned}
&E\left[\left(\sum_{1\leq\ (k,v)<(j,s)<(i,t)\leq\ m_n}\frac{A_{(i,t),(j,s)}A_{(i,t),(k,v)}\phi_{(i,t),n}\left\{u_{(j,s),n}\varepsilon_{(k,v),n}+\varepsilon_{(j,s),n}u_{(k,v),n}\right\}}{K_{2,n}^2}\right)^2 \mid \mathcal{F}_n^W\right] \\
=\ & o_{a.s.}(1)
\end{aligned}$$

Moreover, by the conditional version of the Markov's inequality, we deduce for any $\epsilon > 0$

$$\begin{aligned}
&\Pr\left(\left|\sum_{1\leq\ (k,v)<(j,s)<(i,t)\leq\ m_n}\frac{A_{(i,t),(j,s)}A_{(i,t),(k,v)}\phi_{(i,t),n}\left\{u_{(j,s),n}\varepsilon_{(k,v),n}+\varepsilon_{(j,s)}u_{(k,v),n}\right\}}{K_{2,n}}\right| \geq \epsilon \mid \mathcal{F}_n^W\right) \\
\to\ & 0 \quad a.s.
\end{aligned}$$

Since

$$\begin{aligned}
&\sup_n E\left[\left|\Pr\left(\left|\sum_{\substack{1\leq\ (k,v)<(j,s)\\ <(i,t)\leq\ m_n}}\frac{A_{(i,t),(j,s)}A_{(i,t),(k,v)}\phi_{(i,t),n}\left\{u_{(j,s),n}\varepsilon_{(k,v),n}+\varepsilon_{(j,s),n}u_{(k,v),n}\right\}}{K_{2,n}}\right| \geq \epsilon \mid \mathcal{F}_n^W\right)\right|^2\right] \\
<\ & \infty,
\end{aligned}$$

it further follows by a version of the dominated convergence theorem, as given by Theorem 25.12 of Billingsley (1986), that as $n \to \infty$

$$
\begin{aligned}
& \Pr\left(\left|\sum_{1\leq\ (k,v)<(j,s)<(i,t)\leq\ m_n} \frac{A_{(i,t),(j,s)}A_{(i,t),(k,v)}\phi_{(i,t),n}\left\{u_{(j,s),n}\varepsilon_{(k,v),n}+\varepsilon_{(j,s),n}u_{(k,v),n}\right\}}{K_{2,n}}\right| \geq \epsilon\right) \\
= \;& E\left[\Pr\left(\left|\sum_{1\leq\ (k,v)<(j,s)<(i,t)\leq\ m_n} \frac{A_{(i,t),(j,s)}A_{(i,t),(k,v)}\phi_{(i,t),n}\left\{u_{(j,s),n}\varepsilon_{(k,v),n}+\varepsilon_{(j,s),n}u_{(k,v),n}\right\}}{K_{2,n}}\right| \geq \epsilon \mid \mathcal{F}_n^W\right)\right] \\
\to \;& 0,
\end{aligned}
$$

as required for part (b).

It is easily seen that parts (c) and (d) can be proved in essentially the same way as part (b); hence, to avoid redundancy, we do not provide detailed arguments here. □

Lemma S2-9: Let

$$
\widehat{\Delta}\left(\delta_0\right) = -\frac{\left(y - X\delta_0\right)' M^{(Z_1,Q)}\left(y - X\delta_0\right)}{2} \frac{\partial}{\partial\delta}\left\{\frac{\left(y - X\delta\right)' A\left(y - X\delta\right)}{\left(y - X\delta\right)' M^{(Z_1,Q)}\left(y - X\delta\right)}\right\}\Bigg|_{\delta=\delta_0}.
$$

Suppose that Assumptions 1-8 hold; then, $D_\mu^{-1}\widehat{\Delta}\left(\delta_0\right) = \Gamma' M^{(Z_1,Q)}\varepsilon/\sqrt{n} + D_\mu^{-1}\underline{U}' A\varepsilon + o_p\left(1\right)$, where $\underline{U} = U - \varepsilon\rho'$ and where $\rho = \lim_{n\to\infty} E\left[U' M^Q \varepsilon\right]/E\left[\varepsilon' M^Q \varepsilon\right]$.

Proof of Lemma S2-9: To proceed, note first that we can write

$$
\begin{aligned}
& \widehat{\Delta}\left(\delta_0\right) \\
= \;& -\frac{\left(y - X\delta_0\right)' M^{(Z_1,Q)}\left(y - X\delta_0\right)}{2} \frac{\partial}{\partial\delta}\left\{\frac{\left(y - X\delta\right)' A\left(y - X\delta\right)}{\left(y - X\delta\right)' M^{(Z_1,Q)}\left(y - X\delta\right)}\right\}\Bigg|_{\delta=\delta_0} \\
= \;& -\frac{\left(y - X\delta_0\right)' M^{(Z_1,Q)}\left(y - X\delta_0\right)}{2} \\
& \times\left\{\frac{-2X' A\left(y - X\delta_0\right)}{\left(y - X\delta_0\right)' M^{(Z_1,Q)}\left(y - X\delta_0\right)} + \frac{\left(y - X\delta_0\right)' A\left(y - X\delta_0\right) 2X' M^{(Z_1,Q)}\left(y - X\delta_0\right)}{\left[\left(y - X\delta_0\right)' M^{(Z_1,Q)}\left(y - X\delta_0\right)\right]^2}\right\} \\
= \;& X' A\left(y - X\delta_0\right) - \frac{\left(y - X\delta_0\right)' A\left(y - X\delta_0\right)}{\left(y - X\delta_0\right)' M^{(Z_1,Q)}\left(y - X\delta_0\right)} X' M^{(Z_1,Q)}\left(y - X\delta_0\right) \\
= \;& X' A\left(y - X\delta_0\right) - \widehat{\ell}\left(\delta_0\right) X' M^{(Z_1,Q)}\left(y - X\delta_0\right)
\end{aligned}
$$

where $A = P^{\perp} - M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}$. It follows from making use of the fact that $\rho_n = E\left[U' M^{(Z_1,Q)}\varepsilon\right]/E\left[\varepsilon' M^{(Z_1,Q)}\varepsilon\right]$ and from applying parts (a), (b), and (e) of Lemma OA-6 and part (c) of Lemma OA-5 that

$$
\begin{aligned}
& D_\mu^{-1} \widehat{\Delta}\left(\delta_0\right) \\
= \;& D_\mu^{-1}\left[X' A\left(y-X \delta_0\right)-\widehat{\ell}\left(\delta_0\right) X' M^{\left(Z_1, Q\right)}\left(y-X \delta_0\right)\right] \\
= \;& D_\mu^{-1} X' A\left(y-X \delta_0\right)-\widehat{\ell}\left(\delta_0\right) D_\mu^{-1} X' M^{\left(Z_1, Q\right)}\left(y-X \delta_0\right) \\
= \;& \frac{\Gamma' M^{\left(Z_1, Q\right)} \varepsilon}{\sqrt{n}}+D_\mu^{-1} U' A \varepsilon+O_p\left(\frac{1}{K_n^{\rho_\gamma}}\right)+O_p\left(\frac{\kappa_n^{\max }}{\mu_n^{\min }} \frac{1}{K_{1, n}^{\varrho_f}}\right)+O_p\left(\frac{\tau_n}{K_{1, n}^{\varrho_g}}\right) \\
& -\frac{\varepsilon' A \varepsilon}{\varepsilon' M^Q \varepsilon}\left[1+O_p\left(\max \left\{\frac{\tau_n}{\sqrt{K_{2, n}} K_{1, n}^{\varrho_g}}, \frac{K_{1, n}}{n}\right\}\right)\right] \\
& \times\left[\frac{\Gamma' M^{\left(Z_1, Q\right)} \varepsilon}{\sqrt{n}}+D_\mu^{-1} U' M^{\left(Z_1, Q\right)} \varepsilon+O_p\left(\frac{\kappa_n^{\max }}{\mu_n^{\min }} \frac{1}{K_{1, n}^{\varrho_f}}\right)+O_p\left(\frac{\tau_n}{K_{1, n}^{\varrho_g}}\right)\right] \\
= \;& \frac{\Gamma' M^{\left(Z_1, Q\right)} \varepsilon}{\sqrt{n}}+D_\mu^{-1}\left(U-\varepsilon \rho'\right)' A \varepsilon \\
& -\varepsilon' A \varepsilon D_\mu^{-1}\left[\frac{U' M^{\left(Z_1, Q\right)} \varepsilon}{\varepsilon' M^Q \varepsilon}-\rho\right]\left[1+O_p\left(\max \left\{\frac{\tau_n}{\sqrt{K_{2, n}} K_{1, n}^{\varrho_g}}, \frac{K_{1, n}}{n}\right\}\right)\right] \\
& -\frac{\sqrt{K_{2, n}}}{n} \frac{\varepsilon' A \varepsilon / \sqrt{K_{2, n}}}{\varepsilon' M^Q \varepsilon / n} \frac{\Gamma' M^{\left(Z_1, Q\right)} \varepsilon}{\sqrt{n}}+O_p\left(\frac{1}{K_{2, n}^{\varrho_\gamma}}\right)+O_p\left(\frac{\kappa_n^{\max }}{\mu_n^{\min }} \frac{1}{K_{1, n}^{\varrho_f}}\right)+O_p\left(\frac{\tau_n}{K_{1, n}^{\varrho_g}}\right) \\
& +O_p\left(\max \left\{\frac{\tau_n}{n K_{1, n}^{\varrho_g}}, \frac{K_{1, n} \sqrt{K_{2, n}}}{n^2}\right\}\right)+O_p\left(\max \left\{\frac{\tau_n}{\left(\mu_n^{\min }\right) K_{1, n}^{\varrho_g}}, \frac{K_{1, n} \sqrt{K_{2, n}}}{\left(\mu_n^{\min }\right) n}\right\}\right) \\
& +O_p\left(\frac{\kappa_n^{\max }}{\mu_n^{\min }} \frac{\sqrt{K_{2, n}}}{n K_{1, n}^{\varrho_f}}\right)+O_p\left(\frac{\tau_n \sqrt{K_{2, n}}}{n K_{1, n}^{\varrho_g}}\right) \\
= \;& \frac{\Gamma' M^{\left(Z_1, Q\right)} \varepsilon}{\sqrt{n}}+D_\mu^{-1}\left(U-\varepsilon \rho'\right)' A \varepsilon \\
& -\varepsilon' A \varepsilon D_\mu^{-1}\left[\frac{U' M^{\left(Z_1, Q\right)} \varepsilon}{\varepsilon' M^Q \varepsilon}-\rho\right]\left[1+O_p\left(\max \left\{\frac{\tau_n}{\sqrt{K_{2, n}} K_{1, n}^{\varrho_g}}, \frac{K_{1, n}}{n}\right\}\right)\right] \\
& +O_p\left(\frac{\sqrt{K_{2, n}}}{n}\right)+O_p\left(\frac{1}{K_{2, n}^{\varrho_\gamma}}\right)+O_p\left(\frac{\kappa_n^{\max }}{\mu_n^{\min }} \frac{1}{K_{1, n}^{\varrho_f}}\right)+O_p\left(\frac{\tau_n}{K_{1, n}^{\varrho_g}}\right) \\
& +O_p\left(\max \left\{\frac{\tau_n}{\left(\mu_n^{\min }\right) K_{1, n}^{\varrho_g}}, \frac{K_{1, n} \sqrt{K_{2, n}}}{\left(\mu_n^{\min }\right) n}\right\}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\Gamma' M^{(Z_1,Q)}\varepsilon}{\sqrt{n}} + D_\mu^{-1}\left(U - \varepsilon\rho'\right)' A\varepsilon + O_p\left(\frac{1}{\mu_n^{\min}} \max\left\{\sqrt{\frac{K_{2,n}}{n}}, \frac{K_{1,n}\sqrt{K_{2,n}}}{n}\right\}\right) \\
&\quad + O_p\left(\frac{\sqrt{K_{2,n}}}{n}\right) + O_p\left(\frac{1}{K_{2,n}^{\varrho_\gamma}}\right) + O_p\left(\frac{\kappa_n^{\max}}{\mu_n^{\min}}\frac{1}{K_{1,n}^{\varrho_f}}\right) \\
&\quad + O_p\left(\frac{\tau_n}{K_{1,n}^{\varrho_g}}\right) + O_p\left(\max\left\{\frac{\tau_n}{\left(\mu_n^{\min}\right) K_{1,n}^{\varrho_g}}, \frac{K_{1,n}\sqrt{K_{2,n}}}{\left(\mu_n^{\min}\right) n}\right\}\right) \\
&= \frac{\Gamma' M^{(Z_1,Q)}\varepsilon}{\sqrt{n}} + D_\mu^{-1}\underline{U}' A\varepsilon \\
&\quad + O_p\left(\max\left\{\frac{1}{\mu_n^{\min}}\sqrt{\frac{K_{2,n}}{n}}, \frac{K_{1,n}\sqrt{K_{2,n}}}{\left(\mu_n^{\min}\right) n}, \frac{1}{K_{2,n}^{\varrho_\gamma}}, \frac{\kappa_n^{\max}}{\mu_n^{\min}}\frac{1}{K_{1,n}^{\varrho_f}}, \frac{\tau_n}{K_{1,n}^{\varrho_g}}\right\}\right). \\
&= \frac{\Gamma' M^{(Z_1,Q)}\varepsilon}{\sqrt{n}} + D_\mu^{-1}\underline{U}' A\varepsilon + o_p(1). \ \square
\end{aligned}
$$

Lemma S2-10: Suppose that Assumptions 1-7 are satisfied. Let $\overline{\delta}_n$ be any estimator such that, as $n \to \infty$, $D_\mu\left(\overline{\delta}_n - \delta_0\right)/\mu_n^{\min} = o_p(1)$. Then, $-D_\mu^{-1}\left(\partial\widehat{\Delta}\left(\overline{\delta}_n\right)/\partial\delta'\right) D_\mu^{-1} = H_n + o_p(1)$, where $H_n = \Gamma' M^{(Z_1,Q)}\Gamma/n$ and where

$$
\begin{aligned}
\widehat{\Delta}(\delta) &= -\left[(y - X\delta)' M^{(Z_1,Q)}(y - X\delta)/2\right]\left[\partial\widehat{Q}_{FELIM}(\delta)/\partial\delta\right] \\
&= X'A(y - X\delta) - \widehat{\ell}(\delta) X' M^{(Z_1,Q)}(y - X\delta),
\end{aligned}
$$

with

$$
\widehat{\ell}(\delta) = (y - X\delta)' A (y - X\delta) / \left[(y - X\delta)' M^{(Z_1,Q)}(y - X\delta)\right].
$$

In addition, we also have

$$
D_\mu^{-1} X'\left[A - \widehat{\ell}\left(\overline{\delta}_n\right) M^{(Z_1,Q)}\right] X D_\mu^{-1} = H_n + o_p(1). \tag{19}
$$

Proof of Lemma S2-10:

Taking derivative of $-\widehat{\Delta}(\delta)$ with respect to δ, we obtain

$$
\begin{aligned}
& -\frac{\partial \widehat{\Delta}(\delta)}{\partial \delta'} \\
= \; & X'AX - \frac{(y - X\delta)' A (y - X\delta)}{(y - X\delta)' M^{(Z_1,Q)} (y - X\delta)} X' M^{(Z_1,Q)} X - \frac{2X' M^{(Z_1,Q)} (y - X\delta)(y - X\delta)' AX}{(y - X\delta)' M^{(Z_1,Q)} (y - X\delta)} \\
& + 2X' M^{(Z_1,Q)} (y - X\delta)(y - X\delta)' M^{(Z_1,Q)} X \frac{(y - X\delta)' A (y - X\delta)}{\left[(y - X\delta)' M^{(Z_1,Q)} (y - X\delta)\right]^2} \\
= \; & X'AX - \widehat{\ell}(\delta) X' M^{(Z_1,Q)} X \\
& - \frac{2X' M^{(Z_1,Q)} (y - X\delta)}{(y - X\delta)' M^{(Z_1,Q)} (y - X\delta)} \left\{ (y - X\delta)' AX - \widehat{\ell}(\delta)(y - X\delta)' M^{(Z_1,Q)} X \right\} \\
= \; & X'AX - \widehat{\ell}(\delta) X' M^{(Z_1,Q)} X - \frac{2X' M^{(Z_1,Q)} (y - X\delta)}{(y - X\delta)' M^{(Z_1,Q)} (y - X\delta)} \widehat{\Delta}(\delta)'
\end{aligned}
$$

so that evaluating $-\partial \widehat{\Delta}(\delta)/\partial \delta'$ at $\delta = \overline{\delta}_n$, we have

$$
\begin{aligned}
-\frac{\partial \widehat{\Delta}(\overline{\delta}_n)}{\partial \delta'} &= X'AX - \widehat{\ell}(\overline{\delta}_n) X' M^{(Z_1,Q)} X - \frac{2X' M^{(Z_1,Q)} (y - X\overline{\delta}_n)}{(y - X\overline{\delta}_n)' M^{(Z_1,Q)} (y - X\overline{\delta}_n)} \widehat{\Delta}(\overline{\delta}_n)' \\
&= X' \left[A - \widehat{\ell}(\overline{\delta}_n) M^{(Z_1,Q)} \right] X - \frac{2X' M^{(Z_1,Q)} (y - X\overline{\delta}_n)}{(y - X\overline{\delta}_n)' M^{(Z_1,Q)} (y - X\overline{\delta}_n)} \widehat{\Delta}(\overline{\delta}_n)'
\end{aligned}
$$

Pre-multiplying and post-multiplying the above expression by D_μ^{-1}, we then obtain

$$
\begin{aligned}
& -D_\mu^{-1} \left(\frac{\partial \widehat{\Delta}(\overline{\delta}_n)}{\partial \delta'} \right) D_\mu^{-1} \\
= \; & D_\mu^{-1} X' \left[A - \widehat{\ell}(\overline{\delta}_n) M^{(Z_1,Q)} \right] X D_\mu^{-1} - \frac{2D_\mu^{-1} X' M^{(Z_1,Q)} (y - X\overline{\delta}_n)}{(y - X\overline{\delta}_n)' M^{(Z_1,Q)} (y - X\overline{\delta}_n)} \widehat{\Delta}(\overline{\delta}_n)' D_\mu^{-1}
\end{aligned}
$$

Now, part (b) of Lemma S2-2 gives

$$
D_\mu^{-1} X' AX D_\mu^{-1} = \frac{\Gamma' M^{(Z_1,Q)} \Gamma}{n} + o_p(1) = H_n + o_p(1), \tag{20}
$$

and, by part (a) of Lemma S2-2 as well as parts (c) and (d) of Lemma OA-7 given section 3 below, we have

$$
\begin{aligned}
& \widehat{\ell}(\overline{\delta}_n) D_\mu^{-1} X' M^{(Z_1,Q)} X D_\mu^{-1} \\
= \; & \frac{(y - X\overline{\delta}_n)' A (y - X\overline{\delta}_n) / (\mu_n^{\min})^2}{(y - X\overline{\delta}_n)' M^{(Z_1,Q)} (y - X\overline{\delta}_n) / n} \left((\mu_n^{\min})^2 \frac{D_\mu^{-1} X' M^{(Z_1,Q)} X D_\mu^{-1}}{n} \right) \\
= \; & o_p(1) O_p(1) = o_p(1).
\end{aligned} \tag{21}
$$

It follows from expressions (20) and (21) that

$$
\begin{aligned}
D_\mu^{-1} X' \left[A - \widehat{\ell}\left(\overline{\delta}_n\right) M^{(Z_1,Q)} \right] X D_\mu^{-1} &= D_\mu^{-1} X' A X D_\mu^{-1} - \widehat{\ell}\left(\overline{\delta}_n\right) D_\mu^{-1} X' M^{(Z_1,Q)} X D_\mu^{-1} \\
&= H_n + o_p(1).
\end{aligned}
$$

where $H_n = \Gamma' M^{(Z_1,Q)} \Gamma / n$. Moreover, making use of parts (a)-(d) of Lemma OA-7, we further obtain

$$
\begin{aligned}
& \frac{2 D_\mu^{-1} X' M^{(Z_1,Q)} \left(y - X\overline{\delta}_n\right)}{\left(y - X\overline{\delta}_n\right)' M^{(Z_1,Q)} \left(y - X\overline{\delta}_n\right)} \widehat{\Delta}\left(\overline{\delta}_n\right)' D_\mu^{-1} \\
= \; & \frac{2 D_\mu^{-1} X' M^{(Z_1,Q)} \left(y - X\overline{\delta}_n\right)/n}{\left(y - X\overline{\delta}_n\right)' M^{(Z_1,Q)} \left(y - X\overline{\delta}_n\right)/n} \\
& \times \left[\left(y - X\overline{\delta}_n\right)' A X D_\mu^{-1} - \frac{\left(y - X\overline{\delta}_n\right)' A \left(y - X\overline{\delta}_n\right)}{\left(y - X\overline{\delta}_n\right)' M^{(Z_1,Q)} \left(y - X\overline{\delta}_n\right)/n} \frac{\left(y - X\overline{\delta}_n\right)' M^{(Z_1,Q)} X D_\mu^{-1}}{n} \right] \\
= \; & O_p\left(\frac{1}{\mu_n^{\min}}\right) \left[o_p\left(\mu_n^{\min}\right) - o_p\left(\left[\mu_n^{\min}\right]^2\right) O_p\left(\frac{1}{\mu_n^{\min}}\right) \right] = o_p(1)
\end{aligned}
$$

so that

$$
-D_\mu^{-1} \left(\frac{\partial \widehat{\Delta}\left(\overline{\delta}_n\right)}{\partial \delta'} \right) D_\mu^{-1} = \frac{\Gamma' M^{(Z_1,Q)} \Gamma}{n} + o_p(1) = H_n + o_p(1). \; \square
$$

Lemma S2-11: Let $\widehat{\ell}_L = Q\left(\widetilde{\beta}\right) = \min_{\beta \in \overline{B}} Q(\beta)$, where $Q(\beta)$ is as defined in Assumption 9. Then, $\widehat{\ell}_L$ is also the smallest root of the determinantal equation $\det\left[\overline{X}' A \overline{X} - \ell \overline{X}' M^{(Z_1,Q)} \overline{X}\right] = 0$, where $\overline{X} = [y, X]$. Suppose in addition that condition (11) in Assumption 9 is satisfied; then, $\widehat{\ell}_L$ has the representation

$$
\widehat{\ell}_L = \frac{\left(y - X\widehat{\delta}_L\right)' A \left(y - X\widehat{\delta}_L\right)}{\left(y - X\widehat{\delta}_L\right)' M^{(Z_1,Q)} \left(y - X\widehat{\delta}_L\right)}, \tag{22}
$$

where $\widehat{\delta}_L$ denotes the FELIM estimator. Moreover, $\overline{X}' A \left(y - X\widehat{\delta}_L\right) - \widehat{\ell}_L \overline{X}' M^{(Z_1,Q)} \left(y - X\widehat{\delta}_L\right)$ $= 0$. In particular, this implies that $\widehat{\Delta}\left(\widehat{\delta}_L\right) = 0$, where $\widehat{\Delta}(\delta) = -\left[(y - X\delta)' M^{(Z_1,Q)} (y - X\delta)/2\right] \left(\partial \widehat{Q}_{FELIM}(\delta)/\partial \delta\right)$, so that $\widehat{\delta}_L$ satisfies the set of (normalized) first-order conditions for minimizing the variance ratio objective function $\widehat{Q}_{FELIM}(\delta) = (y - X\delta)' A (y - X\delta) / \left[(y - X\delta)' M^{(Z_1,Q)} (y - X\delta)\right]$.

Proof of Lemma S2-11:

Note that the first-order condition for minimizing the objective function $Q(\beta)$ can be written as $\partial Q\left(\widetilde{\beta}\right)/\partial \beta = 2\overline{X}' A \overline{X} \widetilde{\beta} / \left(\widetilde{\beta}' \overline{X}' M^{(Z_1,Q)} \overline{X} \widetilde{\beta}\right)$ $- \widetilde{\beta}' \overline{X}' A \overline{X} \widetilde{\beta} \left(2 \overline{X}' M^{(Z_1,Q)} \overline{X} \widetilde{\beta}\right) / \left[\widetilde{\beta}' \overline{X}' M^{(Z_1,Q)} \overline{X} \widetilde{\beta}\right]^2 = 0$. Pre-multiplying this first order condition by the factor $\frac{1}{2}\widetilde{\beta}'\overline{X}'M^{(Z_1,Q)}\overline{X}\widetilde{\beta}$, we then obtain

$$0 = \left[\overline{X}'A\overline{X} - \widehat{\ell}_L\overline{X}'M^{(Z_1,Q)}\overline{X}\right]\widetilde{\beta} \tag{23}$$

where $\widehat{\ell}_L = Q\left(\widetilde{\beta}\right) = \widetilde{\beta}'\overline{X}'A\overline{X}\widetilde{\beta}/\widetilde{\beta}'\overline{X}'M^{(Z_1,Q)}\overline{X}\widetilde{\beta}$. It is clear that in order for there to be a nontrivial solution, i.e., $\widetilde{\beta} \neq 0$ such that equation (23) is true, $\widehat{\ell}_L$ must be a root of the determinantal equation $\det\left[\overline{X}'A\overline{X} - \ell\overline{X}'M^{(Z_1,Q)}\overline{X}\right] = 0$. Moreover, since our goal is to minimize the value of the objective function $Q(\beta)$, this implies that we should choose $\widehat{\ell}_L$ to be the smallest root of this determinantal equation. Now, define $\widetilde{\delta} = -\widetilde{\beta}_2/\widetilde{\beta}_1$, and rewrite the first-order conditions given by expression (23) as $0 = \overline{X}'A\overline{X}\widetilde{\beta} - \widehat{\ell}_L\overline{X}'M^{(Z_1,Q)}\overline{X}\widetilde{\beta} = \widetilde{\beta}_1\left\{\overline{X}'A\left(y - X\widetilde{\delta}\right) - \widehat{\ell}_L\overline{X}'M^{(Z_1,Q)}\left(y - X\widetilde{\delta}\right)\right\}$, so that, given the condition that $\left|\widetilde{\beta}_1\right| \geq \underline{C} > 0$ $a.s.n.$ for some constant $\underline{C}$ (as stated in Assumption 9), we must have

$$\overline{X}'A\left(y - X\widetilde{\delta}\right) - \widehat{\ell}_L\overline{X}'M^{(Z_1,Q)}\left(y - X\widetilde{\delta}\right) = 0 \tag{24}$$

Since $\overline{X} = [y, X]$, we can partition (24) into two sets of equations

$$\begin{aligned} 0 &= y'A\left(y - X\widetilde{\delta}\right) - \widehat{\ell}_L y'M^{(Z_1,Q)}\left(y - X\widetilde{\delta}\right), & (25)\\ 0 &= X'A\left(y - X\widetilde{\delta}\right) - \widehat{\ell}_L X'M^{(Z_1,Q)}\left(y - X\widetilde{\delta}\right). & (26) \end{aligned}$$

Solving (26) for $\widetilde{\delta}$, we obtain $\widetilde{\delta} = \left(X'\left[A - \widehat{\ell}_L M^{(Z_1,Q)}\right]X\right)^{-1}X'\left[A - \widehat{\ell}_L M^{(Z_1,Q)}\right]y = \widehat{\delta}_L$, so that the FELIM estimator $\widehat{\delta}_L$ is a solution to the second set of equations given by (26). In addition , note that, under condition (11) in Assumption 9, we have

$$\widehat{\ell}_L = \frac{\widetilde{\beta}'\overline{X}'A\overline{X}\widetilde{\beta}}{\widetilde{\beta}'\overline{X}'M^{(Z_1,Q)}\overline{X}\widetilde{\beta}} = \frac{\widetilde{\beta}_1\left(y - X\widetilde{\delta}\right)'A\left(y - X\widetilde{\delta}\right)\widetilde{\beta}_1}{\widetilde{\beta}_1\left(y - X\widetilde{\delta}\right)M^{(Z_1,Q)}\left(y - X\widetilde{\delta}\right)\widetilde{\beta}_1} = \frac{\left(y - X\widehat{\delta}_L\right)'A\left(y - X\widehat{\delta}_L\right)}{\left(y - X\widehat{\delta}_L\right)'M^{(Z_1,Q)}\left(y - X\widehat{\delta}_L\right)}.$$

which shows (22). Furthermore, note that $\widehat{\delta}_L$ also satisfies equation (25) since

$$\begin{aligned} & y'A\left(y - X\widehat{\delta}_L\right) - \widehat{\ell}_L y'M^{(Z_1,Q)}\left(y - X\widehat{\delta}_L\right) \\ = & \left(y - X\widehat{\delta}_L\right)'\left[A - \widehat{\ell}_L M^{(Z_1,Q)}\right]\left(y - X\widehat{\delta}_L\right) + \widehat{\delta}_L'X'\left[A - \widehat{\ell}_L M^{(Z_1,Q)}\right]\left(y - X\widehat{\delta}_L\right) \\ = & \left(y - X\widehat{\delta}_L\right)'A\left(y - X\widehat{\delta}_L\right) - \frac{\left(y - X\widehat{\delta}_L\right)'A\left(y - X\widehat{\delta}_L\right)\left(y - X\widehat{\delta}_L\right)'M^{(Z_1,Q)}\left(y - X\widehat{\delta}_L\right)}{\left(y - X\widehat{\delta}_L\right)M^{(Z_1,Q)}\left(y - X\widehat{\delta}_L\right)} \\ & + \widehat{\delta}_L'X'\left[A - \widehat{\ell}_L M^{(Z_1,Q)}\right]y \\ & - \widehat{\delta}_L'X'\left[A - \widehat{\ell}_L M^{(Z_1,Q)}\right]X\left(X'\left[A - \widehat{\ell}_L M^{(Z_1,Q)}\right]X\right)^{-1}X'\left[A - \widehat{\ell}_L M^{(Z_1,Q)}\right]y \\ = & \; 0 \end{aligned}$$

from which we further deduce that $\widehat{\delta}_L$ is a solution of the complete set of first-order conditions given by (24). Finally, since $\widehat{\Delta}\left(\widehat{\delta}_L\right) = X'A\left(y - X\widehat{\delta}_L\right) - \widehat{\ell}_L X' M^{(Z_1,Q)}\left(y - X\widehat{\delta}_L\right)$, the fact that $\widehat{\delta}_L$ is a solution of (26) directly imply that $\widehat{\delta}_L$ satisfies the set of (normalized) first-order conditions for minimizing the variance ratio objective function. $\square$

Lemma S2-12: Suppose that Assumptions 1-7 are satisfied. Then, $D_\mu^{-1}X'\left[A - \widehat{\ell}_{F,n}M^{(Z_1,Q)}\right]XD_\mu^{-1} = H_n + o_p(1)$, where $H_n = \Gamma' M^{(Z_1,Q)}\Gamma/n$, $\widehat{\ell}_{F,n} = \left[\widehat{\ell}_{L,n} - \left(1 - \widehat{\ell}_{L,n}\right)(C/m_n)\right] / \left[1 - \left(1 - \widehat{\ell}_{L,n}\right)(C/m_n)\right]$, and $\widehat{\ell}_{L,n}$ is smallest root of the determinantal equation $\det\left\{\overline{X}'A\overline{X} - \ell\overline{X}'M^{(Z_1,Q)}\overline{X}\right\} = 0$, with $\overline{X} = \left[\begin{array}{cc} y & X \end{array}\right]$.

Proof of Lemma S2-12: The result follows directly from applying part (b) of Lemma S2-7 and parts (a) and (b) of Lemma S2-2. $\square$

Lemma S2-13: Suppose that Assumptions 1-8 hold. Then, $D_\mu^{-1}X'\left[A - \widehat{\ell}_{F,n}M^{(Z_1,Q)}\right](y - X\delta_0) = \mathcal{Y}_n\left[1 + o_p(1)\right]$, where $\mathcal{Y}_n = \Gamma' M^{(Z_1,Q)}\varepsilon/\sqrt{n} + D_\mu^{-1}\underline{U}'A\varepsilon$ with $\underline{U} = U - \varepsilon\rho'$ and $\rho = \lim_{n\to\infty} E\left[U'M^Q\varepsilon\right]/E\left[\varepsilon' M^Q \varepsilon\right]$.

Proof of Lemma S2-13:

Note that, by Lemma S2-11 above, $\widehat{\ell}_{L,n}$ has the representation

$$\widehat{\ell}_{L,n} = \frac{\left(y - X\widehat{\delta}_{L,n}\right)' A \left(y - X\widehat{\delta}_{L,n}\right)}{\left(y - X\widehat{\delta}_{L,n}\right)' M^{(Z_1,Q)} \left(y - X\widehat{\delta}_{L,n}\right)}.$$

Next, from expression (10), we have $\widehat{\ell}_{F,n} = \widehat{\ell}_{L,n} + O_p\left(n^{-1}\right)$. It then follows, by tedious but straightforward calculations[2] and by making use of Assumptions 1, 2(i), 3-6, 7(i)-(iii), and 8 that

$$\begin{aligned}
& D_\mu^{-1}X'\left[A - \widehat{\ell}_{F,n}M^{(Z_1,Q)}\right](y - X\delta_0) \\
= \;& D_\mu^{-1}X'A(y - X\delta_0) - \widehat{\ell}_{L,n}D_\mu^{-1}X'M^{(Z_1,Q)}(y - X\delta_0) + O_p\left(\frac{1}{n}\right)O_p\left(\frac{n}{\mu_n^{\min}}\right) \\
= \;& \frac{\Gamma' M^{(Z_1,Q)}\varepsilon}{\sqrt{n}} + D_\mu^{-1}\left(U - \varepsilon\rho'\right)' A\varepsilon\left[1 + o_p(1)\right] + O_p\left(\frac{1}{\mu_n^{\min}}\max\left\{\sqrt{\frac{K_{2,n}}{n}}, \frac{K_{1,n}\sqrt{K_{2,n}}}{n}\right\}\right) \\
& + O_p\left(\frac{\sqrt{K_{2,n}}}{n}\right) + O_p\left(\frac{1}{K_{2,n}^{\varrho_\gamma}}\right) + O_p\left(\frac{\kappa_n^{\max}}{\mu_n^{\min}}\frac{1}{K_{1,n}^{\varrho_f}}\right) + O_p\left(\frac{\tau_n}{K_{1,n}^{\varrho_g}}\right) + O_p\left(\frac{1}{\mu_n^{\min}}\right) \\
= \;& \frac{\Gamma' M^{(Z_1,Q)}\varepsilon}{\sqrt{n}} + D_\mu^{-1}\underline{U}'A\varepsilon\left[1 + o_p(1)\right], \text{ where } \underline{U} = U - \varepsilon\rho'. \;\square
\end{aligned}$$

Lemma S2-14: For any $a \in \mathbb{R}^d$ such that $\|a\| = 1$, define $b_{1n} = \Sigma_n^{-1/2}a$, $\underline{u}_{2,(i,t),n} = b_{2n}'\underline{U}_{(i,t)}$

[2] Further details are available from the authors upon request.

$= \sqrt{K_{2,n}} a' \Sigma_n^{-1/2} D_\mu^{-1} \underline{U}_{(i,t)}$, $\sigma^2_{(i,t),n} = E\left[\varepsilon^2_{(i,t)} | \mathcal{F}_n^W\right]$, $\widetilde{\psi}_{(i,t),n} = E\left[\underline{u}_{2,(i,t),n} \varepsilon_{(i,t)} | \mathcal{F}_n^W\right]$, and $\widetilde{\omega}^2_{(i,t)} = E\left[\underline{u}^2_{2,(i,t),n} | \mathcal{F}_n^W\right]$. Suppose that Assumptions 1-2 and 5-6 are satisfied. Then, the following statements are true.

(a) $\sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} \left[b'_{1n} \Gamma' M^{(Z_1,Q)} e_{(i,t)} / \sqrt{n}\right] \left(A_{(i,t),(j,s)} / \sqrt{K_{2,n}}\right) \left\{\varepsilon_{(j,s)} \widetilde{\psi}_{(i,t),n} + \underline{u}_{2,(j,s)} \sigma^2_{(i,t),n}\right\} = O_p\left(K_{2,n}^{1/4} / \mu_n^{\min}\right) = o_p(1)$.

(b) $\sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} \left(A^2_{(i,t),(j,s)} / K_{2,n}\right) \left(\varepsilon^2_{(j,s)} - \sigma^2_{(j,s),n}\right) \widetilde{\omega}^2_{(i,t),n} = O_p\left(K_{2,n} \left(\mu_n^{\min}\right)^{-2} n^{-1/2}\right) = o_p(1)$.

(c) $\sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} \left(A^2_{(i,t),(j,s)} / K_{2,n}\right) \left(\underline{u}^2_{2,(j,s),n} - \widetilde{\omega}^2_{(j,s),n}\right) \sigma^2_{(i,t),n} = O_p\left(K_{2,n} \left(\mu_n^{\min}\right)^{-2} n^{-1/2}\right) = o_p(1)$.

Proof of Lemma S2-14:

To show part (a), first let $\mathfrak{W}_n = \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} \left[b'_{1n} \Gamma' M^{(Z_1,Q)} e_{(i,t)} / \sqrt{n}\right] \left(A_{(i,t),(j,s)} / \sqrt{K_{2,n}}\right) \times \left\{\varepsilon_{(j,s)} \widetilde{\psi}_{(i,t),n} + \underline{u}_{2,(j,s),n} \sigma^2_{(i,t),n}\right\}$. By taking expectation and applying the triangle inequality, we obtain $E\left[\mathfrak{W}_n^2 | \mathcal{F}_n^W\right] \leq \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4$, where $\mathcal{H}_1 = \sum_{(i,t),(k,v)=2}^{m_n} \left| n^{-1} \left[b'_{1n} \Gamma' M^{(Z_1,Q)} e_{(i,t)}\right] \times \left[b'_{1n} \Gamma' M^{(Z_1,Q)} e_{(k,v)}\right] \widetilde{\psi}_{(i,t),n} \widetilde{\psi}_{(k,v),n} \sum_{(j,s)=1}^{\min\{(i,t),(k,v)\}-1} \left(A_{(i,t),(j,s)} A_{(k,v),(j,s)} \sigma^2_{(j,s),n} / K_{2,n}\right) \right|$,
$\mathcal{H}_2 = \sum_{(i,t),(k,v)=2}^{m_n} \left| \left(\left[b'_{1n} \Gamma' M^{(Z_1,Q)} e_{(i,t)}\right] \left[b'_{1n} \Gamma' M^{(Z_1,Q)} e_{(k,v)}\right] / n \right) \sigma^2_{(i,t),n} \widetilde{\psi}_{(k,v),n} \times \sum_{(j,s)=1}^{\min\{(i,t),(k,v)\}-1} \left(A_{(i,t),(j,s)} A_{(k,v),(j,s)} \widetilde{\psi}_{(j,s),n} / K_{2,n}\right) \right|$, $\mathcal{H}_3 = \sum_{(i,t),(k,v)=2}^{m_n} \left| n^{-1} \left[b'_{1n} \Gamma' M^{(Z_1,Q)} e_{(i,t)}\right] \times \left[b'_{1n} \Gamma' M^{(Z_1,Q)} e_{(k,v)}\right] \sigma^2_{(k,v),n} \widetilde{\psi}_{(i,t),n} \sum_{(j,s)=1}^{\min\{(i,t),(k,v)\}-1} \left(A_{(i,t),(j,s)} A_{(k,v),(j,s)} \widetilde{\psi}_{(j,s),n} / K_{2,n}\right) \right|$,
$\mathcal{H}_4 = \sum_{(i,t),(k,v)=2}^{m_n} \left| \left(\left[b'_{1n} \Gamma' M^{(Z_1,Q)} e_{(i,t)}\right] \left[b'_{1n} \Gamma' M^{(Z_1,Q)} e_{(k,v)}\right] / n \right) \sigma^2_{(i,t),n} \sigma^2_{(k,v),n} \times \sum_{(j,s)=1}^{\min\{(i,t),(k,v)\}-1} \left(A_{(i,t),(j,s)} A_{(k,v),(j,s)} \widetilde{\omega}^2_{(j,s),n} / K_{2,n}\right) \right|$. Focusing first on $\mathcal{H}_1$, we obtain, by applying the CS inequality,

$$
\begin{aligned}
&\mathcal{H}_1 \\
&\leq \sqrt{\sum_{(i,t),((k,v))=2}^{m_n} \frac{b'_{1n} \Gamma' M^{(Z_1,Q)} e_{(i,t)} e'_{(i,t)} M^{(Z_1,Q)} \Gamma b_{1n} \widetilde{\psi}^2_{(i,t),n}}{n} \frac{b'_{1n} \Gamma' M^{(Z_1,Q)} e_{(k,v)} e'_{(k,v)} M^{(Z_1,Q)} \Gamma b_{1n} \widetilde{\psi}^2_{(k,v),n}}{n}} \\
&\quad \times \sqrt{\sum_{(i,t)=2}^{m_n} \sum_{(k,v)=2}^{m_n} \left(\sum_{(j,s)=1}^{\min\{(i,t),(k,v)\}-1} \frac{A_{(i,t),(j,s)} A_{(k,v),(j,s)} \sigma^2_{(j,s),n}}{K_{2,n}} \right)^2}
\end{aligned}
$$

Applying the CS inequality, Assumption 2(i), part (d) of Lemma S2-3, and Lemma S2-6, we obtain

$$
\begin{aligned}
\max_{1\leq(i,t)\leq m_n}\left|\widetilde{\psi}_{(i,t),n}\right| &= \max_{1\leq(i,t)\leq m_n}\sqrt{K_{2,n}}E\left[\left|\varepsilon_{(i,t)}\underline{U}'_{(i,t)}D_{\mu}^{-1}\Sigma_n^{-1/2}a\right||\mathcal{F}_n^W\right] \\
&\leq \frac{\sqrt{K_{2,n}}}{(\mu_n^{\min})}\sqrt{a'\Sigma_n^{-1}a}\sqrt{\max_{1\leq(i,t)\leq m_n}E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right]}\sqrt{\max_{1\leq(i,t)\leq m_n}E\left[\left\|\underline{U}_{(i,t)}\right\|_2^2|\mathcal{F}_n^W\right]} \\
&= O_{a.s.}\left(\frac{\sqrt{K_{2,n}}}{(\mu_n^{\min})}\right) \qquad (27)
\end{aligned}
$$

Moreover, by direct calculations,
$\sum_{(i,t)=2}^{m_n}\sum_{(k,v)=2}^{m_n}\left(\sum_{(j,s)=1}^{\min\{(i,t),(k,v)\}-1}A_{(i,t),(j,s)}A_{(k,v),(j,s)}\sigma_{(j,s),n}^2/K_{2,n}\right)^2 = K_{2,n}^{-2}tr\left\{LD_{\sigma^2}L'LD_{\sigma^2}L'\right\}$,
where L is the lower triangular matrix such that $L_{(i,t),(j,s)} = A_{(i,t),(j,s)}\mathbb{I}\left\{(i,t)>(j,s)\right\}$ and $D_{\sigma^2} = diag\left(\sigma_{(1,1),n}^2,....,\sigma_{(n,T_n),n}^2\right) = diag\left(\sigma_{1,n}^2,....,\sigma_{m_n,n}^2\right)$ and where $\sigma_{(i,t),n}^2 = E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right]$ for $(i,t) = 1,...,m_n$. In addition, by the result shown in Lemma OA-12 (given in section 3 of this appendix), we have $\|LL'\|_F = O_{a.s.}\left(\sqrt{K_{2,n}}\right)$. Using these results, we further deduce, by applying the CS inequality, Assumptions 2 and 3(iii), and part (d) of Lemma S2-3 that

$$
\begin{aligned}
\mathcal{H}_1 &\leq \left(\max_{1\leq(i,t)\leq m_n}\left|\widetilde{\psi}_{(i,t),n}\right|\right)^2\left(\frac{b_{1n}'\Gamma'\Gamma b_{1n}}{n}\right)\frac{1}{K_{2,n}}\sqrt{tr\left\{L'LD_{\sigma^2}L'LD_{\sigma^2}\right\}} \\
&\leq \left(\max_{1\leq(i,t)\leq m_n}\left|\widetilde{\psi}_{(i,t),n}\right|\right)^2\left(\frac{b_{1n}'\Gamma'\Gamma b_{1n}}{n}\right)\frac{1}{K_{2,n}}\left(tr\left\{L'LD_{\sigma^2}^2L'L\right\}\right)^{1/4}\left(tr\left\{D_{\sigma^2}L'LL'LD_{\sigma^2}\right\}\right)^{1/4} \\
&\leq \left(\max_{1\leq(i,t)\leq m_n}\left|\widetilde{\psi}_{(i,t),n}\right|\right)^2\left(\max_{1\leq(i,t)\leq m_n}\sigma_{(i,t),n}^2\right)\left(\frac{b_{1n}'\Gamma'\Gamma b_{1n}}{n}\right)\frac{1}{K_{2,n}}\left\|LL'\right\|_F \\
&= O_{a.s.}\left(\frac{\sqrt{K_{2,n}}}{(\mu_n^{\min})^2}\right). \qquad (28)
\end{aligned}
$$

Similarly, let $D_{\widetilde{\psi}} = diag\left(\widetilde{\psi}_{(1,1),n},....,\widetilde{\psi}_{(n,T_n),n}\right) = diag\left(\widetilde{\psi}_{1,n},....,\widetilde{\psi}_{m_n,n}\right)$, we can also show

$$
\begin{aligned}
\mathcal{H}_2 &\leq \left(\max_{1\leq(i,t)\leq m_n}\left|\widetilde{\psi}_{(i,t),n}\right|\right)\left(\max_{1\leq(i,t)\leq m_n}\sigma_{(i,t),n}^2\right)\left(\frac{b_{1n}'\Gamma'\Gamma b_{1n}}{n}\right)\frac{1}{K_{2,n}}\sqrt{tr\left\{L'LD_{\widetilde{\psi}}L'LD_{\widetilde{\psi}}\right\}} \\
&\leq \left(\max_{1\leq(i,t)\leq m_n}\left|\widetilde{\psi}_{(i,t),n}\right|\right)^2\left(\max_{1\leq(i,t)\leq m_n}\sigma_{(i,t),n}^2\right)\left(\frac{b_{1n}'\Gamma'\Gamma b_{1n}}{n}\right)\frac{1}{K_{2,n}}\left\|LL'\right\|_F \\
&= O_{a.s.}\left(\frac{\sqrt{K_{2,n}}}{(\mu_n^{\min})^2}\right), \qquad (29)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{H}_3 &\leq \left(\max_{1\leq(i,t)\leq m_n}\left|\widetilde{\psi}_{(i,t),n}\right|\right)^2\left(\max_{1\leq(i,t)\leq m_n}\sigma^2_{(i,t),n}\right)\left(\frac{b_{1n}'\Gamma'\Gamma b_{1n}}{n}\right)\frac{1}{K_{2,n}}\left\|LL'\right\|_F \\
&= O_{a.s.}\left(\frac{\sqrt{K_{2,n}}}{\left(\mu_n^{\min}\right)^2}\right). \qquad (30)
\end{aligned}
$$

Moreover, let $D_{\widetilde{\omega}^2}=diag\left(\widetilde{\omega}^2_{(1,1),n},....,\widetilde{\omega}^2_{(n,T_n),n}\right)=diag\left(\widetilde{\omega}^2_{1,n},....,\widetilde{\omega}^2_{m_n,n}\right)$, and note that

$$
\begin{aligned}
\mathcal{H}_4 &\leq \left(\max_{1\leq(i,t)\leq m_n}\sigma^2_{(i,t),n}\right)^2\left(\frac{b_{1n}'\Gamma'\Gamma b_{1n}}{n}\right)\frac{1}{K_{2,n}}\sqrt{tr\left\{L'LD_{\widetilde{\omega}^2}L'LD_{\widetilde{\omega}^2}\right\}} \\
&\leq \left(\max_{1\leq(i,t)\leq m_n}\sigma^2_{(i,t),n}\right)^2\left(\frac{b_{1n}'\Gamma'\Gamma b_{1n}}{n}\right)\left(\max_{1\leq(i,t)\leq m_n}\widetilde{\omega}^2_{(i,t),n}\right)\frac{\left\|LL'\right\|_F}{K_{2,n}} \\
&= O_{a.s.}\left(\frac{\sqrt{K_{2,n}}}{\left(\mu_n^{\min}\right)^2}\right), \qquad (31)
\end{aligned}
$$

where the order of magnitude is calculated by applying Assumptions 2 and 3(iii), part (d) of Lemma S2-3, and the fact that $\left\|LL'\right\|_F=O_{a.s.}\left(\sqrt{K_{2,n}}\right)$ and by making use of the result

$$
\begin{aligned}
\max_{1\leq(i,t)\leq m_n}\widetilde{\omega}^2_{(i,t),n} &= \max_{1\leq(i,t)\leq m_n}K_{2,n}a'\Sigma_n^{-1/2}D_\mu^{-1}E\left[\underline{U}_{(i,t)}\underline{U}_{(i,t)}'|\mathcal{F}_n^W\right]D_\mu^{-1}\Sigma_n^{-1/2}a \\
&\leq \left(\max_{1\leq(i,t)\leq m_n}E\left[\left\|\underline{U}_{(i,t)}\right\|_2^2|\mathcal{F}_n^W\right]\right)\frac{K_{2,n}a'\Sigma_n^{-1}a}{\left(\mu_n^{\min}\right)^2} \\
&= O_{a.s.}\left(\frac{K_{2,n}}{\left(\mu_n^{\min}\right)^2}\right) \qquad (32)
\end{aligned}
$$

which can be easily deduced from part (d) of Lemma S2-3 and Lemma S2-6. Combining (28)-(31), we obtain $E\left[\mathfrak{W}_n^2|\mathcal{F}_n^W\right]=O_{a.s.}\left(\sqrt{K_{2,n}}\left(\mu_n^{\min}\right)^{-2}\right)$. Hence, by the law of iterated expectations and by Theorem 16.1 of Billingsley (1995), there exists a constant $\overline{C}<\infty$ such that, for all n sufficiently large, $E\left(\left[\left(\mu_n^{\min}\right)^2/\sqrt{K_{2,n}}\right]\mathfrak{W}_n^2\right)=E_{W_n}\left(E\left\{\left[\left(\mu_n^{\min}\right)^2/\sqrt{K_{2,n}}\right]\mathfrak{W}_n^2|\mathcal{F}_n^W\right\}\right)$ $\leq\overline{C}$. It follows from the Markov's inequality that $\mathfrak{W}_n=\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}\left[b_{1n}'\Gamma'M^{(Z_1,Q)}e_{(i,t)}/\sqrt{n}\right]\left(A_{(i,t),(j,s)}/\sqrt{K_{2,n}}\right)\left\{\varepsilon_{(j,s)}\widetilde{\psi}_{(i,t),n}+\underline{u}_{2,(j,s),n}\sigma^2_{(i,t),n}\right\}$ $=O_p\left(K_{2,n}^{1/4}/\left(\mu_n^{\min}\right)\right)=o_p(1)$.

For part (b), note that, by Assumption 2, the symmetry of A, part (c) of Lemma S2-1, and expression (32) above; we obtain

$$
\begin{aligned}
& E\left\{\left(\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}\frac{A^2_{(i,t),(j,s)}}{K_{2,n}}\left(\varepsilon^2_{(j,s)}-\sigma^2_{(j,s),n}\right)\widetilde{\omega}^2_{(i,t),n}\right)^2|\mathcal{F}^W_n\right\} \\
= & \frac{1}{K^2_{2,n}}\sum_{(i,t)=2}^{m_n}\sum_{(k,\upsilon)=2}^{m_n}\sum_{(j,s)=1}^{\min\{(i,t)-1,(k,\upsilon)-1\}}A^2_{(i,t),(j,s)}A^2_{(k,\upsilon),(j,s)}\widetilde{\omega}^2_{(i,t),n}\widetilde{\omega}^2_{(k,\upsilon)}E\left[\left(\varepsilon^2_{(j,s)}-\sigma^2_{(j,s),n}\right)^2|\mathcal{F}^W_n\right] \\
\leq & \frac{1}{K^2_{2,n}}\sum_{(j,s)=1}^{m_n}\sum_{\substack{(i,t),(k,\upsilon)=1\\(i,t)\neq(j,s),(k,\upsilon)\neq(j,s)}}^{m_n}A^2_{(i,t),(j,s)}A^2_{(k,\upsilon),(j,s)}\widetilde{\omega}^2_{(i,t),n}\widetilde{\omega}^2_{(k,\upsilon),n}\left\{E\left[\varepsilon^4_{(j,s)}|\mathcal{F}^W_n\right]+\sigma^4_{(j,s),n}\right\} \\
= & O_{a.s.}\left(\frac{K^2_{2,n}}{\left(\mu^{\min}_n\right)^4 n}\right)
\end{aligned}
$$

Hence, there exists a constant $\overline{C}<\infty$ such that for all n sufficiently large

$$
\begin{aligned}
& E\left[\frac{\left(\mu^{\min}_n\right)^4 n}{K^2_{2,n}}\left(\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}\frac{A^2_{(i,t),(j,s)}}{K_{2,n}}\left(\varepsilon^2_{(j,s)}-\sigma^2_{(j,s),n}\right)\widetilde{\omega}^2_{(i,t),n}\right)^2\right] \\
= & E\left[\frac{\left(\mu^{\min}_n\right)^4 n}{K^2_{2,n}}E\left\{\left(\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}\frac{A^2_{(i,t),(j,s)}}{K_{2,n}}\left(\varepsilon^2_{(j,s)}-\sigma^2_{(j,s),n}\right)\widetilde{\omega}^2_{(i,t),n}\right)^2|\mathcal{F}^W_n\right\}\right] \\
\leq & \overline{C}
\end{aligned}
$$

It follows from the Markov's inequality that for any $\epsilon>0$,

$$
\begin{aligned}
& \Pr\left(\frac{\left(\mu^{\min}_n\right)^2\sqrt{n}}{K_{2,n}}\left|\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}\frac{A^2_{(i,t),(j,s)}}{K_{2,n}}\left(\varepsilon^2_{(j,s)}-\sigma^2_{(j,s),n}\right)\widetilde{\omega}^2_{(i,t),n}\right|\geq\sqrt{\frac{\overline{C}}{\epsilon}}\right) \\
= & \Pr\left(\frac{\left(\mu^{\min}_n\right)^4 n}{K^2_{2,n}}\left(\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}\frac{A^2_{(i,t),(j,s)}}{K_{2,n}}\left(\varepsilon^2_{(j,s)}-\sigma^2_{(j,s),n}\right)\widetilde{\omega}^2_{(i,t),n}\right)^2\geq\frac{\overline{C}}{\epsilon}\right) \\
\leq & \frac{\epsilon}{\overline{C}}E\left[\frac{\left(\mu^{\min}_n\right)^4 n}{K^2_{2,n}}\left(\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}\frac{A^2_{(i,t),(j,s)}}{K_{2,n}}\left(\varepsilon^2_{(j,s)}-\sigma^2_{(j,s),n}\right)\widetilde{\omega}^2_{(i,t),n}\right)^2\right] \\
\leq & \epsilon
\end{aligned}
$$

for all n sufficiently large, which shows that

$$
\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}\frac{A^2_{(i,t),(j,s)}}{K_{2,n}}\left(\varepsilon^2_{(j,s)}-\sigma^2_{(j,s),n}\right)\widetilde{\omega}^2_{(i,t),n}=O_p\left(\frac{K_{2,n}}{\left(\mu^{\min}_n\right)^2\sqrt{n}}\right)=o_p(1)
$$

For part (c), note that, by Assumption 2, the symmetry of A, part (c) of Lemma S2-1, and expression (32) above; we get

$$
\begin{aligned}
& E\left\{\left(\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}\frac{A^2_{(i,t),(j,s)}}{K_{2,n}}\left(\underline{u}^2_{2,(j,s)}-\widetilde{\omega}^2_{(j,s),n}\right)\sigma^2_{(i,t),n}\right)^2|\mathcal{F}^W_n\right\} \\
= & \frac{1}{K^2_{2,n}}\sum_{(i,t)=2}^{m_n}\sum_{(k,\upsilon)=2}^{m_n}\sum_{(j,s)=1}^{\min\{(i,t)-1,(k,\upsilon)-1\}} A^2_{(i,t),(j,s)}A^2_{(k,\upsilon),(j,s)}\sigma^2_{(i,t),n}\sigma^2_{(k,\upsilon),n}E\left[\left(\underline{u}^2_{2,(j,s),n}-\widetilde{\omega}^2_{(j,s),n}\right)^2|\mathcal{F}^W_n\right] \\
\leq & \frac{1}{K^2_{2,n}}\sum_{(j,s)=1}^{m_n}\sum_{\substack{(i,t),(k,\upsilon)=1 \\ (i,t)\neq(j,s),(k,\upsilon)\neq(j,s)}}^{m_n} A^2_{(i,t),(j,s)}A^2_{(k,\upsilon),(j,s)}\sigma^2_{(i,t),n}\sigma^2_{(k,\upsilon),n}\left\{E\left[\underline{u}^4_{2,(j,s),n}|\mathcal{F}^W_n\right]+\widetilde{\omega}^4_{(j,s),n}\right\} \\
= & O_{a.s.}\left(\frac{K^2_{2,n}}{\left(\mu^{\min}_n\right)^4 n}\right)
\end{aligned}
$$

where, in obtaining the result above, we have also calculated the almost sure order of magnitude of $E\left[\underline{u}^4_{2,(j,s),n}|\mathcal{F}^W_n\right]$ as follows

$$
\begin{aligned}
E\left[\underline{u}^4_{2,(j,s),n}|\mathcal{F}^W_n\right] &= E\left[\left(\sqrt{K_{2,n}}a'\Sigma_n^{-1/2}D_\mu^{-1}\underline{U}_{(i,t)}\right)^4|\mathcal{F}^W_n\right] \\
&= K^2_{2,n}E\left[\left(a'\Sigma_n^{-1/2}D_\mu^{-1}\underline{U}_{(i,t)}\underline{U}'_{(i,t)}D_\mu^{-1}\Sigma_n^{-1/2}a\right)^2|\mathcal{F}^W_n\right] \\
&\leq E\left[\left\|\underline{U}_{(i,t)}\right\|_2^4|\mathcal{F}^W_n\right]\left[a'\Sigma_n^{-1}a\right]^2\frac{K^2_{2,n}}{\left(\mu^{\min}_n\right)^4} \\
&= O_{a.s.}\left(\frac{K^2_{2,n}}{\left(\mu^{\min}_n\right)^4}\right). \qquad (33)
\end{aligned}
$$

using part (d) of Lemma S2-3 and Lemma S2-6. Hence, there exists a constant $\overline{C}<\infty$ such that for all n sufficiently large

$$
\begin{aligned}
& E\left[\frac{\left(\mu^{\min}_n\right)^4 n}{K^2_{2,n}}\left(\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}\frac{A^2_{(i,t),(j,s)}}{K_{2,n}}\left(\underline{u}^2_{2,(j,s)}-\widetilde{\omega}^2_{(j,s),n}\right)\sigma^2_{(i,t),n}\right)^2\right] \\
= & E\left[\frac{\left(\mu^{\min}_n\right)^4 n}{K^2_{2,n}}E\left\{\left(\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}\frac{A^2_{(i,t),(j,s)}}{K_{2,n}}\left(\underline{u}^2_{2,(j,s)}-\widetilde{\omega}^2_{(j,s),n}\right)\sigma^2_{(i,t),n}\right)^2|\mathcal{F}^W_n\right\}\right] \\
\leq & \overline{C}
\end{aligned}
$$

It follows from the Markov's inequality that for any $\epsilon > 0$,

$$\begin{aligned}
& \Pr\left(\frac{\left(\mu_n^{\min}\right)^2\sqrt{n}}{K_{2,n}}\left|\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}\frac{A_{(i,t),(j,s)}^2}{K_{2,n}}\left(\underline{u}_{2,(j,s)}^2-\widetilde{\omega}_{(j,s),n}^2\right)\sigma_{(i,t),n}^2\right|\geq\sqrt{\frac{\overline{C}}{\epsilon}}\right) \\
= & \Pr\left(\frac{\left(\mu_n^{\min}\right)^4 n}{K_{2,n}^2}\left(\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}\frac{A_{(i,t),(j,s)}^2}{K_{2,n}}\left(\underline{u}_{2,(j,s)}^2-\widetilde{\omega}_{(j,s),n}^2\right)\sigma_{(i,t),n}^2\right)^2\geq\frac{\overline{C}}{\epsilon}\right) \\
\leq & \frac{\epsilon}{\overline{C}}E\left[\frac{\left(\mu_n^{\min}\right)^4 n}{K_{2,n}^2}\left(\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}\frac{A_{(i,t),(j,s)}^2}{K_{2,n}}\left(\underline{u}_{2,(j,s)}^2-\widetilde{\omega}_{(j,s),n}^2\right)\sigma_{(i,t),n}^2\right)^2\right] \\
\leq & \epsilon
\end{aligned}$$

for all n sufficiently large, which shows that

$$\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}\frac{A_{(i,t),(j,s)}^2}{K_{2,n}}\left(\underline{u}_{2,(j,s),n}^2-\widetilde{\omega}_{(j,s),n}^2\right)\sigma_{(i,t),n}^2=O_p\left(\frac{K_{2,n}}{\left(\mu_n^{\min}\right)^2\sqrt{n}}\right)=o_p\left(1\right),$$

as required. □

Lemma S2-15: Let $\{X_{i,n},\mathcal{F}_{i,n},1\leq i\leq k_n,n\geq 1\}$ be a square integrable martingale difference array. Suppose that for all $\epsilon>0$

$$\sum_{i=1}^{k_n}E\left[X_{i,n}^2\mathbb{I}\left\{|X_{i,n}|>\epsilon\right\}|\mathcal{F}_{i-1,n}\right]\xrightarrow{p}0 \tag{34}$$

and

$$\sum_{i=1}^{k_n}E\left[X_{i,n}^2|\mathcal{F}_{i-1,n}\right]\xrightarrow{p}1. \tag{35}$$

Then, $\sum_{i=1}^{k_n}X_{i,n}\xrightarrow{d}N\left(0,1\right)$.

Proof of Lemma S2-15: The proof of this central limit theorem for square integrable martingale difference array is given in Gänsler and Stute (1977). See also Corollary 3.1 in Hall and Heyde (1980).

Remark: Note that a sufficient condition for condition (34), which we will verify in lieu of (34) in the proof of Theorems 2 and 3 in Appendix S1, is the following

$$\sum_{i=1}^{k_n}E\left[\left|X_{i,n}\right|^{2+\delta}\right]\xrightarrow{p}0\text{, for some }\delta>0. \tag{36}$$

Lemma S2-16: Let $\widetilde{L}_n$ be a sequence of $l\times d$, nonrandom matrices (with $l\leq d$) such that $\left\|\widetilde{L}_n\right\|_F^2\leq\overline{C}<\infty$ for some constant $\overline{C}$, and let $\Sigma_{2,n}=VC\left(D_\mu^{-1}\underline{U}'A\varepsilon|\mathcal{F}_n^W\right)$

$= D_{\mu}^{-1} V C \left(\underline{U}' A \varepsilon | \mathcal{F}_n^W \right) D_{\mu}^{-1}$. Suppose that there exists a positive constant $\underline{C}$ such that $\lambda_{\min} \left(\left(\mu_n^{\min} \right)^2 \widetilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \widetilde{L}_n' / K_{2,n} \right) \geq \underline{C} > 0$ $a.s.n.$ Furthermore, let $a \in \mathbb{R}^d$ such that $\|a\|_2 = 1$ and let $\underline{u}_{a,(i,t),n} = a' \left(\left(\mu_n^{\min} \right)^2 \widetilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \widetilde{L}_n' / K_{2,n} \right)^{-1/2} \widetilde{L}_n D_{\mu}^{-1} \underline{U}_{(i,t)}$. Suppose that Assumptions 1-2 and 5-6 are satisfied and that $\left(\mu_n^{\min} \right)^2 / K_{2,n} = o(1)$ but $\sqrt{K_{2,n}} / \left(\mu_n^{\min} \right)^2 \to 0$. Under these conditions, the following statements are true:

(a) $\left[\left(\mu_n^{\min} \right)^2 / K_{2,n} \right] \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 \left(\underline{u}_{a,(j,s),n}^2 - E \left[\underline{u}_{a,(j,s),n}^2 | \mathcal{F}_n^W \right] \right) E \left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W \right]$
$= O_p \left(n^{-1/2} \right) = o_p(1)$;

(b) $\left[\left(\mu_n^{\min} \right)^2 / K_{2,n} \right] \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 \left(\varepsilon_{(j,s)}^2 - E \left[\varepsilon_{(j,s)}^2 | \mathcal{F}_n^W \right] \right) E \left[\underline{u}_{a,(i,t),n}^2 | \mathcal{F}_n^W \right]$
$= O_p \left(n^{-1/2} \right) = o_p(1)$.

Proof of Lemma S2-16:

To proceed, note first that Lemma S2-6 along with the assumptions on $\lambda_{\min} \left(\left(\mu_n^{\min} \right)^2 \widetilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \widetilde{L}_n' / K_{2,n} \right)$ and $\left\| \widetilde{L}_n \right\|_F^2$ together imply that

$$\begin{aligned}
E \left[\underline{u}_{a,(i,t),n}^2 | \mathcal{F}_n^W \right] &= a' \left(\frac{\left(\mu_n^{\min} \right)^2 \widetilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \widetilde{L}_n'}{K_{2,n}} \right)^{-1/2} \widetilde{L}_n D_{\mu}^{-1} E \left[\underline{U}_{(i,t)} \underline{U}_{(i,t)}' | \mathcal{F}_n^W \right] \\
&\quad D_{\mu}^{-1} \widetilde{L}_n' \left(\frac{\left(\mu_n^{\min} \right)^2 \widetilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \widetilde{L}_n'}{K_{2,n}} \right)^{-1/2} a \\
&\leq \frac{1}{\left(\mu_n^{\min} \right)^2} \frac{\max_{1 \leq (i,t) \leq m_n} E \left[\left\| \underline{U}_{(i,t)} \right\|_2^2 | \mathcal{F}_n^W \right] \left\| \widetilde{L}_n \right\|_F^2}{\lambda_{\min} \left(\left(\mu_n^{\min} \right)^2 \widetilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \widetilde{L}_n' / K_{2,n} \right)} \\
&= O_{a.s.} \left(\frac{1}{\left(\mu_n^{\min} \right)^2} \right) \qquad (37)
\end{aligned}$$

and that

$$\begin{aligned}
E \left[\underline{u}_{a,(i,t),n}^4 | \mathcal{F}_n^W \right] &= E \left[\left(a' \left(\frac{\left(\mu_n^{\min} \right)^2 \widetilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \widetilde{L}_n'}{K_{2,n}} \right)^{-1/2} \widetilde{L}_n D_{\mu}^{-1} \underline{U}_{(i,t)} \underline{U}_{(i,t)}' D_{\mu}^{-1} \widetilde{L}_n' \right. \right. \\
&\quad \left. \left. \times \left(\frac{\left(\mu_n^{\min} \right)^2 \widetilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \widetilde{L}_n'}{K_{2,n}} \right)^{-1/2} a \right)^2 | \mathcal{F}_n^W \right] \\
&\leq \frac{1}{\left(\mu_n^{\min} \right)^4} \frac{\max_{1 \leq (i,t) \leq m_n} E \left[\left\| \underline{U}_{(i,t)} \right\|_2^4 | \mathcal{F}_n^W \right] \left\| \widetilde{L}_n \right\|_F^4}{\left[\lambda_{\min} \left(\left(\mu_n^{\min} \right)^2 \widetilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \widetilde{L}_n' / K_{2,n} \right) \right]^2} \\
&= O_{a.s.} \left(\frac{1}{\left(\mu_n^{\min} \right)^4} \right) \qquad (38)
\end{aligned}$$

For part (a), define $\mathfrak{Z}_n = \left[\left(\mu_n^{\min}\right)^2 / K_{2,n}\right] \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 \left(\underline{u}_{a,(j,s),n}^2 - E\left[\underline{u}_{a,(j,s),n}^2 | \mathcal{F}_n^W\right]\right) E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W\right]$, and note that we can apply Assumption 2(i), part (c) of Lemma S2-1, and the upper bounds given by (37) and (38) above to obtain

$$
\begin{aligned}
E\left[\mathfrak{Z}_n^2 | \mathcal{F}_n^W\right] &\leq \frac{\left(\mu_n^{\min}\right)^4}{K_{2,n}^2} \sum_{(j,s)=1}^{m_n} \sum_{\substack{(i,t),(k,v)=1 \\ (i,t)\neq(j,s),(k,v)\neq(j,s)}}^{m_n} \left(A_{(i,t),(j,s)}^2 A_{(k,v),(j,s)}^2 E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W\right] E\left[\varepsilon_{(k,v)}^2 | \mathcal{F}_n^W\right]\right. \\
&\qquad \left. \times \left\{E\left[\underline{u}_{a,(j,s),n}^4 | \mathcal{F}_n^W\right] + \left(E\left[\underline{u}_{a,(j,s),n}^2 | \mathcal{F}_n^W\right]\right)^2\right\}\right) \\
&\leq O_{a.s.}\left(\frac{\left(\mu_n^{\min}\right)^4}{K_{2,n}^2}\right) O_{a.s.}\left(\frac{K_{2,n}^2}{n}\right) O_{a.s.}(1)\, O_{a.s.}(1)\, O_{a.s.}\left(\frac{1}{\left(\mu_n^{\min}\right)^4}\right) = O_{a.s.}\left(\frac{1}{n}\right)
\end{aligned}
$$

Hence, the law of iterated expectations and Theorem 16.1 of Billingsley (1995) imply that there exists a positive constant $\overline{C} < \infty$ such that, for all n sufficiently large, $E\left(n\mathfrak{Z}_n^2\right) = E_{W_n}\left(nE\left[\mathfrak{Z}_n^2 | \mathcal{F}_n^W\right]\right) \leq \overline{C}$. Application of the Markov's inequality then implies that $\mathfrak{Z}_n = O_p\left(n^{-1/2}\right) = o_p(1)$, which shows part (a).

For part (b), we can apply Assumption 2, part (c) of Lemma S2-1, and the result given in expression (37) above to obtain

$$
\begin{aligned}
& E\left\{\left(\frac{\left(\mu_n^{\min}\right)^2}{K_{2,n}} \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 \left(\varepsilon_{(j,s)}^2 - E\left[\varepsilon_{(j,s)}^2 | \mathcal{F}_n^W\right]\right) E\left[\underline{u}_{a,(i,t),n}^2 | \mathcal{F}_n^W\right]\right)^2 | \mathcal{F}_n^W\right\} \\
= & \frac{\left(\mu_n^{\min}\right)^4}{K_{2,n}^2} \sum_{(i,t)=2}^{m_n} \sum_{(k,v)=2}^{m_n} \sum_{(j,s)=1}^{\min\{(i,t)-1,(k,v)-1\}} \left(A_{(i,t),(j,s)}^2 A_{(k,v),(j,s)}^2\right. \\
& \qquad \left. \times E\left[\underline{u}_{a,(i,t),n}^2 | \mathcal{F}_n^W\right] E\left[\underline{u}_{a,(k,v),n}^2 | \mathcal{F}_n^W\right] E\left[\left(\varepsilon_{(j,s)}^2 - E\left[\varepsilon_{(j,s)}^2 | \mathcal{F}_n^W\right]\right)^2 | \mathcal{F}_n^W\right]\right) \\
\leq & \frac{\left(\mu_n^{\min}\right)^4}{K_{2,n}^2} \sum_{(j,s)=1}^{m_n} \sum_{\substack{(i,t),(k,v)=1 \\ (i,t)\neq(j,s),(k,v)\neq(j,s)}}^{m_n} \left(A_{(i,t),(j,s)}^2 A_{(k,v),(j,s)}^2 E\left[\underline{u}_{a,(i,t),n}^2 | \mathcal{F}_n^W\right] E\left[\underline{u}_{a,(k,v),n}^2 | \mathcal{F}_n^W\right]\right. \\
& \qquad \left. \times \left\{E\left[\varepsilon_{(j,s)}^4 | \mathcal{F}_n^W\right] + \left(E\left[\varepsilon_{(j,s)}^2 | \mathcal{F}_n^W\right]\right)^2\right\}\right) \\
= & O_{a.s.}\left(\frac{\left(\mu_n^{\min}\right)^4}{K_{2,n}^2}\right) O_{a.s.}\left(\frac{K_{2,n}^2}{n}\right) O_{a.s.}\left(\frac{1}{\left(\mu_n^{\min}\right)^2}\right) O_{a.s.}\left(\frac{1}{\left(\mu_n^{\min}\right)^2}\right) \\
= & O_{a.s.}\left(\frac{1}{n}\right)
\end{aligned}
$$

Hence, the law of iterated expectations and Theorem 16.1 of Billingsley (1995) imply that there exists a positive constant $\overline{C} < \infty$ such that for all n sufficiently large

$$\begin{aligned}
& nE\left[\left(\frac{\left(\mu_n^{\min}\right)^2}{K_{2,n}}\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}A_{(i,t),(j,s)}^2\left(\varepsilon_{(j,s)}^2-E\left[\varepsilon_{(j,s)}^2|\mathcal{F}_n^W\right]\right)E\left[\underline{u}_{a,(i,t),n}^2|\mathcal{F}_n^W\right]\right)^2\right] \\
= \; & E_W\left\{nE\left[\left(\frac{\left(\mu_n^{\min}\right)^2}{K_{2,n}}\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}A_{(i,t),(j,s)}^2\left(\varepsilon_{(j,s)}^2-E\left[\varepsilon_{(j,s)}^2|\mathcal{F}_n^W\right]\right)E\left[\underline{u}_{a,(i,t),n}^2|\mathcal{F}_n^W\right]\right)^2|\mathcal{F}_n^W\right]\right\} \\
\leq \; & \overline{C}
\end{aligned}$$

For any $\epsilon > 0$, set $\overline{C}_\epsilon = \sqrt{\overline{C}/\epsilon}$, and it follows by Markov's inequality that, for all n sufficiently large,

$$\begin{aligned}
& \Pr\left(\sqrt{n}\left|\frac{\left(\mu_n^{\min}\right)^2}{K_{2,n}}\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}A_{(i,t),(j,s)}^2\left(\varepsilon_{(j,s)}^2-E\left[\varepsilon_{(j,s)}^2|\mathcal{F}_n^W\right]\right)E\left[\underline{u}_{a,(i,t),n}^2|\mathcal{F}_n^W\right]\right|\geq\overline{C}_\epsilon\right) \\
= \; & \Pr\left(n\left(\frac{\left(\mu_n^{\min}\right)^2}{K_{2,n}}\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}A_{(i,t),(j,s)}^2\left(\varepsilon_{(j,s)}^2-E\left[\varepsilon_{(j,s)}^2|\mathcal{F}_n^W\right]\right)E\left[\underline{u}_{a,(i,t),n}^2|\mathcal{F}_n^W\right]\right)^2\geq\overline{C}_\epsilon^2\right) \\
\leq \; & \frac{1}{\overline{C}_\epsilon^2}nE\left[\left(\frac{\left(\mu_n^{\min}\right)^2}{K_{2,n}}\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}A_{(i,t),(j,s)}^2\left(\varepsilon_{(j,s)}^2-E\left[\varepsilon_{(j,s)}^2|\mathcal{F}_n^W\right]\right)E\left[\underline{u}_{a,(i,t),n}^2|\mathcal{F}_n^W\right]\right)^2\right] \\
\leq \; & \frac{\overline{C}}{\overline{C}/\epsilon}=\epsilon
\end{aligned}$$

which shows that

$$\begin{aligned}
& \frac{\left(\mu_n^{\min}\right)^2}{K_{2,n}}\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}A_{(i,t),(j,s)}^2\left(\varepsilon_{(j,s)}^2-E\left[\varepsilon_{(j,s)}^2|\mathcal{F}_n^W\right]\right)E\left[\underline{u}_{a,(i,t),n}^2|\mathcal{F}_n^W\right] \\
= \; & O_p\left(\frac{1}{\sqrt{n}}\right)=o_p\left(1\right). \; \square
\end{aligned}$$

Lemma S2-17 Under Assumptions 1-7, $D_\mu^{-1}X'AD\left(\varepsilon\circ\varepsilon\right)AXD_\mu^{-1}=\Sigma_{1,n}$ $+\sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n}A_{(i,t),(j,s)}^2\sigma_{(i,t)}^2D_\mu^{-1}\Psi_{(j,s)}D_\mu^{-1}+o_p\left(1\right)$, where $\Sigma_{1,n}=\Gamma'M^{(Z_1,Q)}D_{\sigma^2}M^{(Z_1,Q)}\Gamma/n$, $\sigma_{(i,t)}^2=E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right]$, $D_{\sigma^2}=diag\left(\sigma_{(1,1)}^2,...,\sigma_{(n,T_n)}^2\right)$, and $\Psi_{(j,s)}=E\left[U_{(j,s)}U_{(j,s)}'|\mathcal{F}_n^W\right]$.

Proof of Lemma S2-17: To proceed, using the fact that $AQ = 0$, we can write

$$
\begin{aligned}
D_\mu^{-1} X' AD\left(\varepsilon \circ \varepsilon\right) AXD_\mu^{-1} &= D_\mu^{-1}\left(\frac{1}{\sqrt{n}}D_\mu \Gamma' + \frac{1}{\sqrt{n}}D_\kappa F' + \Xi' Q' + U'\right) AD\left(\varepsilon \circ \varepsilon\right) A \\
&\quad \times \left(\frac{1}{\sqrt{n}}\Gamma D_\mu + \frac{1}{\sqrt{n}}FD_\kappa + Q\Xi + U\right) D_\mu^{-1} \\
&= \frac{\Gamma' AD\left(\varepsilon \circ \varepsilon\right) A\Gamma}{n} + \frac{D_\mu^{-1} D_\kappa F' AD\left(\varepsilon \circ \varepsilon\right) AFD_\kappa D_\mu^{-1}}{n} \\
&\quad + D_\mu^{-1} U' AD\left(\varepsilon \circ \varepsilon\right) AUD_\mu^{-1} + \frac{D_\mu^{-1} D_\kappa F' AD\left(\varepsilon \circ \varepsilon\right) A\Gamma}{n} \\
&\quad + \frac{\Gamma' AD\left(\varepsilon \circ \varepsilon\right) AFD_\kappa D_\mu^{-1}}{n} + \frac{\Gamma' AD\left(\varepsilon \circ \varepsilon\right) AUD_\mu^{-1}}{\sqrt{n}} \\
&\quad + \frac{D_\mu^{-1} U' AD\left(\varepsilon \circ \varepsilon\right) A\Gamma}{\sqrt{n}} + \frac{D_\mu^{-1} D_\kappa F' AD\left(\varepsilon \circ \varepsilon\right) AUD_\mu^{-1}}{\sqrt{n}} \\
&\quad + \frac{D_\mu^{-1} U' AD\left(\varepsilon \circ \varepsilon\right) AFD_\kappa D_\mu^{-1}}{\sqrt{n}}
\end{aligned}
$$

Hence, by applying part (f) of Lemma OA-9 as well as (a)-(e) of Lemma OA-10, we obtain

$$
\begin{aligned}
& D_\mu^{-1} X' AD\left(\varepsilon \circ \varepsilon\right) AXD_\mu^{-1} - \frac{\Gamma' M^{(Z_1,Q)} D_{\sigma^2} M^{(Z_1,Q)} \Gamma}{n} \\
& - \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^2 E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W\right] D_\mu^{-1} E\left[U_{(j,s)} U_{(j,s)}' | \mathcal{F}_n^W\right] D_\mu^{-1} \\
= & \frac{\Gamma' AD\left(\varepsilon \circ \varepsilon\right) A\Gamma}{n} - \frac{\Gamma' M^{(Z_1,Q)} D_{\sigma^2} M^{(Z_1,Q)} \Gamma}{n} \\
& + D_\mu^{-1} U' AD\left(\varepsilon \circ \varepsilon\right) AUD_\mu^{-1} - \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^2 E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W\right] D_\mu^{-1} E\left[U_{(j,s)} U_{(j,s)}' | \mathcal{F}_n^W\right] D_\mu^{-1} \\
& + \frac{D_\mu^{-1} D_\kappa F' AD\left(\varepsilon \circ \varepsilon\right) AFD_\kappa D_\mu^{-1}}{n} + \frac{D_\mu^{-1} D_\kappa F' AD\left(\varepsilon \circ \varepsilon\right) A\Gamma}{n} \\
& + \frac{\Gamma' AD\left(\varepsilon \circ \varepsilon\right) AFD_\kappa D_\mu^{-1}}{n} + \frac{\Gamma' AD\left(\varepsilon \circ \varepsilon\right) AUD_\mu^{-1}}{\sqrt{n}} \\
& + \frac{D_\mu^{-1} U' AD\left(\varepsilon \circ \varepsilon\right) A\Gamma}{\sqrt{n}} + \frac{D_\mu^{-1} U' AD\left(\varepsilon \circ \varepsilon\right) AFD_\kappa D_\mu^{-1}}{\sqrt{n}} \\
& + \frac{D_\mu^{-1} D_\kappa F' AD\left(\varepsilon \circ \varepsilon\right) AUD_\mu^{-1}}{\sqrt{n}} \\
= & o_p(1). \ \square
\end{aligned}
$$

Lemma S2-18 Suppose that Assumptions 1-8 are satisfied, and let $\left\{\widehat{\delta}_n\right\}$ be any sequence of estimators such that $\left\|\widehat{\delta}_n - \delta_0\right\|_2 \xrightarrow{p} 0$ as $n \to \infty$, as long as $\sqrt{K_{2,n}}/\left(\mu_n^{\min}\right)^2 \to 0$. Also, define

the following notations: let $\widehat{\varepsilon} = M^{(Z,Q)}\left(y - X\widehat{\delta}_n\right)$, $J = \left[M^Q \circ M^Q\right]^{-1}$, $S_1 = X'AD\left(J\left[\widehat{\varepsilon}\circ\widehat{\varepsilon}\right]\right)AX$, $S_2 = \left(\widehat{\varepsilon}\circ\widehat{\varepsilon}\right)' J\left(A\circ A\right)J\left(\widehat{\varepsilon}\iota_d' \circ M^{(Z,Q)}X\right)$,
$\underline{S}_2 = \left(\widehat{\varepsilon}\circ\widehat{\varepsilon}\right)' J\left(A\circ A\right)J\left(\widehat{\varepsilon}\iota_d' \circ \underline{\widehat{U}}\right)$ with $\underline{\widehat{U}} = M^{(Z,Q)}X - \widehat{\varepsilon}\widehat{\rho}_n'$, $S_3 = \left(\widehat{\varepsilon}\circ\widehat{\varepsilon}\right)' J\left(A\circ A\right)J\left(\widehat{\varepsilon}\circ\widehat{\varepsilon}\right)$,
$S_4 = \left(\widehat{\varepsilon}\iota_d' \circ M^{(Z,Q)}X\right)' J\left(A\circ A\right)J\left(\widehat{\varepsilon}\iota_d' \circ M^{(Z,Q)}X\right)$, $\underline{S}_4 = \left(\widehat{\varepsilon}\iota_d' \circ \underline{\widehat{U}}\right)' J\left(A\circ A\right)J\left(\widehat{\varepsilon}\iota_d' \circ \underline{\widehat{U}}\right)$, and $\Sigma_{1,n} = \Gamma' M^{(Z_1,Q)} D_{\sigma^2} M^{(Z_1,Q)}\Gamma/n$. In addition, define $\sigma^2_{(i,t)} = E\left[\varepsilon^2_{(i,t)}|\mathcal{F}^W_n\right]$,
$D_{\sigma^2} = diag\left(\sigma^2_{(1,1)}, ..., \sigma^2_{(n,T_n)}\right)$, $\phi_{(i,t)} = E\left[U_{(i,t)}\varepsilon_{(i,t)}|\mathcal{F}^W_n\right]$, $\Psi_{(i,t)} = E\left[U_{(i,t)}U'_{(i,t)}|\mathcal{F}^W_n\right]$,
$\underline{\phi}_{(i,t)} = E\left[\underline{U}_{(i,t)}\varepsilon_{(i,t)}|\mathcal{F}^W_n\right]$, and $\underline{\Psi}_{(i,t)} = E\left[\underline{U}_{(i,t)}\underline{U}'_{(i,t)}|\mathcal{F}^W_n\right]$ where $\underline{U}_{(i,t)} = U_{(i,t)} - \rho\varepsilon_{(i,t)}$ and where for notational convenience we suppress the dependence of $\sigma^2_{(i,t)}$, $\phi_{(i,t)}$, $\Psi_{(i,t)}$, $\underline{\phi}_{(i,t)}$, and $\underline{\Psi}_{(i,t)}$ on $\mathcal{F}^W_n = \sigma\left(W_n\right)$. Then, under the above conditions, the following statements are true.

(a) $D_\mu^{-1}S_1D_\mu^{-1} = \Sigma_{1,n} + \sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A^2_{(i,t),(j,s)}\sigma^2_{(i,t)}D_\mu^{-1}\Psi_{(j,s)}D_\mu^{-1}$
$+o_p\left(\max\left\{1, K_{2,n}\left(\mu_n^{\min}\right)^{-2}\right\}\right)$.

(b) $S_3/K_{2,n} - K_{2,n}^{-1}\sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A^2_{(i,t),(j,s)}\sigma^2_{(i,t)}\sigma^2_{(j,s)} = o_p\left(1\right)$.

(c) $D_\mu^{-1}S_4D_\mu^{-1} - \sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A^2_{(i,t),(j,s)}D_\mu^{-1}\phi_{(i,t)}\phi'_{(j,s)}D_\mu^{-1} = o_p\left(K_{2,n}\left(\mu_n^{\min}\right)^{-2}\right)$.

(d) $\left(\mu_n^{\min}/K_{2,n}\right)S_2D_\mu^{-1} - \left(\mu_n^{\min}/K_{2,n}\right)\sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A^2_{(i,t),(j,s)}\sigma^2_{(i,t)}\phi'_{(j,s)}D_\mu^{-1} = o_p\left(1\right)$.

(e) $D_\mu^{-1}\widehat{\rho}_n = O_p\left(\left(\mu_n^{\min}\right)^{-1}\right)$ and $D_\mu^{-1}\left(\widehat{\rho}_n - \rho\right) = o_p\left(\left(\mu_n^{\min}\right)^{-1}\right)$, where $\rho = \lim_{n\to\infty}\rho_n = \lim_{n\to\infty}\left(E\left[U'M^Q\varepsilon\right]/n\right)/\left(E\left[\varepsilon'M^Q\varepsilon\right]/n\right)$.

(f) $D_\mu^{-1}\underline{S}_4D_\mu^{-1} - \sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A^2_{(i,t),(j,s)}D_\mu^{-1}\underline{\phi}_{(i,t)}\underline{\phi}'_{(j,s)}D_\mu^{-1} = o_p\left(K_{2,n}\left(\mu_n^{\min}\right)^{-2}\right)$.

(g) $\left(\mu_n^{\min}/K_{2,n}\right) - \left(\mu_n^{\min}/K_{2,n}\right)\sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A^2_{(i,t),(j,s)}\sigma^2_{(i,t)}\underline{\phi}'_{(j,s)}D_\mu^{-1} = o_p(1)$.

Proof of Lemma S2-18:

To show part (a), note first that, by making use of the decomposition $M^{(Z,Q)} = M^Q - P^{Z^\perp}$, where $P^{Z^\perp} = M^QZ\left(Z'M^QZ\right)^{-1}Z'M^Q$, we can, after some straightforward algebra, write $\widehat{\varepsilon} = -M^{(Z,Q)}X\left(\widehat{\delta}_n - \delta_0\right) + M^{(Z,Q)}\varphi_n + M^Q\varepsilon - P^{Z^\perp}\varepsilon$, from which we obtain

$$\begin{aligned} J\left[\widehat{\varepsilon}\circ\widehat{\varepsilon}\right] &= J\left[M^{(Z,Q)}X\left(\widehat{\delta}_n - \delta_0\right)\circ M^{(Z,Q)}X\left(\widehat{\delta}_n - \delta_0\right)\right] + J\left[M^{(Z,Q)}\varphi_n \circ M^{(Z,Q)}\varphi_n\right] \\ &\quad + J\left[M^Q\varepsilon\circ M^Q\varepsilon\right] + J\left[P^{Z^\perp}\varepsilon\circ P^{Z^\perp}\varepsilon\right] - 2J\left[M^{(Z,Q)}\varphi_n \circ M^{(Z,Q)}X\left(\widehat{\delta}_n - \delta_0\right)\right] \\ &\quad - 2J\left[M^Q\varepsilon\circ M^{(Z,Q)}X\left(\widehat{\delta}_n - \delta_0\right)\right] + 2J\left[P^{Z^\perp}\varepsilon\circ M^{(Z,Q)}X\left(\widehat{\delta}_n - \delta_0\right)\right] \\ &\quad + 2J\left[M^Q\varepsilon\circ M^{(Z,Q)}\varphi_n\right] - 2J\left[P^{Z^\perp}\varepsilon\circ M^{(Z,Q)}\varphi_n\right] - 2J\left[M^Q\varepsilon\circ P^{Z^\perp}\varepsilon\right] \end{aligned} \quad (39)$$

where $J = \left[M^Q \circ M^Q\right]^{-1}$. Substituting the right-hand side of (39) into covariance matrix estimator $D_\mu^{-1} X' A D \left(J \left[\widehat{\varepsilon} \circ \widehat{\varepsilon}\right]\right) A X D_\mu^{-1}$, we get

$$\begin{aligned} & D_\mu^{-1} S_1 D_\mu^{-1} - D_\mu^{-1} X' A D \left(\varepsilon \circ \varepsilon\right) A X D_\mu^{-1} \\ = \quad & \mathcal{T}_{1,n} + \mathcal{T}_{2,n} + \mathcal{T}_{3,n} + \mathcal{T}_{4,n} + \mathcal{T}_{5,n} + \mathcal{T}_{6,n} + \mathcal{T}_{7,n} + \mathcal{T}_{8,n} + \mathcal{T}_{9,n} + \mathcal{T}_{10,n}, \end{aligned}$$

where $\mathcal{T}_{1,n} = D_\mu^{-1} X' A D \left(J \left[M^{(Z,Q)} X \left(\widehat{\delta}_n - \delta_0\right) \circ M^{(Z,Q)} X \left(\widehat{\delta}_n - \delta_0\right)\right]\right) A X D_\mu^{-1}$,
$\mathcal{T}_{2,n} = D_\mu^{-1} X' A D \left(J \left[M^{(Z,Q)} \varphi_n \circ M^{(Z,Q)} \varphi_n\right]\right) A X D_\mu^{-1}$,
$\mathcal{T}_{3,n} = D_\mu^{-1} X' A D \left(J \left[M^Q \varepsilon \circ M^Q \varepsilon\right]\right) A X D_\mu^{-1} - D_\mu^{-1} X' A D \left(\varepsilon \circ \varepsilon\right) A X D_\mu^{-1}$,
$\mathcal{T}_{4,n} = D_\mu^{-1} X' A D \left(J \left[P^{Z^\perp} \varepsilon \circ P^{Z^\perp} \varepsilon\right]\right) A X D_\mu^{-1}$,
$\mathcal{T}_{5,n} = -2 D_\mu^{-1} X' A D \left(J \left[M^{(Z,Q)} \varphi_n \circ M^{(Z,Q)} X \left(\widehat{\delta}_n - \delta_0\right)\right]\right) A X D_\mu^{-1}$,
$\mathcal{T}_{6,n} = -2 D_\mu^{-1} X' A D \left(J \left[M^Q \varepsilon \circ M^{(Z,Q)} X \left(\widehat{\delta}_n - \delta_0\right)\right]\right) A X D_\mu^{-1}$,
$\mathcal{T}_{7,n} = 2 D_\mu^{-1} X' A D \left(J \left[P^{Z^\perp} \varepsilon \circ M^{(Z,Q)} X \left(\widehat{\delta}_n - \delta_0\right)\right]\right) A X D_\mu^{-1}$,
$\mathcal{T}_{8,n} = 2 D_\mu^{-1} X' A D \left(J \left[M^Q \varepsilon \circ M^{(Z,Q)} \varphi_n\right]\right) A X D_\mu^{-1}$,
$\mathcal{T}_{9,n} = -2 D_\mu^{-1} X' A D \left(J \left[P^{Z^\perp} \varepsilon \circ M^{(Z,Q)} \varphi_n\right]\right) A X D_\mu^{-1}$, and
$\mathcal{T}_{10,n} = -2 D_\mu^{-1} X' A D \left(J \left[M^Q \varepsilon \circ P^{Z^\perp} \varepsilon\right]\right) A X D_\mu^{-1}$.

Consider the term $\mathcal{T}_{1,n}$. Let $\mathcal{G}_i = \{(\ell, h) : \ell = i \text{ and } h = 1, ..., T_i\}$; and note that for any $a, b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$, we have

$$\begin{aligned} \left|a' \mathcal{T}_{1,n} b\right| &= \Bigg| \sum_{(i,t)=1}^{m_n} \sum_{(j,s) \neq (i,t)} \sum_{(k,v) \neq (i,t)} \sum_{(p,q)=1}^{m_n} a' D_\mu^{-1} X' e_{(j,s)} A_{(i,t),(j,s)} J_{(i,t),(p,q)} \\ &\quad \times e_{(p,q)}' M^{(Z,Q)} X \left(\widehat{\delta}_n - \delta_0\right) \left(\widehat{\delta}_n - \delta_0\right)' X' M^{(Z,Q)} e_{(p,q)} A_{(i,t),(k,v)} e_{(k,v)}' X D_\mu^{-1} b \Bigg| \\ &= \Bigg| \sum_{(i,t)=1}^{m_n} \sum_{(j,s) \neq (i,t)} a' D_\mu^{-1} X' e_{(j,s)} A_{(i,t),(j,s)} \\ &\quad \times \left(\sum_{(p,q)=1}^{m_n} J_{(i,t),(p,q)} \mathbb{I}\left\{(p,q) \in \mathcal{G}_i\right\} e_{(p,q)}' M^{(Z,Q)} X \left(\widehat{\delta}_n - \delta_0\right) \left(\widehat{\delta}_n - \delta_0\right)' X' M^{(Z,Q)} e_{(p,q)} \right) \\ &\quad \times \sum_{(k,v) \neq (i,t)} A_{(i,t),(k,v)} e_{(k,v)}' X D_\mu^{-1} b \Bigg| \end{aligned}$$

where $J_{(i,t),(p,q)}$ is the element in the $(i,t)^{th}$ row and the $(p,q)^{th}$ column of the matrix J for $(i,t), (p,q) \in \{1, ..., m_n\}$, where $\mathbb{I}\{\cdot\}$ denotes an indicator function, where $e_{(j,s)}$ is an $m_n \times 1$ elementary vector whose $(j,s)^{th}$ component is 1 and all other components are 0, and where the second equality above follows from the fact that $J_{(i,t),(p,q)} = 0$ for $p \neq i$ due to the sparsity (or block diagonal nature) of J. Now, let $D^{\Sigma J}\left(M^{(Z,Q)} X X' M^{(Z,Q)}\right)$ be a diagonal matrix whose $(i,t)^{th}$ diagonal element is given by $\sum_{q=1}^{T_i} \left|J_{(i,t),(i,q)}\right| e_{(i,q)}' M^{(Z,Q)} X X' M^{(Z,Q)} e_{(i,q)}$. Applying the triangle inequality and the inequality $|XY| \leq (1/2)\, X^2 + (1/2)\, Y^2$, we then obtain

$$
\begin{aligned}
\left|a'\mathcal{T}_{1,n}b\right| &= \Bigg|\sum_{(i,t)=1}^{m_n} a'D_\mu^{-1}X'Ae_{(i,t)}\sum_{q=1}^{T_i} J_{(i,t),(i,q)}e'_{(i,q)}M^{(Z,Q)}X\left(\widehat{\delta}_n-\delta_0\right)\left(\widehat{\delta}_n-\delta_0\right)' \\
&\quad \times X'M^{(Z,Q)}e_{(i,q)}e'_{(i,t)}AXD_\mu^{-1}b\Bigg| \\
&\leq \frac{1}{2}\sum_{(i,t)=1}^{m_n}\sum_{q=1}^{T_i}\Big\{\left|J_{(i,t),(i,q)}\right|e'_{(i,q)}M^{(Z,Q)}X\left(\widehat{\delta}_n-\delta_0\right)\left(\widehat{\delta}_n-\delta_0\right)'X'M^{(Z,Q)}e_{(i,q)} \\
&\qquad \times a'D_\mu^{-1}X'Ae_{(i,t)}e'_{(i,t)}AXD_\mu^{-1}a\Big\} \\
&\quad +\frac{1}{2}\sum_{(i,t)=1}^{m_n}\sum_{q=1}^{T_i}\Big\{\left|J_{(i,t),(i,q)}\right|e'_{(i,q)}M^{(Z,Q)}X\left(\widehat{\delta}_n-\delta_0\right)\left(\widehat{\delta}_n-\delta_0\right)'X'M^{(Z,Q)}e_{(i,q)} \\
&\qquad \times b'D_\mu^{-1}X'Ae_{(i,t)}e'_{(i,t)}AXD_\mu^{-1}b\Big\} \\
&\leq \frac{1}{2}\left\|\widehat{\delta}_n-\delta_0\right\|_2^2 a'D_\mu^{-1}X'AD^{\Sigma J}\left(M^{(Z,Q)}XX'M^{(Z,Q)}\right)AXD_\mu^{-1}a \\
&\quad +\frac{1}{2}\left\|\widehat{\delta}_n-\delta_0\right\|_2^2 b'D_\mu^{-1}X'AD^{\Sigma J}\left(M^{(Z,Q)}XX'M^{(Z,Q)}\right)AXD_\mu^{-1}b \qquad (40)
\end{aligned}
$$

By tedious but straightforward calculations, we can show that $D_\mu^{-1}X'AD^{\Sigma J}\left(M^{(Z,Q)}XX'M^{(Z,Q)}\right)AXD_\mu^{-1} = O_p\left(\max\left\{1, K_{2,n}\left(\mu_n^{\min}\right)^{-2}\right\}\right)$. Hence, given the assumption that $\left\|\widehat{\delta}_n-\delta_0\right\|_2^2 \xrightarrow{p} 0$, it follows from expression (40) that $|a'\mathcal{T}_{1,n}b| = o_p\left(\max\left\{1, K_{2,n}\left(\mu_n^{\min}\right)^{-2}\right\}\right)$. Since the above argument holds for all $a, b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$, we further deduce that $\mathcal{T}_{1,n} = o_p\left(\max\left\{1, K_{2,n}\left(\mu_n^{\min}\right)^{-2}\right\}\right)$. By following a similar method of proof, we can also show that $\mathcal{T}_{k,n} = o_p\left(\max\left\{1, K_{2,n}\left(\mu_n^{\min}\right)^{-2}\right\}\right)$ for $k = 2, ..., 10$.

The fact that $\mathcal{T}_{k,n} = o_p\left(\max\left\{1, K_{2,n}\left(\mu_n^{\min}\right)^{-2}\right\}\right)$ for each $k = \{1, ..., 10\}$ further implies that

$$
D_\mu^{-1}S_1D_\mu^{-1} - D_\mu^{-1}X'AD\left(\varepsilon\circ\varepsilon\right)AXD_\mu^{-1} = \sum_{k=1}^{10}\mathcal{T}_{k,n} = o_p\left(\max\left\{1, \frac{K_{2,n}}{\left(\mu_n^{\min}\right)^2}\right\}\right) \qquad (41)
$$

Moreover, by the result of Lemma S2-17,

$$
D_\mu^{-1}X'AD\left(\varepsilon\circ\varepsilon\right)AXD_\mu^{-1} = \Sigma_{1,n} + \sum_{\substack{(i,t),(j,s)=1\\(i,t)\neq(j,s)}}^{m_n} A^2_{(i,t),(j,s)}\sigma^2_{(i,t)}D_\mu^{-1}\Psi_{(j,s)}D_\mu^{-1} + o_p(1). \qquad (42)
$$

Combining (41) and (42), we further obtain $D_\mu^{-1}S_1D_\mu^{-1} = \Sigma_{1,n} + \sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A^2_{(i,t),(j,s)}\sigma^2_{(i,t)}D_\mu^{-1}\Psi_{(j,s)}D_\mu^{-1} + o_p\left(\max\left\{1, K_{2,n}\left(\mu_n^{\min}\right)^{-2}\right\}\right)$, which shows part (a).

To show part (b), write $S_3/K_{2,n} - K_{2,n}^{-1}\sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A_{(i,t),(j,s)}^2 \sigma_{(i,t)}^2 \sigma_{(j,s)}^2 = \mathcal{A} + \mathfrak{A}$, where $\mathcal{A} = S_3/K_{2,n} - K_{2,n}^{-1}\sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A_{(i,t),(j,s)}^2 \varepsilon_{(i,t)}^2 \varepsilon_{(j,s)}^2$ and where $\mathfrak{A} = K_{2,n}^{-1}\sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A_{(i,t),(j,s)}^2 \left(\varepsilon_{(i,t)}^2 \varepsilon_{(j,s)}^2 - \sigma_{(i,t)}^2 \sigma_{(j,s)}^2\right)$. To analyze the term $\mathcal{A}$, note that, by direct calculation, we can obtain the following decomposition

$$
\begin{aligned}
\mathcal{A} &= \frac{1}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^2 \left\{ \sum_{h=1}^{T_i} J_{(i,t),(i,h)} \left[e_{(i,h)}' M^{(Z,Q)} \left(y - X\widehat{\delta}_n \right) \right]^2 \right. \\
&\quad \left. \times \sum_{v=1}^{T_j} J_{(j,s),(j,v)} \left[\left(y - X\widehat{\delta}_n \right)' M^{(Z,Q)} e_{(j,v)} \right]^2 \right\} - \frac{1}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^2 \varepsilon_{(i,t)}^2 \varepsilon_{(j,s)}^2 \\
&= \sum_{\ell=1}^{6} \mathcal{A}_\ell \qquad (43)
\end{aligned}
$$

where $\mathcal{A}_1 = 4K_{2,n}^{-1}\sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A_{(i,t),(j,s)}^2$
$\times \sum_{h=1}^{T_i} J_{(i,t),(i,h)} \left(e_{(i,h)}' \left[P^{Z^\perp}\varepsilon - M^{(Z,Q)}\varphi_n \right] + e_{(i,h)}' M^{(Z,Q)} X \left[\widehat{\delta}_n - \delta_0 \right] \right) \left(e_{(i,h)}' M^Q \varepsilon \right)$
$\times \sum_{v=1}^{T_j} J_{(j,s),(j,v)} \left(e_{(j,v)}' \left[P^{Z^\perp}\varepsilon - M^{(Z,Q)}\varphi_n \right] + e_{(j,v)}' M^{(Z,Q)} X \left[\widehat{\delta}_n - \delta_0 \right] \right) \left(e_{(j,v)}' M^Q \varepsilon \right)$,
$\mathcal{A}_2 = K_{2,n}^{-1}\sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A_{(i,t),(j,s)}^2$
$\times \sum_{h=1}^{T_i} J_{(i,t),(i,h)} \left(e_{(i,h)}' \left[P^{Z^\perp}\varepsilon - M^{(Z,Q)}\varphi_n \right] + e_{(i,h)}' M^{(Z,Q)} X \left[\widehat{\delta}_n - \delta_0 \right] \right)^2$
$\times \sum_{v=1}^{T_j} J_{(j,s),(j,v)} \left(e_{(j,v)}' \left[P^{Z^\perp}\varepsilon - M^{(Z,Q)}\varphi_n \right] + e_{(j,v)}' M^{(Z,Q)} X \left[\widehat{\delta}_n - \delta_0 \right] \right)^2$,
$\mathcal{A}_3 = -4K_{2,n}^{-1}\sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A_{(i,t),(j,s)}^2 \sum_{v=1}^{T_j} J_{(j,s),(j,v)} \left(e_{(j,v)}' M^Q \varepsilon \right)^2$
$\times \sum_{h=1}^{T_i} J_{(i,t),(i,h)} \left(e_{(i,h)}' \left[P^{Z^\perp}\varepsilon - M^{(Z,Q)}\varphi_n \right] + e_{(i,h)}' M^{(Z,Q)} X \left[\widehat{\delta}_n - \delta_0 \right] \right) \left(e_{(i,h)}' M^Q \varepsilon \right)$,
$\mathcal{A}_4 = 2K_{2,n}^{-1}\sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A_{(i,t),(j,s)}^2 \sum_{v=1}^{T_j} J_{(j,s),(j,v)} \left(e_{(j,v)}' M^Q \varepsilon \right)^2$
$\times \sum_{h=1}^{T_i} J_{(i,t),(i,h)} \left(e_{(i,h)}' \left[P^{Z^\perp}\varepsilon - M^{(Z,Q)}\varphi_n \right] + e_{(i,h)}' M^{(Z,Q)} X \left[\widehat{\delta}_n - \delta_0 \right] \right)^2$
, $\mathcal{A}_5 = -4K_{2,n}^{-1}\sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A_{(i,t),(j,s)}^2$
$\times \sum_{h=1}^{T_i} J_{(i,t),(i,h)} \left(e_{(i,h)}' \left[P^{Z^\perp}\varepsilon - M^{(Z,Q)}\varphi_n \right] + e_{(i,h)}' M^{(Z,Q)} X \left[\widehat{\delta}_n - \delta_0 \right] \right)^2$
$\times \sum_{v=1}^{T_j} J_{(j,s),(j,v)} \left(e_{(j,v)}' \left[P^{Z^\perp}\varepsilon - M^{(Z,Q)}\varphi_n \right] + e_{(j,v)}' M^{(Z,Q)} X \left[\widehat{\delta}_n - \delta_0 \right] \right) \left(e_{(j,v)}' M^Q \varepsilon \right)$,
and $\mathcal{A}_6 = K_{2,n}^{-1}\sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} \left\{ A_{(i,t),(j,s)}^2 \right.$
$\left. \times \sum_{h=1}^{T_i} J_{(i,t),(i,h)} \left(e_{(i,h)}' M^Q \varepsilon \right)^2 \sum_{v=1}^{T_j} J_{(j,s),(j,v)} \left(e_{(j,v)}' M^Q \varepsilon \right)^2 \right\}$
$-K_{2,n}^{-1}\sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A_{(i,t),(j,s)}^2 \varepsilon_{(i,t)}^2 \varepsilon_{(j,s)}^2$.

Consider now the $\mathcal{A}_1$ term. By applying the triangle inequality and the inequality $|XY| \leq (1/2)\,X^2 + (1/2)\,Y^2$ and collecting like terms, we get

$$\begin{aligned}
|\mathcal{A}_1| \leq\; & \frac{12}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A^2_{(i,t),(j,s)} \sum_{h=1}^{T_i} \sum_{v=1}^{T_i} \left|J_{(i,t),(i,h)}\right| \left|J_{(j,s),(j,v)}\right| \left(e'_{(j,v)} M^Q \varepsilon\right)^2 \left(e'_{(i,h)} P^{Z^\perp} \varepsilon\right)^2 \\
& + \frac{12}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A^2_{(i,t),(j,s)} \sum_{h=1}^{T_i} \sum_{v=1}^{T_i} \left|J_{(i,t),(i,h)}\right| \left|J_{(j,s),(j,v)}\right| \Bigg\{ \left(e'_{(i,h)} M^Q \varepsilon\right)^2 \\
& \qquad \times \left(e'_{(j,v)} M^{(Z,Q)} X \left[\widehat{\delta}_n - \delta_0\right]\right)^2 \Bigg\} \\
& + \frac{12}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A^2_{(i,t),(j,s)} \sum_{h=1}^{T_i} \sum_{v=1}^{T_i} \left|J_{(i,t),(i,h)}\right| \left|J_{(j,s),(j,v)}\right| \left(e'_{(i,h)} M^Q \varepsilon\right)^2 \left(e'_{(j,v)} M^{(Z,Q)} \varphi_n\right)^2 \\
=\; & \mathcal{A}_{1,1} + \mathcal{A}_{1,2} + \mathcal{A}_{1,3}, \;\; (say).
\end{aligned}$$

Clearly $\mathcal{A}_{1,k} \geq 0$ for $k = 1, 2, 3$. Next, note that, after some straightforward moment calculations and after applying the triangle inequality as well as Assumptions 1, 2(i), 5, and 6 and part (a) of

Lemma S2-1; we obtain

$$
\begin{aligned}
E\left[\mathcal{A}_{1,1}|\mathcal{F}_n^W\right] &= \frac{12}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A^2_{(i,t),(j,s)} \sum_{h=1}^{T_i}\sum_{v=1}^{T_j} \left|J_{(i,t),(i,h)}\right| \left|J_{(j,s),(j,v)}\right| \sum_{q=1}^{T_j} M^Q_{(j,v),(j,q)} \\
&\quad \times \sum_{g=1}^{T_j} M^Q_{(j,v),(j,g)} \sum_{(l,r)=1}^{m_n} P^{Z^\perp}_{(i,h),(l,r)} \sum_{(k,c)=1}^{m_n} P^{Z^\perp}_{(i,h),(k,c)} E\left[\varepsilon_{(l,r)}\varepsilon_{(k,c)}\varepsilon_{(j,q)}\varepsilon_{(j,g)}|\mathcal{F}_n^W\right] \\
&\leq \frac{12\overline{T}^3}{K_{2,n}} \left(\max_{1\leq(i,t),(j,s)\leq m_n} \left|M^Q_{(i,t),(j,s)}\right|\right)^2 \left(\max_{1\leq(j,s)\leq m_n} E\left[\varepsilon^4_{(j,s)}|\mathcal{F}_n^W\right]\right) \\
&\quad \times \left(\max_{1\leq(i,t)\leq m_n} P^{Z^\perp}_{(i,t),(i,t)}\right)^2 \left(\max_{1\leq(i,t),(j,s)\leq m_n} \left|J_{(i,t),(j,s)}\right|\right)^2 \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A^2_{(i,t),(j,s)} \\
&\quad + \frac{12\overline{T}^3}{K_{2,n}} \left(\max_{1\leq(i,t),(j,s)\leq m_n} \left|M^Q_{(i,t),(j,s)}\right|\right)^2 \left(\max_{1\leq(i,t),(j,s)\leq m_n} \left|J_{(i,t),(j,s)}\right|\right)^2 \\
&\quad \times \left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon^2_{(i,t)}|\mathcal{F}_n^W\right]\right)^2 \left(\max_{1\leq(i,t)\leq m_n} P^{Z^\perp}_{(i,t),(i,t)}\right) \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A^2_{(i,t),(j,s)} \\
&\quad + 24\overline{T}^4 \left(\max_{1\leq(i,t),(j,s)\leq m_n} \left|J_{(i,t),(j,s)}\right|\right)^2 \left(\max_{1\leq(i,t),(j,s)\leq m_n} \left|M^Q_{(i,t),(j,s)}\right|\right)^2 \\
&\qquad \times \left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon^2_{(i,t)}|\mathcal{F}_n^W\right]\right)^2 \left(\max_{1\leq(i,t)\leq m_n} P^{Z^\perp}_{(i,t),(i,t)}\right)^2 \frac{1}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A^2_{(i,t),(j,s)} \\
&= O_{a.s.}\left(\frac{K_n^2}{n^2}\right) + O_{a.s.}\left(\frac{K_n}{n}\right) + O_{a.s.}\left(\frac{K_n^2}{n^2}\right) = O_{a.s.}\left(\frac{K_n}{n}\right). \qquad (44)
\end{aligned}
$$

It follows from the law of iterated expectations and Theorem 16.1 of Billingsley (1995) that there exists a constant $\overline{C} > 0$ such that for all n sufficiently large $E\left[(n/K_n)\left|\mathcal{A}_{1,1}\right|\right]$
$= E_W\left(\frac{n}{K_n} E\left[\left|\mathcal{A}_{1,1}\right| |\mathcal{F}_n^W\right]\right) \leq \overline{C} < \infty$. Application of Markov's inequality then allows us to further deduce that

$$\mathcal{A}_{1,1} = O_p\left(\frac{K_n}{n}\right) = o_p(1). \qquad (45)$$

Consider next the subterm $\mathcal{A}_{1,2}$. Here, by applying the CS inequality, we have

$$\begin{aligned}
\mathcal{A}_{1,2} &= \frac{12}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A^2_{(i,t),(j,s)} \sum_{h=1}^{T_i} \sum_{v=1}^{T_i} \left|J_{(i,t),(i,h)}\right| \left|J_{(j,s),(j,v)}\right| \left(e'_{(i,h)} M^Q \varepsilon\right)^2 \left(e'_{(j,v)} M^{(Z,Q)} X \left[\widehat{\delta}_n - \delta_0\right]\right)^2 \\
&\leq \frac{12 \left\|\widehat{\delta}_n - \delta_0\right\|_2^2}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A^2_{(i,t),(j,s)} \sum_{h=1}^{T_i} \sum_{v=1}^{T_j} \left\{ \left|J_{(i,t),(i,h)}\right| \left|J_{(j,s),(j,v)}\right| \left(e'_{(i,h)} M^Q \varepsilon\right)^2 \right. \\
&\qquad \left. \times e'_{(j,v)} M^{(Z,Q)} X X' M^{(Z,Q)} e_{(j,v)} \right\} \\
&\leq \left\|\widehat{\delta}_n - \delta_0\right\|_2^2 \mathcal{A}^*_{1,2},
\end{aligned}$$

where $\mathcal{A}^*_{1,2} = 12 K_{2,n}^{-1} \sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A^2_{(i,t),(j,s)} \sum_{h=1}^{T_i} \left|J_{(i,t),(i,h)}\right| \left(e'_{(i,h)} M^Q \varepsilon\right)^2$ $\times \sum_{v=1}^{T_j} \left|J_{(j,s),(j,v)}\right| e'_{(j,v)} M^{(Z,Q)} X X' M^{(Z,Q)} e_{(j,v)}$. Next, by applying the inequality $(x+y)'(x+y) \leq 2x'x + 2y'y$ twice, we obtain

$$\begin{aligned}
0 &\leq \mathcal{A}^*_{1,2} \\
&\leq \frac{24}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A^2_{(i,t),(j,s)} \sum_{h=1}^{T_i} \sum_{v=1}^{T_j} \left|J_{(i,t),(i,h)}\right| \left|J_{(j,s),(j,v)}\right| \left(e'_{(i,h)} M^Q \varepsilon\right)^2 \\
&\quad \times e'_{(j,v)} M^{(Z,Q)} X_1 X_1' M^{(Z,Q)} e_{(j,v)} \\
&\quad + \frac{48}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A^2_{(i,t),(j,s)} \sum_{h=1}^{T_i} \sum_{v=1}^{T_j} \left|J_{(i,t),(i,h)}\right| \left|J_{(j,s),(j,v)}\right| \left(e'_{(i,h)} M^Q \varepsilon\right)^2 e'_{(j,v)} M^Q U U' M^Q e_{(j,v)} \\
&\quad + \frac{48}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A^2_{(i,t),(j,s)} \sum_{h=1}^{T_i} \sum_{v=1}^{T_j} \left|J_{(i,t),(i,h)}\right| \left|J_{(j,s),(j,v)}\right| \left(e'_{(i,h)} M^Q \varepsilon\right)^2 e'_{(j,v)} P^{Z^\perp} U U' P^{Z^\perp} e_{(j,v)} \\
&= \mathcal{A}^*_{1,2,1} + \mathcal{A}^*_{1,2,2} + \mathcal{A}^*_{1,2,3}, \ \ (say),
\end{aligned}$$

where $X_1 = \Gamma D_\mu / \sqrt{n} + F D_\kappa / \sqrt{n}$, $M^{(Z,Q)} = M^Q - P^{Z^\perp}$, $P^{Z^\perp} = M^Q Z \left(Z' M^Q Z\right)^{-1} Z' M^Q$, $M^Q = I_{m_n} - Q \left(Q'Q\right)^{-1} Q'$, and $F = \left(f\left(W_{1,(1,1)}\right), .., f\left(W_{1,(1,T_1)}\right), ..., f\left(W_{1,(n,1)}\right), .., f\left(W_{1,(n,T_n)}\right)\right)'$. By applying Assumptions 3(i), 4(i), 5, 7(ii), and 7(iii) and part (a) of Lemma S2-1, it is straightforward to show that $\sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A^2_{(i,t),(j,s)}$ $\times \sum_{v=1}^{T_j} e'_{(j,v)} M^{(Z,Q)} X_1 X_1' M^{(Z,Q)} e_{(j,v)} = O_{a.s.}\left(K_{2,n} \max\left\{\left(\mu_n^{\max}\right)^2 K_{2,n}^{-2\varrho_\gamma}, \left(\kappa_n^{\max}\right)^2 K_{1,n}^{-2\varrho_f}\right\}\right)$. Making use of this result as well as Assumptions 1, 2(i), 5, and 6; we then obtain

$$
\begin{aligned}
0 &\leq E\left[\mathcal{A}_{1,2,1}^{*}|\mathcal{F}_n^W\right] \\
&\leq \frac{24}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^2 \sum_{h=1}^{T_i}\sum_{v=1}^{T_j} \left|J_{(i,t),(i,h)}\right| \left|J_{(j,s),(j,v)}\right| \\
&\quad \times e_{(j,v)}' M^{(Z,Q)} X_1 X_1' M^{(Z,Q)} e_{(j,v)} e_{(i,h)}' M^Q E\left[\varepsilon\varepsilon'|\mathcal{F}_n^W\right] M^Q e_{(i,h)} \\
&\leq 24\overline{T}\left(\max_{1\leq(i,t),(j,s)\leq m_n}\left|J_{(i,t),(j,s)}\right|\right)^2 \left(\max_{1\leq(i,t)\leq m_n}\left|M_{(i,t),(i,t)}^Q\right|\right)\left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right]\right) \\
&\quad \times \frac{1}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^2 \sum_{v=1}^{T_j} e_{(j,v)}' M^{(Z,Q)} X_1 X_1' M^{(Z,Q)} e_{(j,v)} \\
&= O_{a.s.}\left(\max\left\{\frac{(\mu_n^{\max})^2}{K_{2,n}^{2\varrho_\gamma}}, \frac{(\kappa_n^{\max})^2}{K_{1,n}^{2\varrho_f}}\right\}\right) = O_{a.s.}(1) \text{ (given Assumption 7(ii) and (iii)).}
\end{aligned}
$$

Moreover, applying part (a) of Lemma S2-1 and Assumptions 2(i), 5, and 6; we get

$$
\begin{aligned}
0 &\leq E\left[\mathcal{A}_{1,2,2}^{*}|\mathcal{F}_n^W\right] \\
&\leq \frac{48}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^2 \sum_{h=1}^{T_i}\sum_{v=1}^{T_j} \left|J_{(i,t),(i,h)}\right| \left|J_{(j,s),(j,v)}\right| \sum_{g=1}^{T_i}\sum_{r=1}^{T_i}\left|M_{(i,h),(i,g)}^Q\right|\left|M_{(i,h),(i,r)}^Q\right| \\
&\quad \times \sum_{q=1}^{T_j}\sum_{c=1}^{T_j}\left|M_{(j,v),(j,q)}^Q\right|\left|M_{(j,v),(j,c)}^Q\right| E\left[\left|\varepsilon_{(i,g)}\varepsilon_{(i,r)}U_{(j,q)}'U_{(j,c)}\right||\mathcal{F}_n^W\right] \\
&\leq 48\overline{T}^6\left(\max_{1\leq(i,t),(j,s)\leq m_n}\left|J_{(i,t),(j,s)}\right|\right)^2\left(\max_{1\leq(i,t),(j,s)\leq m_n}\left|M_{(i,t),(j,s)}^Q\right|\right)^4\left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon_{(i,t)}^4|\mathcal{F}_n^W\right]\right)^{1/2} \\
&\quad \times\left(\max_{1\leq(i,t)\leq m_n} E\left[\left\|U_{(i,t)}\right\|_2^4|\mathcal{F}_n^W\right]\right)^{1/2} \frac{1}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^2 \\
&= O_{a.s.}(1).
\end{aligned}
$$

In addition, for $E\left[\mathcal{A}_{1,2,3}^{*}|\mathcal{F}_n^W\right]$, we have, by straightforward calculations using Assumption 1 and the triangle inequality,

$$
\begin{aligned}
0 &\leq E\left[\mathcal{A}^*_{1,2,3}|\mathcal{F}^W_n\right] \\
&\leq \frac{48}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A^2_{(i,t),(j,s)} \left\{ \sum_{h=1}^{T_i} \sum_{v=1}^{T_j} \left|J_{(i,t),(i,h)}\right| \left|J_{(j,s),(j,v)}\right| \right. \\
&\qquad \left. \times \sum_{g=1}^{T_i} \left(M^Q_{(i,h),(i,g)}\right)^2 \left(P^{Z^\perp}_{(j,v),(i,g)}\right)^2 E\left[\varepsilon^2_{(i,g)} U'_{(i,g)} U_{(i,g)}|\mathcal{F}^W_n\right] \right\} \\
&\quad + \frac{48}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A^2_{(i,t),(j,s)} \left\{ \sum_{h=1}^{T_i} \sum_{v=1}^{T_j} \left|J_{(i,t),(i,h)}\right| \left|J_{(j,s),(j,v)}\right| \sum_{g=1}^{T_i} \left(M^Q_{(i,h),(i,g)}\right)^2 \right. \\
&\qquad \left. \times \sum_{(k,q)=1}^{m_n} \left(P^{Z^\perp}_{(j,v),(k,q)}\right)^2 E\left[\varepsilon^2_{(i,g)}|\mathcal{F}^W_n\right] E\left[U'_{(k,q)} U_{(k,q)}|\mathcal{F}^W_n\right] \right\} \\
&\quad + \frac{96}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A^2_{(i,t),(j,s)} \left\{ \sum_{h=1}^{T_i} \left|J_{(i,t),(i,h)}\right| \sum_{g=1}^{T_i} \sum_{r=1}^{T_i} \left|M^Q_{(i,h),(i,g)}\right| \left|M^Q_{(i,h),(i,r)}\right| \right. \\
&\qquad \left. \times \sum_{v=1}^{T_j} \left|J_{(j,s),(j,v)}\right| \left|P^{Z^\perp}_{(j,v),(i,g)}\right| \left|P^{Z^\perp}_{(j,v),(i,r)}\right| E\left[\left|\varepsilon_{(i,g)}\varepsilon_{(i,r)} U'_{(i,g)} U_{(i,r)}\right| |\mathcal{F}^W_n\right] \right\} \qquad (46)
\end{aligned}
$$

Applying the CS inequality to (46) and making use of part (a) of Lemma S2-1 as well as Assumptions 2(i), 5, and 6; we further obtain

$$
\begin{aligned}
& E\left[\mathcal{A}^*_{1,2,3}|\mathcal{F}^W_n\right] \\
\leq\ & 48\overline{T}^3 \left(\max_{1\leq(i,t),(j,s)\leq m_n} \left|J_{(i,t),(j,s)}\right| \right)^2 \left(\max_{1\leq(i,t),(j,s)\leq m_n} \left|M^Q_{(i,t),(j,s)}\right| \right)^2 \left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon^4_{(i,t)}|\mathcal{F}^W_n\right] \right)^{1/2} \\
& \times \left(\max_{1\leq(i,t)\leq m_n} E\left[\left\|U_{(i,t)}\right\|_2^4 |\mathcal{F}^W_n\right] \right)^{1/2} \left(\max_{1\leq(i,t)\leq m_n} P^{Z^\perp}_{(i,t),(i,t)} \right)^2 \frac{1}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A^2_{(i,t),(j,s)} \\
& +48\overline{T}^3 \left(\max_{1\leq(i,t),(j,s)\leq m_n} \left|J_{(i,t),(j,s)}\right| \right)^2 \left(\max_{1\leq(i,t),(j,s)\leq m_n} \left|M^Q_{(i,t),(j,s)}\right| \right)^2 \left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon^2_{(i,t)}|\mathcal{F}^W_n\right] \right) \\
& \times \left(\max_{1\leq(i,t)\leq m_n} E\left[\left\|U_{(i,t)}\right\|_2^2 |\mathcal{F}^W_n\right] \right) \left(\max_{1\leq(i,t)\leq m_n} P^{Z^\perp}_{(i,t),(i,t)} \right) \frac{1}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A^2_{(i,t),(j,s)}
\end{aligned}
$$

$$+96\overline{T}^4\left(\max_{1\le(i,t),(j,s)\le m_n}\left|J_{(i,t),(j,s)}\right|\right)^2\left(\max_{1\le(i,t),(j,s)\le m_n}\left|M^{Q}_{(i,t),(j,s)}\right|\right)^2\left(\max_{1\le(i,t)\le m_n}E\left[\varepsilon^4_{(i,t)}|\mathcal{F}^W_n\right]\right)^{1/2}$$
$$\times\left(\max_{1\le(i,t)\le m_n}E\left[\left\|U_{(i,t)}\right\|_2^4|\mathcal{F}^W_n\right]\right)^{1/2}\left(\max_{1\le(i,t)\le m_n}P^{Z^{\perp}}_{(i,t),(i,t)}\right)^2\frac{1}{K_{2,n}}\sum_{\substack{(i,t),(j,s)=1\\(i,t)\neq(j,s)}}^{m_n}A^2_{(i,t),(j,s)}$$
$$= O_{a.s.}\left(\frac{K_n^2}{n^2}\right)+O_{a.s.}\left(\frac{K_n}{n}\right)+O_{a.s.}\left(\frac{K_n^2}{n^2}\right)=O_{a.s.}\left(\frac{K_n}{n}\right)=o_{a.s.}\left(1\right).$$

These calculations imply that $0\le E\left[\mathcal{A}^*_{1,2}|\mathcal{F}^W_n\right]\le E\left[\mathcal{A}^*_{1,2,1}|\mathcal{F}^W_n\right]+E\left[\mathcal{A}^*_{1,2,2}|\mathcal{F}^W_n\right]+E\left[\mathcal{A}^*_{1,2,3}|\mathcal{F}^W_n\right]$ $=O_{a.s.}\left(1\right)+O_{a.s.}\left(1\right)+o_{a.s.}\left(1\right)=O_{a.s.}\left(1\right)$. It follows by the law of iterated expectations and Theorem 16.1 of Billingsley (1995) that there exists a constant $\overline{C}<\infty$ such that for all n sufficiently large $E\left[\mathcal{A}^*_{1,2}\right]=E_{W_n}\left(E\left[\mathcal{A}^*_{1,2,}|\mathcal{F}^W_n\right]\right)\le\overline{C}$. Markov's inequality then implies that

$$\mathcal{A}^*_{1,2}=O_p\left(1\right). \tag{47}$$

from which we further deduce that

$$\mathcal{A}_{1,2}\le\left\|\widehat{\delta}_n-\delta_0\right\|_2^2\mathcal{A}^*_{1,2}=o_p\left(1\right)O_p\left(1\right)=o_p\left(1\right). \tag{48}$$

Turning our attention to $\mathcal{A}_{1,3}$, observe that

$$\begin{aligned}
\mathcal{A}_{1,3} &= \frac{12}{K_{2,n}}\sum_{\substack{(i,t),(j,s)=1\\(i,t)\neq(j,s)}}^{m_n}A^2_{(i,t),(j,s)}\sum_{h=1}^{T_i}\sum_{v=1}^{T_i}\left|J_{(i,t),(i,h)}\right|\left|J_{(j,s),(j,v)}\right|\left(e'_{(i,h)}M^Q\varepsilon\right)^2\left[e'_{(j,v)}M^{(Z,Q)}\varphi_n\right]^2\\
&= \frac{12}{K_{2,n}}\sum_{\substack{(i,t),(j,s)=1\\(i,t)\neq(j,s)}}^{m_n}A^2_{(i,t),(j,s)}\sum_{h=1}^{T_i}\sum_{v=1}^{T_i}\left|J_{(i,t),(i,h)}\right|\left|J_{(j,s),(j,v)}\right|\left(e'_{(i,h)}M^Q\varepsilon\right)^2\\
&\qquad\times\left[\frac{\tau_n}{\sqrt{n}}e'_{(j,v)}M^{(Z,Q)}\left(g-Z_1\theta^{K_{1,n}}\right)\right]^2\\
&\le \frac{m_n\tau_n^2}{n}\left\|g\left(\cdot\right)-\theta^{K_{1,n}\prime}Z_1\left(\cdot\right)\right\|_\infty^2\mathcal{A}^*_{1,3}
\end{aligned}$$

where $\mathcal{A}^*_{1,3}=12K^{-1}_{2,n}\sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n}A^2_{(i,t),(j,s)}\sum_{h=1}^{T_i}\sum_{v=1}^{T_j}\left|J_{(i,t),(i,h)}\right|\left|J_{(j,s),(j,v)}\right|\left(e'_{(i,h)}M^Q\varepsilon\right)^2$ and where $g=\left(g\left(W_{1,(1,1)}\right),..,g\left(W_{1,(1,T_1)}\right),...,g\left(W_{1,(n,1)}\right),..,g\left(W_{1,(n,T_n)}\right)\right)'$. Next, note that, by

Lemma S2-1(a) and Assumptions 1, 2(i), 5, and 6,

$$\begin{aligned}
E\left[\mathcal{A}_{1,3}^{*}|\mathcal{F}_{n}^{W}\right] &= \frac{12}{K_{2,n}}\sum_{\substack{(i,t),(j,s)=1\\(i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^{2}\sum_{h=1}^{T_i}\sum_{v=1}^{T_i}\left|J_{(i,t),(i,h)}\right|\left|J_{(j,s),(j,v)}\right| E\left[\left(e_{(i,h)}^{\prime}M^{Q}\varepsilon\right)^{2}|\mathcal{F}_{n}^{W}\right] \\
&= \frac{12}{K_{2,n}}\sum_{\substack{(i,t),(j,s)=1\\(i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^{2}\sum_{h=1}^{T_i}\sum_{v=1}^{T_j}\left|J_{(i,t),(i,h)}\right|\left|J_{(j,s),(j,v)}\right| e_{(i,h)}^{\prime}M^{Q}E\left[\varepsilon\varepsilon^{\prime}|\mathcal{F}_{n}^{W}\right]M^{Q}e_{(i,h)} \\
&\leq 12\overline{T}^{2}\left(\max_{1\leq(i,t),(j,s)\leq m_n}\left|J_{(i,t),(j,s)}\right|\right)^{2}\left(\max_{1\leq(i,t)\leq m_n}M_{(i,t),(i,t)}^{Q}\right) \\
&\quad\times\left(\max_{1\leq(i,t)\leq m_n}E\left[\varepsilon_{(i,t)}^{2}|\mathcal{F}_{n}^{W}\right]\right)\frac{1}{K_{2,n}}\sum_{\substack{(i,t),(j,s)=1\\(i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^{2} \\
&= O_{a.s.}(1) \qquad (49)
\end{aligned}$$

It follows by the law of iterated expectations and Theorem 16.1 of Billingsley (1995) that there exists a constant $\overline{C}<\infty$ such that for all n sufficiently large $E\left[\mathcal{A}_{1,3}^{*}\right]=E_{W_n}\left(E\left[\mathcal{A}_{1,3}^{*}|\mathcal{F}_{n}^{W}\right]\right)$ $\leq\overline{C}$. Markov's inequality then implies that $\mathcal{A}_{1,3}^{*}=O_p(1)$, from which we further deduce using Assumption 7(i) that

$$\mathcal{A}_{1,3}\leq\frac{m_n}{n}\tau_n^{2}\left\|g(\cdot)-\theta^{K_{1,n}\prime}Z_1(\cdot)\right\|_{\infty}^{2}\mathcal{A}_{1,3}^{*}=O_p\left(\frac{\tau_n^{2}}{K_{1,n}^{2\varrho_g}}\right)=o_p(1). \qquad (50)$$

Putting things together, we note that the results given in expressions (45), (48), and (50) imply that

$$\mathcal{A}_1=\mathcal{A}_{1,1}+\mathcal{A}_{1,2}+\mathcal{A}_{1,3}=O_p\left(\frac{K_n}{n}\right)+o_p(1)+o_p(1)=o_p(1).$$

By a similar method of proof, we can also show that $\mathcal{A}_k=o_p(1)$ for $k=2,...,6$. It then follows from equation (43) that

$$\begin{aligned}
\mathcal{A} &= \frac{S_3}{K_{2,n}}-\frac{1}{K_{2,n}}\sum_{\substack{(i,t),(j,s)=1\\(i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^{2}\varepsilon_{(i,t)}^{2}\varepsilon_{(j,s)}^{2} \\
&= \mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3+\mathcal{A}_4+\mathcal{A}_5+\mathcal{A}_6=o_p(1). \qquad (51)
\end{aligned}$$

Now, for the term $\mathfrak{A}=K_{2,n}^{-1}\sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n}A_{(i,t),(j,s)}^{2}\left(\varepsilon_{(i,t)}^{2}\varepsilon_{(j,s)}^{2}-\sigma_{(i,t)}^{2}\sigma_{(j,s)}^{2}\right)$, it can be shown by straightforward calculation and by using Assumptions 1, 2(i), 5 and 6 and Lemma S2-1(a) that $E\left[\mathfrak{A}^{2}|\mathcal{F}_{n}^{W}\right]=O_{a.s.}\left(n^{-1}\right)$. It then follows by application of the law of iterated expectations, Theorem 16.1 of Billingsley (1995), and the Markov's inequality that

$$\mathfrak{A}=O_p\left(\frac{1}{\sqrt{n}}\right). \qquad (52)$$

Combining (51) and (52), we get

$$\frac{S_3}{K_{2,n}} - \frac{1}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A^2_{(i,t),(j,s)} \sigma^2_{(i,t)} \sigma^2_{(j,s)} = \mathcal{A} + \mathfrak{A} = o_p(1) + O_p\left(\frac{1}{\sqrt{n}}\right) = o_p(1),$$

which shows part (b).

The proofs for parts (c) and (d) are very similar to that of part (b). Hence, to avoid duplication, we will not provide detailed proofs of these parts here.

Turning our attention to part (e), note first that, we can write

$$\begin{aligned}
\widehat{\rho}_n - \rho &= \frac{X' M^{(Z,Q)} \left(y - X\widehat{\delta}_n\right)/n}{\left(y - X\widehat{\delta}_n\right)' M^{(Z,Q)} \left(y - X\widehat{\delta}_n\right)/n} - \rho \\
&= \frac{X' M^{(Z,Q)} \left(y - X\widehat{\delta}_n\right)/n - U' M^{(Z_1,Q)}\varepsilon/n + U' M^{(Z_1,Q)}\varepsilon/n}{\left(y - X\widehat{\delta}_n\right)' M^{(Z,Q)} \left(y - X\widehat{\delta}_n\right)/n - \varepsilon' M^Q \varepsilon/n + \varepsilon' M^Q \varepsilon/n} - \rho
\end{aligned}$$

where $\rho = \lim_{n\to\infty} \rho_n = \lim_{n\to\infty} \left(E\left[U' M^Q \varepsilon\right]/n\right) / \left(E\left[\varepsilon' M^Q \varepsilon\right]/n\right)$. By straightforward asymptotic analysis, we can show that

$$\begin{aligned}
\frac{X' M^{(Z,Q)} \left(y - X\widehat{\delta}_n\right)}{n} - \frac{U' M^{(Z_1,Q)}\varepsilon}{n} &= o_p(1), \\
\frac{\left(y - X\widehat{\delta}_n\right)' M^{(Z,Q)} \left(y - X\widehat{\delta}_n\right)}{n} - \frac{\varepsilon' M^{(Z_1,Q)}\varepsilon}{n} &= o_p(1), \\
\frac{\varepsilon' M^{(Z_1,Q)}\varepsilon}{n} - \frac{\varepsilon' M^Q \varepsilon}{n} &= o_p(1),
\end{aligned}$$

$$\frac{U' M^{(Z_1,Q)}\varepsilon}{\varepsilon' M^Q \varepsilon} - \rho = O_p\left(\max\left\{\frac{1}{\sqrt{n}}, \frac{K_{1,n}}{n}\right\}\right) = o_p(1).$$

Next, note that, under Assumption 6(i), $T_i \geq 3$ for all i, so that $\frac{T_i - 1}{T_i} \geq \frac{2}{3}$ for all i. Hence, by Assumption 2(ii), there exists a positive constant $\underline{C}$ such that $E\left[\varepsilon^2_{(i,t)} | \mathcal{F}^W_n\right] \geq \underline{C} > 0$ *a.s.*, so that

$$\begin{aligned}
\frac{E\left[\varepsilon' M^Q \varepsilon\right]}{n} &= \frac{1}{n} E_{W_n}\left\{\sum_{i=1}^{n}\sum_{t=1}^{T_i}\left[\left(\frac{T_i - 1}{T_i}\right)\right] E\left[\varepsilon^2_{(i,t)} | \mathcal{F}^W_n\right]\right\} \\
&\geq \frac{2}{3n} E_{W_n}\left\{\sum_{i=1}^{n}\sum_{t=1}^{T_i} E\left[\varepsilon^2_{(i,t)} | \mathcal{F}^W_n\right]\right\} \\
&\geq \frac{2}{3n} E_{W_n}\left[\sum_{i=1}^{n}\sum_{t=1}^{T_i}\underline{C}\right] = \frac{2}{3}\frac{m_n}{n}\underline{C} \quad \left(\text{since } m_n = \sum_{i=1}^{n} T_i\right) \\
&\geq \frac{2}{3}\underline{C} > 0
\end{aligned}$$

for all n sufficiently large. It follows from these results that $\widehat{\rho}_n - \rho = \left[U' M^{(Z_1,Q)} \varepsilon / \left(\varepsilon' M^Q \varepsilon\right)\right] - \rho + o_p(1) = o_p(1)$ and $\|\widehat{\rho}_n\|_2 \leq \|\widehat{\rho}_n - \rho\|_2 + \|\rho\|_2 = O_p(1)$.

Now, let $a \in \mathbb{R}^d$ such that $\|a\|_2 = 1$; and note that, by applying the CS inequality, we have

$$\left|a' D_\mu^{-1} \widehat{\rho}_n\right| \leq \frac{1}{(\mu_n^{\min})} \|\widehat{\rho}_n\|_2 = O_p\left(\frac{1}{(\mu_n^{\min})}\right), \tag{53}$$

$$\left|a' D_\mu^{-1} (\widehat{\rho}_n - \rho)\right| \leq \frac{1}{(\mu_n^{\min})} \|\widehat{\rho}_n - \rho\|_2 = o_p\left(\frac{1}{(\mu_n^{\min})}\right), \tag{54}$$

Since the argument above holds for any $a \in \mathbb{R}^d$ such that $\|a\|_2 = 1$, we further deduce that $D_\mu^{-1} \widehat{\rho}_n = O_p\left(\left(\mu_n^{\min}\right)^{-1}\right)$ and $D_\mu^{-1}(\widehat{\rho}_n - \rho) = o_p\left(\left(\mu_n^{\min}\right)^{-1}\right)$, which shows part (e).

Part (f) can be shown by applying the results of parts (b), (c), (d), and (e) of this lemma as well as part (a) of Lemma S2-1 and Assumptions 2(i) and 3(ii). Part (g), on the other hand, can be proved by applying the results of parts (b), (d), and (e) of this lemma. □

Section 2: Proof of Lemma 1 of the Main Paper

Lemma 1: Suppose that Assumptions 5 and 6(i) are satisfied. Then, there exists a positive constant C such that

$$\lambda_{\min}\left(M^{(Z,Q)} \circ M^{(Z,Q)}\right) \geq C > 0 \text{ } a.s.$$

for all n sufficiently large.

Proof of Lemma 1:

To show the required result, we shall apply Geršgorin's theorem (see Theorem 6.1.1 on page 344 of Horn and Johnson, 1985). To apply this theorem to the matrix $M^{(Z,Q)} \circ M^{(Z,Q)}$, we note first that this matrix is symmetric and its elements are real-valued, so that all the eigenvalues of this matrix are real. Now, for any $m_n \times m_n$ matrix A, define

$$R'_{(i,t)}(A) = \sum_{\substack{j=1 \\ j \neq i}}^{m_n} |a_{ij}| \text{ for } i \in \{1, ..., m_n\}.$$

Applying Geršgorin's theorem to the matrix $A = M^{(Z,Q)} \circ M^{(Z,Q)}$, we see that

$$\begin{aligned} &\lambda_{\min}\left(M^{(Z,Q)} \circ M^{(Z,Q)}\right) \\ \in &\bigcup_{(i,t)=1}^{m_n} \left\{z \in \mathbb{R} : \left|z - \left(M^{(Z,Q)}_{(i,t),(i,t)}\right)^2\right| \leq R'_{(i,t)}\left(M^{(Z,Q)} \circ M^{(Z,Q)}\right)\right\}, \end{aligned}$$

so that there exists at least one pair (k, υ) (where $k = 1, ..., n$ and $\upsilon = 1, .., T_k$ given k) such that

$$\begin{aligned} \left| \lambda_{\min}\left(M^{(Z,Q)} \circ M^{(Z,Q)}\right) - \left(M^{(Z,Q)}_{(k,\upsilon),(k,\upsilon)}\right)^2 \right| &\leq R'_{(k,\upsilon)}\left(M^{(Z,Q)} \circ M^{(Z,Q)}\right) \\ &= \sum_{\substack{(j,s)=1 \\ (j,s)\neq(k,\upsilon)}}^{m_n} \left(M^{(Z,Q)}_{(k,\upsilon),(j,s)}\right)^2 \end{aligned}$$

or

$$\begin{aligned} -\sum_{\substack{(j,s)=1 \\ (j,s)\neq(k,\upsilon)}}^{m_n} \left(M^{(Z,Q)}_{(k,\upsilon),(j,s)}\right)^2 &\leq \lambda_{\min}\left(M^{(Z,Q)} \circ M^{(Z,Q)}\right) - \left(M^{(Z,Q)}_{(k,\upsilon),(k,\upsilon)}\right)^2 \\ &\leq \sum_{\substack{(j,s)=1 \\ (j,s)\neq(k,\upsilon)}}^{m_n} \left(M^{(Z,Q)}_{(k,\upsilon),(j,s)}\right)^2 \end{aligned}$$

Using the first inequality above, we have

$$\begin{aligned} \lambda_{\min}\left(M^{(Z,Q)} \circ M^{(Z,Q)}\right) &\geq \left(M^{(Z,Q)}_{(k,\upsilon),(k,\upsilon)}\right)^2 - \sum_{\substack{(j,s)=1 \\ (j,s)\neq(k,\upsilon)}}^{m_n} \left(M^{(Z,Q)}_{(k,\upsilon),(j,s)}\right)^2 \\ &= 2\left(M^{(Z,Q)}_{(k,\upsilon),(k,\upsilon)}\right)^2 - \sum_{(j,s)=1}^{m_n} \left(M^{(Z,Q)}_{(k,\upsilon),(j,s)}\right)^2 \\ &= 2\left(M^{(Z,Q)}_{(k,\upsilon),(k,\upsilon)}\right)^2 - M^{(Z,Q)}_{(k,\upsilon),(k,\upsilon)} \\ &= 2M^{(Z,Q)}_{(k,\upsilon),(k,\upsilon)}\left(M^{(Z,Q)}_{(k,\upsilon),(k,\upsilon)} - \frac{1}{2}\right) \\ &\geq 2\min_{1\leq(k,\upsilon)\leq m_n}\left[M^{(Z,Q)}_{(k,\upsilon),(k,\upsilon)}\left(M^{(Z,Q)}_{(k,\upsilon),(k,\upsilon)} - \frac{1}{2}\right)\right]. \end{aligned}$$

Hence, a sufficient condition that $\lambda_{\min}\left(M^{(Z,Q)} \circ M^{(Z,Q)}\right)$ is bounded away from zero $a.s.n.$ is that, there exists a positive constant $\underline{C}$ such that

$$\min_{1\leq(k,\upsilon)\leq m_n}\left[M^{(Z,Q)}_{(k,\upsilon),(k,\upsilon)}\left(M^{(Z,Q)}_{(k,\upsilon),(k,\upsilon)} - \frac{1}{2}\right)\right] \geq \underline{C} > 0 \ a.s.n. \tag{55}$$

Now, consider the function

$$f(x) = x\left(x - \frac{1}{2}\right) \quad \text{for } 0 \leq x \leq 1$$

and note that

$$f'(x) = 2x - \frac{1}{2} > 0 \text{ for } \frac{1}{4} < x \leq 1,$$

from which we deduce that a sufficient condition for the condition given by (55) is

$$\min_{1\leq(k,\upsilon)\leq m_n} M^{(Z,Q)}_{(k,\upsilon),(k,\upsilon)} \geq \frac{1}{2}+\epsilon_1 \ a.s.n.$$

for some $\epsilon_1 > 0$. In addition, write $M^{(Z,Q)} = M^Q - P^{Z^\perp}$, where $P^{Z^\perp} = M^Q Z\left(Z'M^Q Z\right)^{-1} Z'M^Q$ and where the $(k,\upsilon)^{th}$ diagonal element of $M^Q = I_{m_n} - Q\left(Q'Q\right)^{-1} Q'$ is given by

$$M^Q_{(k,\upsilon),(k,\upsilon)} = 1-\frac{1}{T_k}$$

Under Assumption 5, $P^{Z^\perp}_{(k,\upsilon),(k,\upsilon)} = O_{a.s.}\left(K_n/n\right) = o_{a.s.}\left(1\right)$ for every $(k,\upsilon) \in \{1,....,m_n\}$, so that for any $0<\epsilon_1<1/6$, there exists a positive integer N_{ϵ_1} such that for all $n \geq N_{\epsilon_1}$, $P^{Z^\perp}_{(k,\upsilon),(k,\upsilon)} < \frac{1}{6}-\epsilon_1$ $a.s$. It follows that, under the assumption that $\min_{1\leq k\leq n} T_k \geq 3$, we have for all $n \geq N_{\epsilon_2}$

$$\begin{aligned}
\min_{1\leq(k,\upsilon)\leq m_n} M^{(Z,Q)}_{(k,\upsilon),(k,\upsilon)} &= \min_{1\leq(k,\upsilon)\leq m_n}\left(M^Q_{(k,\upsilon),(k,\upsilon)} - P^{Z^\perp}_{(k,\upsilon),(k,\upsilon)}\right)\\
&= \min_{1\leq(k,\upsilon)\leq m_n}\left(1-\frac{1}{T_k} - P^{Z^\perp}_{(k,\upsilon),(k,\upsilon)}\right)\\
&> \frac{2}{3}-\left(\frac{1}{6}-\epsilon_1\right) = \frac{1}{2}+\epsilon_1 \ a.s.,
\end{aligned}$$

as required. □

Section 3: Statement and Proof of Additional Lemmas

In this section, we state and prove a number of additional supporting lemmas whose results are used to prove some of the lemmas given in section 1 of this online appendix.

Lemma OA-1: Under Assumptions 5 and 6(i), the following statements are true.

(a)

$$tr\left\{D^2_{\widehat{\vartheta}}\right\} = O_{a.s.}\left(\frac{K^2_{2,n}}{n}\right),$$

where $D_{\widehat{\vartheta}} = diag\left(\widehat{\vartheta}_1, ..., \widehat{\vartheta}_{m_n}\right)$.

(b)

$$\max_{1\leq(i,t)\leq m_n}\left|\widehat{\vartheta}_{(i,t)}\right|^2 = O_{a.s.}\left(\frac{K^2_{2,n}}{n^2}\right).$$

Proof of Lemma OA-1:

To show part (a), note that, by the result of Lemma 1, there exists a constant $C > 0$ such that

$$
\begin{aligned}
tr\left\{D_{\widehat{\vartheta}}^2\right\} &= \sum_{(i,t)=1}^{m_n} \widehat{\vartheta}_{(i,t)}^2 \\
&= d_{P^\perp}' \left(M^{(Z,Q)} \circ M^{(Z,Q)}\right)^{-1} \sum_{(i,t)=1}^{m_n} e_{(i,t)} e_{(i,t)}' \left(M^{(Z,Q)} \circ M^{(Z,Q)}\right)^{-1} d_{P^\perp} \\
&= d_{P^\perp}' \left(M^{(Z,Q)} \circ M^{(Z,Q)}\right)^{-2} d_{P^\perp} \\
&\leq \frac{1}{\left[\lambda_{\min}\left(M^{(Z,Q)} \circ M^{(Z,Q)}\right)\right]^2} d_{P^\perp}' d_{P^\perp} \\
&\leq \left(\frac{1}{C}\right)^2 d_{P^\perp}' d_{P^\perp} \quad a.s. \\
&= \left(\frac{1}{C}\right)^2 \sum_{(i,t)=1}^{m_n} \left(P_{(i,t),(i,t)}^\perp\right)^2 \\
&\leq \left(\frac{1}{C}\right)^2 \left(\max_{1 \leq (i,t) \leq m_n} P_{(i,t),(i,t)}^\perp\right) \sum_{(i,t)=1}^{m_n} P_{(i,t),(i,t)}^\perp \\
&= O_{a.s.}\left(\frac{K_{2,n}^2}{n}\right) \quad \text{(by Assumption 5(iv)).}
\end{aligned}
$$

where $e_{(i,t)}$ denotes an $m_n \times 1$ elementary vector whose $(i,t)^{th}$ component is 1 and all other components are 0.

Next, we consider part (b). Note first that, as shown in the proof of Lemma 1, under the assumptions that $\min_{1 \leq i \leq n} T_i \geq 3$ and $K_n/n = o(1)$, the projection matrix $M^{(Z,Q)}$ is strictly diagonally dominate $a.s.n.$, so that there exists a positive constant $\underline{C}$ such that

$$
\min_{1 \leq (i,t) \leq m_n} \left[M_{(i,t),(i,t)}^{(Z,Q)} \left(M_{(i,t),(i,t)}^{(Z,Q)} - \frac{1}{2}\right)\right] \geq \underline{C} > 0 \quad a.s.n.
$$

See, in particular, expression (55) in the proof of Lemma 1. Under these conditions, then, application of Theorem 1 of Varah (1975) to our problem leads to the following inequality

$$
\left\|\left(M^{(Z,Q)} \circ M^{(Z,Q)}\right)^{-1}\right\|_\infty \leq \left(\min_{1 \leq (i,t) \leq m_n} \left[\left(M_{(i,t),(i,t)}^{(Z,Q)}\right)^2 - \sum_{\substack{(k,v)=1 \\ (k,v) \neq (i,t)}}^{m_n} \left(M_{(i,t),(k,v)}^{(Z,Q)}\right)^2\right]\right)^{-1}.
$$

Now, making use of this inequality, we have, almost surely for all n sufficiently large,

$$
\begin{aligned}
& \max_{1\leq(i,t)\leq m_n}\left|\widehat{\vartheta}_{(i,t)}\right|^2 \\
= & \max_{1\leq(i,t)\leq m_n}\left|e'_{(i,t)}\left(M^{(Z,Q)}\circ M^{(Z,Q)}\right)^{-1}d_{P^{\perp}}\right|^2 \\
\leq & \max_{1\leq(i,t)\leq m_n}\left\|\left(M^{(Z,Q)}\circ M^{(Z,Q)}\right)^{-1}e_{(i,t)}\right\|_1^2\left\|d_{P^{\perp}}\right\|_\infty^2 \quad \text{(by Hölder's inequality)} \\
= & \left(\max_{1\leq(i,t)\leq m_n}\left\|\left(M^{(Z,Q)}\circ M^{(Z,Q)}\right)^{-1}e_{(i,t)}\right\|_1\right)^2\left\|d_{P^{\perp}}\right\|_\infty^2 \\
= & \left(\max_{1\leq(i,t)\leq m_n}\sum_{(j,s)=1}^{m_n}\left|\left(M^{(Z,Q)}\circ M^{(Z,Q)}\right)^{-1}_{(j,s),(i,t)}\right|\right)^2\left(\max_{1\leq(i,t)\leq m_n}P^{\perp}_{(i,t),(i,t)}\right)^2 \\
= & \left\|\left(M^{(Z,Q)}\circ M^{(Z,Q)}\right)^{-1}\right\|_1^2\left(\max_{1\leq(i,t)\leq m_n}P^{\perp}_{(i,t),(i,t)}\right)^2 \\
= & \left\|\left(M^{(Z,Q)}\circ M^{(Z,Q)}\right)^{-1}\right\|_\infty^2\left(\max_{1\leq(i,t)\leq m_n}P^{\perp}_{(i,t),(i,t)}\right)^2 \quad \text{(by symmetry of } \left(M^{(Z,Q)}\circ M^{(Z,Q)}\right)^{-1}\text{)} \\
\leq & \left(\min_{1\leq(i,t)\leq m_n}\left[\left(M^{(Z,Q)}_{(i,t),(i,t)}\right)^2-\sum_{\substack{(k,\upsilon)=1\\(k,\upsilon)\neq(i,t)}}^{m_n}\left(M^{(Z,Q)}_{(i,t),(k,\upsilon)}\right)^2\right]\right)^{-2}\left(\max_{1\leq(i,t)\leq m_n}P^{\perp}_{(i,t),(i,t)}\right)^2 \\
& \text{(by Theorem 1 of Varah, 1975)} \\
= & \left(2\min_{1\leq(i,t)\leq m_n}\left[M^{(Z,Q)}_{(i,t),(i,t)}\left(M^{(Z,Q)}_{(i,t),(i,t)}-\frac{1}{2}\right)\right]\right)^{-2}\left(\max_{1\leq(i,t)\leq m_n}P^{\perp}_{(i,t),(i,t)}\right)^2 \\
\leq & \left(\frac{1}{2\underline{C}}\right)^2\left(\max_{1\leq(i,t)\leq m_n}P^{\perp}_{(i,t),(i,t)}\right)^2 \quad \text{(by expression (55))} \\
= & O_{a.s.}\left(\frac{K_{2,n}^2}{n^2}\right). \ \square
\end{aligned}
$$

Lemma OA-2:

Under Assumptions 1, 2 and 5(iii), the following statements are true.

(a)

$$\frac{\Gamma' M^Q Z_1\left(Z_1' M^Q Z_1\right)^{-1} Z_1' M^Q \varepsilon}{\sqrt{n}} = O_p(1),$$

(b)

$$\frac{\Gamma' M^Q \varepsilon}{\sqrt{n}} = O_p(1).$$

(c)

$$\frac{\Gamma' M^{(Z_1,Q)}\varepsilon}{\sqrt{n}} = O_p(1).$$

Proof of Lemma OA-2:

Note first that, under Assumption 5(iii), $Z'M^Q Z$ is positive definite *a.s.n.*, which implies that $Z_1' M^Q Z_1$ is positive definite and, thus, nonsingular *a.s.n.* as well. Now, to show part (a), note that, for any $a \in \mathbb{R}^d$ such that $\|a\|_2 = 1$, we obtain, by applying Assumptions 1, 2(i), and 3(iii),

$$\begin{aligned}
& E\left[\left(\frac{a'\Gamma' M^Q Z_1 \left(Z_1' M^Q Z_1\right)^{-1} Z_1' M^Q \varepsilon}{\sqrt{n}}\right)^2 |\mathcal{F}_n^W\right] \\
= \;& \frac{a'\Gamma' M^Q Z_1 \left(Z_1' M^Q Z_1\right)^{-1} Z_1' M^Q E\left[\varepsilon\varepsilon' |\mathcal{F}_n^W\right] M^Q Z_1 \left(Z_1' M^Q Z_1\right)^{-1} Z_1' M^Q \Gamma a}{n} \\
\leq \;& \left(\max_{1\leq (i,t)\leq m_n} E\left[\varepsilon_{(i,t)}^2 |\mathcal{F}_n^W\right]\right) \frac{a'\Gamma' M^Q Z_1 \left(Z_1' M^Q Z_1\right)^{-1} Z_1' M^Q \Gamma a}{n} \\
\leq \;& \left(\max_{1\leq (i,t)\leq m_n} E\left[\varepsilon_{(i,t)}^2 |\mathcal{F}_n^W\right]\right) \frac{a'\Gamma'\Gamma a}{n} \\
= \;& O_{a.s.}(1).
\end{aligned}$$

Hence, there exists a positive constant $\overline{C} < \infty$ such that for all n sufficiently large

$$\begin{aligned}
& E\left(\left[\frac{a'\Gamma' M^Q Z_1 \left(Z_1' M^Q Z_1\right)^{-1} Z_1' M^Q \varepsilon}{\sqrt{n}}\right]^2\right) \\
= \;& E_{W_n}\left\{E\left(\left[\frac{a'\Gamma' M^Q Z_1 \left(Z_1' M^Q Z_1\right)^{-1} Z_1' M^Q \varepsilon}{\sqrt{n}}\right]^2 |\mathcal{F}_n^W\right)\right\} \\
\leq \;& \overline{C}
\end{aligned}$$

It follows from Markov's inequality that, for any $\epsilon > 0$, we can set $C_\epsilon = \sqrt{\overline{C}/\epsilon}$ so that, for all n sufficiently large,

$$\begin{aligned}
& \Pr\left(\left|\frac{a'\Gamma' M^Q Z_1 \left(Z_1' M^Q Z_1\right)^{-1} Z_1' M^Q \varepsilon}{\sqrt{n}}\right| \geq C_\epsilon\right) \\
= \;& \Pr\left(\left|\frac{a'\Gamma' M^Q Z_1 \left(Z_1' M^Q Z_1\right)^{-1} Z_1' M^Q \varepsilon}{\sqrt{n}}\right|^2 \geq C_\epsilon^2\right) \\
\leq \;& \frac{E\left(\left[a'\Gamma' M^Q Z_1 \left(Z_1' M^Q Z_1\right)^{-1} Z_1' M^Q \varepsilon/\sqrt{n}\right]^2\right)}{C_\epsilon^2} \\
\leq \;& \frac{\overline{C}}{\overline{C}/\epsilon} = \epsilon
\end{aligned}$$

which shows that

$$\frac{a'\Gamma' M^Q Z_1 \left(Z_1' M^Q Z_1\right)^{-1} Z_1' M^Q \varepsilon}{\sqrt{n}} = O_p\left(1\right)$$

Since the above result holds for all $a \in \mathbb{R}^d$ such that $\|a\|_2 = 1$, we further deduce that

$$\frac{\Gamma' M^Q Z_1 \left(Z_1' M^Q Z_1\right)^{-1} Z_1' M^Q \varepsilon}{\sqrt{n}} = O_p\left(1\right),$$

as required for part (a).

Turning our attention to part (b), we again let $a \in \mathbb{R}^d$ such that $\|a\|_2 = 1$; and note that, by Assumptions 1, 2(i), and 3(iii),

$$\begin{aligned}
E\left(\left[\frac{a'\Gamma' M^Q \varepsilon}{\sqrt{n}}\right]^2 |\mathcal{F}_n^W\right) &= \frac{a'\Gamma' M^Q E\left[\varepsilon\varepsilon'|\mathcal{F}_n^W\right] M^Q \Gamma a}{n} \\
&\leq \left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right]\right)\frac{a'\Gamma' M^Q \Gamma a}{n} \\
&\leq \left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right]\right)\frac{a'\Gamma'\Gamma a}{n} \\
&= O_{a.s.}\left(1\right)
\end{aligned}$$

Hence, there exists a positive constant $\overline{C} < \infty$ such that for all n sufficiently large

$$E\left(\left[\frac{a'\Gamma' M^Q \varepsilon}{\sqrt{n}}\right]^2\right) = E_{W_n}\left\{E\left(\left[\frac{a'\Gamma' M^Q \varepsilon}{\sqrt{n}}\right]^2 |\mathcal{F}_n^W\right)\right\} \leq \overline{C}.$$

It follows from Markov's inequality, for any $\epsilon > 0$, we can set $C_\epsilon = \sqrt{\overline{C}/\epsilon}$ so that for all n sufficiently large

$$\begin{aligned}
\Pr\left(\left|\frac{a'\Gamma' M^Q \varepsilon}{\sqrt{n}}\right| \geq C_\epsilon\right) &= \Pr\left(\left|\frac{a'\Gamma' M^Q \varepsilon}{\sqrt{n}}\right|^2 \geq C_\epsilon^2\right) \\
&\leq \frac{E\left(\left[a'\Gamma' M^Q \varepsilon/\sqrt{n}\right]^2\right)}{C_\epsilon^2} \\
&\leq \frac{\overline{C}}{\overline{C}/\epsilon} = \epsilon
\end{aligned}$$

which shows that

$$\frac{a'\Gamma' M^Q \varepsilon}{\sqrt{n}} = O_p\left(1\right).$$

Since the above result holds for all $a \in \mathbb{R}^d$ such that $\|a\|_2 = 1$, we further deduce that

$$\frac{\Gamma' M^Q \varepsilon}{\sqrt{n}} = O_p\left(1\right).$$

Finally, for part (c), note that it follows immediately from parts (a) and (b) above that

$$\frac{\Gamma' M^{(Z_1,Q)}\varepsilon}{\sqrt{n}} = \frac{\Gamma' M^Q \varepsilon}{\sqrt{n}} - \frac{\Gamma' M^Q Z_1 \left(Z_1' M^Q Z_1\right)^{-1} Z_1' M^Q \varepsilon}{\sqrt{n}} = O_p(1),$$

as required. □

Lemma OA-3:

Under Assumptions 1, 2(i), 4(ii), 5, 6(i), and 7(i); the following statements are true

(a)

$$\varphi' A \varphi = O_p\left(\frac{\tau_n^2}{K_{1,n}^{2\varrho_g}}\right)$$

(b)

$$\varphi' A \varepsilon = O_p\left(\frac{\tau_n}{K_{1,n}^{\varrho_g}}\right).$$

(c)

$$\frac{\varepsilon' A \varepsilon}{\sqrt{K_{2,n}}} = O_p(1).$$

Proof of Lemma OA-3:

To show part (a), note that, making use of the symmetry of A and applying the CS inequality and Assumptions 4(ii), 5, 6(i), and 7(i); we obtain

$$
\begin{aligned}
& \left|\varphi' A\varphi\right| \\
= \; & \frac{\tau_n^2}{n}\left|\left(g - Z_1\theta_{K_{1,n}}\right)' A\left(g - Z_1\theta_{K_{1,n}}\right)\right| \\
\leq \; & \tau_n^2\sqrt{\frac{\left(g - Z_1\theta_{K_{1,n}}\right)' A^2\left(g - Z_1\theta_{K_{1,n}}\right)}{n}}\sqrt{\frac{\left(g - Z_1\theta_{K_{1,n}}\right)'\left(g - Z_1\theta_{K_{1,n}}\right)}{n}} \\
= \; & \tau_n^2\sqrt{\frac{\left(g - Z_1\theta_{K_{1,n}}\right)'\left(P^{\perp} - M^{(Z,Q)}D_{\widehat{\vartheta}}M^{(Z,Q)}\right)^2\left(g - Z_1\theta_{K_{1,n}}\right)}{n}} \\
& \qquad \times\sqrt{\frac{\left(g - Z_1\theta_{K_{1,n}}\right)'\left(g - Z_1\theta_{K_{1,n}}\right)}{n}} \\
= \; & \tau_n^2\sqrt{\frac{\left(g - Z_1\theta_{K_{1,n}}\right)'\left(P^{\perp} + M^{(Z,Q)}D_{\widehat{\vartheta}}M^{(Z,Q)}D_{\widehat{\vartheta}}M^{(Z,Q)}\right)\left(g - Z_1\theta_{K_{1,n}}\right)}{n}} \\
& \times\sqrt{\frac{\left(g - Z_1\theta_{K_{1,n}}\right)'\left(g - Z_1\theta_{K_{1,n}}\right)}{n}} \\
\leq \; & \sqrt{1 + \max_{1\leq(i,t)\leq m_n}\left|\widehat{\vartheta}_{(i,t)}\right|^2}\tau_n^2\frac{m_n}{n}\left\|g(\cdot) - \theta_{K_{1,n}}' Z_1(\cdot)\right\|_{\infty}^2 \\
= \; & O_{a.s.}\left(\frac{\tau_n^2}{K_{1,n}^{2\varrho_g}}\right)
\end{aligned}
$$

Hence, $\varphi' A\varphi = O_p\left(\tau_n^2/K_{1,n}^{2\varrho_g}\right) = o_p(1)$.

Next, to show part (b), note that, by the symmetry of A and by Assumptions 1, 2(i), 4(ii), 5, 6(i), and 7(i); we have

$$
\begin{aligned}
& E\left(\left[\varphi' A\varepsilon\right]^2 |\mathcal{F}_n^W\right) \\
= \; & \frac{\tau_n^2}{n} E\left(\left[g' A\varepsilon\right]^2 |\mathcal{F}_n^W\right) \\
= \; & \frac{\tau_n^2}{n}\left(g - Z_1\theta_{K_{1,n}}\right)' AE\left[\varepsilon\varepsilon' |\mathcal{F}_n^W\right] A\left(g - Z_1\theta_{K_{1,n}}\right) \\
\leq \; & \frac{\tau_n^2}{n}\left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon_{(i,t)}^2 |\mathcal{F}_n^W\right]\right)\left(g - Z_1\theta_{K_{1,n}}\right)'\left(P^{\perp} - M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\right)^2\left(g - Z_1\theta_{K_{1,n}}\right) \\
= \; & \tau_n^2\left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon_{(i,t)}^2 |\mathcal{F}_n^W\right]\right)\frac{\left(g - Z_1\theta_{K_{1,n}}\right)' P^{\perp}\left(g - Z_1\theta_{K_{1,n}}\right)}{n} \\
& +\tau_n^2\left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon_{(i,t)}^2 |\mathcal{F}_n^W\right]\right)\frac{\left(g - Z_1\theta_{K_{1,n}}\right)' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\left(g - Z_1\theta_{K_{1,n}}\right)}{n} \\
\leq \; & \tau_n^2\left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon_{(i,t)}^2 |\mathcal{F}_n^W\right]\right)\frac{m_n}{n}\left\|g(\cdot) - \theta_{K_{1,n}}' Z_1(\cdot)\right\|_{\infty}^2 \\
& +\tau_n^2\left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon_{(i,t)}^2 |\mathcal{F}_n^W\right]\right)\left(\max_{1\leq(i,t)\leq m_n}\left|\widehat{\vartheta}_{(i,t)}\right|^2\right)\frac{m_n}{n}\left\|g(\cdot) - \theta_{K_{1,n}}' Z_1(\cdot)\right\|_{\infty}^2 \\
= \; & O_{a.s.}\left(\frac{\tau_n^2}{K_{1,n}^{2\varrho_g}}\right) + O_{a.s.}\left(\frac{\tau_n^2}{K_{1,n}^{2\varrho_g}}\frac{K_{2,n}^2}{n^2}\right) \\
= \; & O_{a.s.}\left(\frac{\tau_n^2}{K_{1,n}^{2\varrho_g}}\right).
\end{aligned}
$$

Hence, there exists a positive constant $\overline{C} < \infty$ such that for all n sufficiently large

$$
E\left(\frac{K_{1,n}^{2\varrho_g}}{\tau_n^2}\left[\varphi' A\varepsilon\right]^2\right) = E_W\left\{\frac{K_{1,n}^{2\varrho_g}}{\tau_n^2} E\left(\left[\varphi' A\varepsilon\right]^2 |\mathcal{F}_n^W\right)\right\} \leq \overline{C}.
$$

It follows from Markov's inequality, for any $\epsilon > 0$, we can set $C_\epsilon = \sqrt{\overline{C}/\epsilon}$ so that for all n sufficiently large

$$
\begin{aligned}
\Pr\left(\left|\frac{K_{1,n}^{\varrho_g}}{\tau_n}\varphi' A\varepsilon\right| \geq C_\epsilon\right) \; & = \; \Pr\left(\left|\frac{K_{1,n}^{\varrho_g}}{\tau_n}\varphi' A\varepsilon\right|^2 \geq C_\epsilon^2\right) \\
& \leq \; \frac{1}{C_\epsilon^2} E\left(\frac{K_{1,n}^{2\varrho_g}}{\tau_n^2}\left[\varphi' A\varepsilon\right]^2\right) \\
& \leq \; \frac{\overline{C}}{\overline{C}/\epsilon} = \epsilon
\end{aligned}
$$

which shows that

$$\varphi' A\varepsilon = O_p\left(\frac{\tau_n}{K_{1,n}^{\varrho_g}}\right).$$

Finally, to show part (c), note that, by the symmetry of A and by Assumptions 1, 2(i), and 5; we have

$$\begin{aligned}
& E\left(\left[\frac{\varepsilon' A\varepsilon}{\sqrt{K_{2,n}}}\right]^2 |\mathcal{F}_n^W\right) \\
= & \frac{1}{K_{2,n}} \sum_{\substack{(i_1,t_1),(j_1,s_1)=1 \\ (i_1,t_1)\neq(j_1,s_1)}}^{m_n} \sum_{\substack{(i_2,t_2),(j_2,s_2)=1 \\ (i_2,t_2)\neq(j_2,s_2)}}^{m_n} A_{(i_1,t_1),(j_1,s_1)} A_{(i_2,t_2),(j_2,s_2)} E\left[\varepsilon_{(i_1,t_1)}\varepsilon_{(j_1,s_1)}\varepsilon_{(i_2,t_2)}\varepsilon_{(j_2,s_2)}|\mathcal{F}_n^W\right] \\
= & \frac{2}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^2 E\left[\varepsilon_{(i,t)}^2\varepsilon_{(j,s)}^2|\mathcal{F}_n^W\right] \\
\leq & 2\left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon_{(i,t)}^4|\mathcal{F}_n^W\right]\right)\frac{1}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^2 \\
= & O_{a.s.}(1)
\end{aligned}$$

Hence, there exists a positive constant $\overline{C} < \infty$ such that for all n sufficiently large

$$E\left(\left[\frac{\varepsilon' A\varepsilon}{\sqrt{K_{2,n}}}\right]^2\right) = E_W\left\{E\left(\left[\frac{\varepsilon' A\varepsilon}{\sqrt{K_{2,n}}}\right]^2 |\mathcal{F}_n^W\right)\right\} \leq \overline{C}.$$

It follows from Markov's inequality, for any $\epsilon > 0$, we can set $C_\epsilon = \sqrt{\overline{C}/\epsilon}$ so that for all n sufficiently large

$$\begin{aligned}
\Pr\left(\left|\frac{\varepsilon' A\varepsilon}{\sqrt{K_{2,n}}}\right| \geq C_\epsilon\right) &= \Pr\left(\left|\frac{\varepsilon' A\varepsilon}{\sqrt{K_{2,n}}}\right|^2 \geq C_\epsilon^2\right) \\
&\leq \frac{E\left(\left[\varepsilon' A\varepsilon/\sqrt{K_{2,n}}\right]^2\right)}{C_\epsilon^2} \\
&\leq \frac{\overline{C}}{\overline{C}/\epsilon} = \epsilon
\end{aligned}$$

which shows that

$$\frac{\varepsilon' A\varepsilon}{\sqrt{K_{2,n}}} = O_p(1). \ \square$$

Lemma OA-4:

Under Assumptions 1, 2(i), 3, 4, 5(i), 7(i), and 7(ii); the following results hold.

(a)

$$\frac{D_{\mu}\Gamma' M^{(Z_1,Q)}\varphi}{n^{3/2}} = O_{a.s.}\left(\frac{\tau_n\left(\mu_n^{\max}\right)}{nK_{1,n}^{\varrho_g}}\right)$$

(b)

$$\frac{D_{\mu}^{-1}D_{\kappa}F' M^{(Z_1,Q)}\varepsilon}{\sqrt{n}} = O_p\left(\frac{\kappa_n^{\max}}{\mu_n^{\min}}\frac{1}{K_{1,n}^{\varrho_f}}\right) = o_p\left(1\right)$$

(c)

$$\varphi' M^{(Z_1,Q)}\varphi = O_p\left(\frac{\tau_n^2}{K_{1,n}^{2\varrho_g}}\right).$$

(d)

$$\varphi' M^{(Z_1,Q)}\varepsilon = O_p\left(\frac{\tau_n}{K_{1,n}^{\varrho_g}}\right).$$

(e)

$$D_{\mu}^{-1}X' M^{(Z_1,Q)}\varphi = O_p\left(\frac{\tau_n}{K_{1,n}^{\varrho_g}}\right)$$

Proof of Lemma OA-4:

To show part (a), let $b \in \mathbb{R}^d$ such that $\|b\|_2 = 1$, and note that, by the CS inequality and Assumptions 3(i), 3(iii), 4(ii), 5(i), and 7(i);

$$\begin{aligned}
& \left|\frac{b' D_{\mu}\Gamma' M^{(Z_1,Q)}\varphi}{n^{3/2}}\right| \\
= \; & \frac{\tau_n}{n^2}\left|b' D_{\mu}\Gamma' M^{(Z_1,Q)} g\right| \\
= \; & \frac{\tau_n}{n^2}\left|b' D_{\mu}\Gamma' M^{(Z_1,Q)}\left(g - Z_1\theta_{K_{1,n}}\right)\right| \\
\leq \; & \frac{\tau_n}{n}\sqrt{\frac{b' D_{\mu}\Gamma'\Gamma D_{\mu} b}{n}}\sqrt{\frac{\left(g - Z_1\theta_{K_{1,n}}\right)' M^{(Z_1,Q)}\left(g - Z_1\theta_{K_{1,n}}\right)}{n}} \\
\leq \; & \frac{\tau_n\left(\mu_n^{\max}\right)}{n}\sqrt{\lambda_{\max}\left(\frac{\Gamma'\Gamma}{n}\right)}\sqrt{\frac{m_n}{n}}\left\|g\left(\cdot\right) - \theta'_{K_{1,n}} Z_1\left(\cdot\right)\right\|_{\infty} \\
= \; & O_{a.s.}\left(\frac{\tau_n\left(\mu_n^{\max}\right)}{nK_{1,n}^{\varrho_g}}\right)
\end{aligned}$$

Since the above argument holds for all $b \in \mathbb{R}^d$ such that $\|b\|_2 = 1$, we further deduce that

$$\frac{D_{\mu}\Gamma' M^{(Z_1,Q)}\varphi}{n^{3/2}} = O_{a.s.}\left(\frac{\tau_n\left(\mu_n^{\max}\right)}{nK_{1,n}^{\varrho_g}}\right).$$

Next, to show part (b), let $b \in \mathbb{R}^d$ such that $\|b\|_2 = 1$; and note that, by making use of the CS inequality and Assumptions 1, 2(i), 3(ii), 4(i), 5(i), and 7(ii); we obtain

$$\begin{aligned}
& E\left[\left(\frac{b' D_\mu^{-1} D_\kappa F' M^{(Z_1,Q)} \varepsilon}{\sqrt{n}}\right)^2 | \mathcal{F}_n^W\right] \\
= & \frac{b' D_\mu^{-1} D_\kappa F' M^{(Z_1,Q)} E\left[\varepsilon\varepsilon' | \mathcal{F}_n^W\right] M^{(Z_1,Q)} F D_\kappa D_\mu^{-1} b}{n} \\
\leq & \left(\max_{1 \leq (i,t) \leq m_n} E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W\right]\right) \frac{b' D_\mu^{-1} D_\kappa \left(F - Z_1 \Theta_{K_{1,n}}\right)' M^{(Z_1,Q)} \left(F - Z_1 \Theta_{K_{1,n}}\right) D_\kappa D_\mu^{-1} b}{n} \\
\leq & \left(\max_{1 \leq (i,t) \leq m_n} E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W\right]\right) \frac{m_n d}{n} \left\| f(\cdot) - \Theta_{K_{1,n}}' Z_1(\cdot) \right\|_{\infty,d}^2 b' D_\mu^{-1} D_\kappa^2 D_\mu^{-1} b \\
\leq & \left(\max_{1 \leq (i,t) \leq m_n} E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W\right]\right) \frac{\left(\kappa_n^{\max}\right)^2}{\left(\mu_n^{\min}\right)^2} \frac{m_n d}{n} \left\| f(\cdot) - \Theta_{K_{1,n}}' Z_1(\cdot) \right\|_{\infty,d}^2 \\
= & O_{a.s.}\left(\frac{\left(\kappa_n^{\max}\right)^2}{\left(\mu_n^{\min}\right)^2} \frac{1}{K_{1,n}^{2\varrho_f}}\right)
\end{aligned}$$

Hence, there exists a positive constant $\overline{C} < \infty$ such that for all n sufficiently large

$$\begin{aligned}
& E\left\{\frac{\left(\mu_n^{\min}\right)^2 K_{1,n}^{2\varrho_f}}{\left(\kappa_n^{\max}\right)^2} \left(\frac{b' D_\mu^{-1} D_\kappa F' M^{(Z_1,Q)} \varepsilon}{\sqrt{n}}\right)^2\right\} \\
= & E_W\left\{\frac{\left(\mu_n^{\min}\right)^2 K_{1,n}^{2\varrho_f}}{\left(\kappa_n^{\max}\right)^2} E\left\{\left(\frac{b' D_\mu^{-1} D_\kappa F' M^{(Z_1,Q)} \varepsilon}{\sqrt{n}}\right)^2 | \mathcal{F}_n^W\right\}\right\} \leq \overline{C}.
\end{aligned}$$

It follows from Markov's inequality, for any $\epsilon > 0$, we can set $C_\epsilon = \sqrt{\overline{C}/\epsilon}$ so that for all n sufficiently large

$$\begin{aligned}
& \Pr\left(\left|\frac{\left(\mu_n^{\min}\right) K_{1,n}^{\varrho_f}}{\left(\kappa_n^{\max}\right)} \frac{b' D_\mu^{-1} D_\kappa F' M^{(Z_1,Q)} \varepsilon}{\sqrt{n}}\right| \geq C_\epsilon\right) \\
= & \Pr\left(\left|\frac{\left(\mu_n^{\min}\right) K_{1,n}^{\varrho_f}}{\left(\kappa_n^{\max}\right)} \frac{b' D_\mu^{-1} D_\kappa F' M^{(Z_1,Q)} \varepsilon}{\sqrt{n}}\right|^2 \geq C_\epsilon^2\right) \\
\leq & \frac{1}{C_\epsilon^2} E\left\{\frac{\left(\mu_n^{\min}\right)^2 K_{1,n}^{2\varrho_f}}{\left(\kappa_n^{\max}\right)^2} \left(\frac{b' D_\mu^{-1} D_\kappa F' M^{(Z_1,Q)} \varepsilon}{\sqrt{n}}\right)^2\right\} \\
\leq & \frac{\overline{C}}{\overline{C}/\epsilon} = \epsilon
\end{aligned}$$

which shows that

$$\left|\frac{b' D_\mu^{-1} D_\kappa F' M^{(Z_1,Q)} \varepsilon}{\sqrt{n}}\right| = O_p\left(\frac{\kappa_n^{\max}}{\mu_n^{\min}} \frac{1}{K_{1,n}^{\varrho_f}}\right).$$

Since the above argument holds for all $b \in \mathbb{R}^d$ such that $\|b\|_2 = 1$, we further deduce that

$$\frac{D_\mu^{-1} D_\kappa F' M^{(Z_1,Q)} \varepsilon}{\sqrt{n}} = O_p\left(\frac{\kappa_n^{\max}}{\mu_n^{\min}} \frac{1}{K_{1,n}^{\varrho_f}}\right) = o_p(1)$$

For part (c), note that, by the fact that $M^{(Z_1,Q)}$ is symmetric, idempotent, and positive semi-definite and by applying Assumptions 4(ii), 5(i), and 7(i); we obtain

$$\begin{aligned}
\varphi' M^{(Z_1,Q)} \varphi &= \frac{\tau_n^2}{n}\left(g - Z_1\theta_{K_{1,n}}\right)' M^{(Z_1,Q)} \left(g - Z_1\theta_{K_{1,n}}\right) \\
&\leq \tau_n^2 \frac{\left(g - Z_1\theta_{K_{1,n}}\right)'\left(g - Z_1\theta_{K_{1,n}}\right)}{n} \\
&\leq \tau_n^2 \frac{m_n}{n}\left\|g(\cdot) - \theta_{K_{1,n}}' Z_1(\cdot)\right\|_\infty^2 \\
&= O_{a.s.}\left(\frac{\tau_n^2}{K_{1,n}^{2\varrho_g}}\right),
\end{aligned}$$

which, of course, implies that $\varphi' M^{(Z_1,Q)} \varphi = O_p\left(\tau_n^2 / K_{1,n}^{2\varrho_g}\right)$.

Now, consider part (d). In this case, note that, by the fact that $M^{(Z_1,Q)}$ is symmetric and idempotent and by Assumptions 1, 2(i), 4(ii), 5(i), and 7(i); we have

$$\begin{aligned}
& E\left(\left[\varphi' M^{(Z_1,Q)} \varepsilon\right]^2 | \mathcal{F}_n^W\right) \\
=\ & \frac{\tau_n^2}{n} E\left(\left[g' M^{(Z_1,Q)} \varepsilon\right]^2 | \mathcal{F}_n^W\right) \\
=\ & \frac{\tau_n^2}{n}\left(g - Z_1\theta_{K_{1,n}}\right)' M^{(Z_1,Q)} E\left[\varepsilon\varepsilon' | \mathcal{F}_n^W\right] M^{(Z_1,Q)}\left(g - Z_1\theta_{K_{1,n}}\right) \\
\leq\ & \tau_n^2\left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W\right]\right)\frac{\left(g - Z_1\theta_{K_{1,n}}\right)' M^{(Z_1,Q)}\left(g - Z_1\theta_{K_{1,n}}\right)}{n} \\
\leq\ & \tau_n^2\left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W\right]\right)\frac{\left(g - Z_1\theta_{K_{1,n}}\right)'\left(g - Z_1\theta_{K_{1,n}}\right)}{n} \\
\leq\ & \tau_n^2\left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W\right]\right)\frac{m_n}{n}\left\|g(\cdot) - \theta_{K_{1,n}}' Z_1(\cdot)\right\|_\infty^2 \\
=\ & O_{a.s.}\left(\frac{\tau_n^2}{K_{1,n}^{2\varrho_g}}\right)
\end{aligned}$$

Hence, there exists a positive constant $\overline{C} < \infty$ such that for all n sufficiently large

$$E\left(\frac{K_{1,n}^{2\varrho_g}}{\tau_n^2}\left[\varphi' M^{(Z_1,Q)}\varepsilon\right]^2\right) = E_W\left\{\frac{K_{1,n}^{2\varrho_g}}{\tau_n^2}E\left(\left[\varphi' M^{(Z_1,Q)}\varepsilon\right]^2 |\mathcal{F}_n^W\right)\right\} \leq \overline{C}.$$

It follows from Markov's inequality, for any $\epsilon > 0$, we can set $C_\epsilon = \sqrt{\overline{C}/\epsilon}$ so that for all n sufficiently large

$$\begin{aligned}
\Pr\left(\left|\frac{K_{1,n}^{\varrho_g}}{\tau_n}\varphi' M^{(Z_1,Q)}\varepsilon\right| \geq C_\epsilon\right) &= \Pr\left(\left|\frac{K_{1,n}^{\varrho_g}}{\tau_n}\varphi' M^{(Z_1,Q)}\varepsilon\right|^2 \geq C_\epsilon^2\right) \\
&\leq \frac{1}{C_\epsilon^2}E\left(\frac{K_{1,n}^{2\varrho_g}}{\tau_n^2}\left[\varphi' M^{(Z_1,Q)}\varepsilon\right]^2\right) \\
&\leq \frac{\overline{C}}{\overline{C}/\epsilon} = \epsilon
\end{aligned}$$

which shows that

$$\varphi' M^{(Z_1,Q)}\varepsilon = O_p\left(\frac{\tau_n}{K_{1,n}^{\varrho_g}}\right).$$

Finally, to show part (e), first let $b \in \mathbb{R}^d$ such that $\|b\| = 1$ and write

$$\begin{aligned}
b' D_\mu^{-1} X' M^{(Z_1,Q)}\varphi &= b' D_\mu^{-1}\left(\Upsilon + \Phi + Q\Xi + U\right)' M^{(Z_1,Q)}\varphi \\
&= D_\mu^{-1}\left(\Upsilon + \Phi + U\right)' M^{(Z_1,Q)}\varphi \\
&= b' D_\mu^{-1}\Upsilon' M^{(Z_1,Q)}\varphi + b' D_\mu^{-1}\Phi' M^{(Z_1,Q)}\varphi \\
&\quad + b' D_\mu^{-1} U' M^{(Z_1,Q)}\varphi
\end{aligned}$$

where the second equality above follows from the fact that $Q' M^{(Z_1,Q)}\varphi = 0$. We will analyze each term on the right-hand side of the expression above in turn. Consider first the term $D_\mu^{-1}\Upsilon' M^{(Z_1,Q)}\varphi$.

Note that, by applying the CS inequality and Assumptions 3(iii), 4(ii), 5(i), and 7(i); we have

$$\begin{aligned}
& \left|b' D_\mu^{-1} \Upsilon' M^{(Z_1,Q)} \varphi\right| \\
= \;& \frac{\tau_n}{n}\left|b' D_\mu^{-1} D_\mu \Gamma' M^{(Z_1,Q)} g\right| \\
= \;& \frac{\tau_n}{n}\left|b' \Gamma' M^{(Z_1,Q)}\left(g - Z_1 \theta_{K_{1,n}}\right)\right| \\
\leq \;& \tau_n \sqrt{\frac{b'\Gamma'\Gamma b}{n}} \sqrt{\frac{\left(g - Z_1\theta_{K_{1,n}}\right)' M^{(Z_1,Q)}\left(g - Z_1\theta_{K_{1,n}}\right)}{n}} \\
\leq \;& \tau_n \sqrt{\frac{b'\Gamma'\Gamma b}{n}} \sqrt{\frac{m_n}{n}} \left\|g(\cdot) - \theta'_{K_{1,n}} Z_1(\cdot)\right\|_\infty \\
= \;& O_{a.s.}\left(\frac{\tau_n}{K_{1,n}^{\varrho_g}}\right)
\end{aligned}$$

Next, consider $b' D_\mu^{-1} \Phi' M^{(Z_1,Q)} \varphi$. Here, note that, by the CS inequality and by Assumptions 3(ii), 4, 5(i), 7(i), and 7(ii);

$$\begin{aligned}
& \left|b' D_\mu^{-1} \Phi' M^{(Z_1,Q)} \varphi\right| \\
= \;& \frac{\tau_n}{n}\left|b' D_\mu^{-1} D_\kappa \left(F - Z_1\Theta_{K_{1,n}}\right)' M^{(Z_1,Q)}\left(g - Z_1\theta_{K_{1,n}}\right)\right| \\
\leq \;& \tau_n \sqrt{\frac{b' D_\mu^{-1} D_\kappa \left(F - Z_1\Theta_{K_{1,n}}\right)' M^{(Z_1,Q)}\left(F - Z_1\Theta_{K_{1,n}}\right) D_\kappa D_\mu^{-1} b}{n}} \\
& \times \sqrt{\frac{\left(g - Z_1\theta_{K_{1,n}}\right)' M^{(Z_1,Q)}\left(g - Z_1\theta_{K_{1,n}}\right)}{n}} \\
\leq \;& \frac{\tau_n\left(\kappa_n^{\max}\right)}{\left(\mu_n^{\min}\right)} \frac{m_n\sqrt{d}}{n} \left\|f(\cdot) - \Theta'_{K_{1,n}} Z_1(\cdot)\right\|_{\infty,d} \left\|g(\cdot) - \theta'_{K_{1,n}} Z_1(\cdot)\right\|_\infty \\
= \;& O_{a.s.}\left(\frac{\tau_n\left(\kappa_n^{\max}\right)}{\left(\mu_n^{\min}\right) K_{1,n}^{\varrho_f+\varrho_g}}\right) = o_{a.s.}(1)
\end{aligned}$$

Now, consider the term $b' D_\mu^{-1} U' M^{(Z_1,Q)} \varphi$. Let $u_b = U D_\mu^{-1} b$, where $b \in \mathbb{R}^d$ such that $\|b\| = 1$, and note that

$$\begin{aligned}
E\left[u_b u_b' | \mathcal{F}_n^W\right] &= E\left[U D_\mu^{-1} b b' D_\mu^{-1} U' | \mathcal{F}_n^W\right] \leq \frac{1}{\left(\mu_n^{\min}\right)^2} E\left[UU' | \mathcal{F}_n^W\right] \\
&\leq \frac{\max_{1\leq(i,t)\leq m_n} tr\left\{E\left[U_{(i,t)} U'_{(i,t)} | \mathcal{F}_n^W\right]\right\}}{\left(\mu_n^{\min}\right)^2} I_{m_n}
\end{aligned}$$

where we take $A \leq B$ for two square matrices A and B to mean that $A - B$ is negative semi-definite, or, alternatively, $B - A$ is positive semidefinite. Note further that, by applying Assumptions 1, 2(i), 3(ii), 4(ii), and 7(i);

$$
\begin{aligned}
& E\left(\left[b'D_{\mu}^{-1}U'M^{(Z_1,Q)}\varphi\right]^2|\mathcal{F}_n^W\right) \\
= \;& \frac{\tau_n^2}{n}E\left[g'M^{(Z_1,Q)}UD_{\mu}^{-1}bb'D_{\mu}^{-1}U'M^{(Z_1,Q)}g|\mathcal{F}_n^W\right] \\
= \;& \frac{\tau_n^2}{n}g'M^{(Z_1,Q)}E\left[u_b u_b'|\mathcal{F}_n^W\right]M^{(Z_1,Q)}g \\
\leq \;& \frac{\tau_n^2}{n}\frac{\max_{1\leq(i,t)\leq m_n} tr\left\{E\left[U_{(i,t)}U_{(i,t)}'|\mathcal{F}_n^W\right]\right\}}{(\mu_n^{\min})^2}g'M^{(Z_1,Q)}g \\
= \;& \frac{\tau_n^2}{(\mu_n^{\min})^2}\left(\max_{1\leq(i,t)\leq m_n} E\left[\left\|U_{(i,t)}\right\|_2^2|\mathcal{F}_n^W\right]\right)^2 \\
& \times\frac{\left(g-Z_1\theta_{K_{1,n}}\right)'M^{(Z_1,Q)}\left(g-Z_1\theta_{K_{1,n}}\right)}{n} \\
\leq \;& C\frac{\tau_n^2}{(\mu_n^{\min})^2}\frac{m_n}{n}\left\|g(\cdot)-\theta_{K_{1,n}}'Z_1(\cdot)\right\|_\infty^2 \\
= \;& O_{a.s.}\left(\frac{\tau_n^2}{(\mu_n^{\min})^2 K_{1,n}^{2\varrho_g}}\right)
\end{aligned}
$$

Hence, there exists a positive constant $\overline{C}<\infty$ such that for all n sufficiently large

$$
\begin{aligned}
& E\left(\frac{\left(\mu_n^{\min}\right)^2 K_{1,n}^{2\varrho_g}}{\tau_n^2}\left[b'D_{\mu}^{-1}U'M^{(Z_1,Q)}\varphi\right]^2\right) \\
= \;& E_W\left\{\frac{\left(\mu_n^{\min}\right)^2 K_{1,n}^{2\varrho_g}}{\tau_n^2}E\left(\left[b'D_{\mu}^{-1}U'M^{(Z_1,Q)}\varphi\right]^2|\mathcal{F}_n^W\right)\right\}\leq\overline{C}.
\end{aligned}
$$

It follows from Markov's inequality, for any $\epsilon>0$, we can set $C_\epsilon=\sqrt{\overline{C}/\epsilon}$ so that for all n sufficiently large

$$
\begin{aligned}
& \Pr\left(\left|\left(\frac{\left(\mu_n^{\min}\right)K_{1,n}^{\varrho_g}}{\tau_n}\right)b'D_{\mu}^{-1}U'M^{(Z_1,Q)}\varphi\right|\geq C_\epsilon\right) \\
= \;& \Pr\left(\left|\left(\frac{\left(\mu_n^{\min}\right)K_{1,n}^{\varrho_g}}{\tau_n}\right)b'D_{\mu}^{-1}U'M^{(Z_1,Q)}\varphi\right|^2\geq C_\epsilon^2\right) \\
\leq \;& \frac{1}{C_\epsilon^2}E\left(\frac{\left(\mu_n^{\min}\right)^2 K_{1,n}^{2\varrho_g}}{\tau_n^2}\left[b'D_{\mu}^{-1}U'M^{(Z_1,Q)}\varphi\right]^2\right) \\
\leq \;& \frac{\overline{C}}{\overline{C}/\epsilon}=\epsilon
\end{aligned}
$$

which shows that $b'D_\mu^{-1}U'M^{(Z_1,Q)}\varphi = O_p\left(\tau_n / \left[\left(\mu_n^{\min}\right) K_{1,n}^{\varrho_g}\right]\right)$. Putting the above results together, we get

$$\begin{aligned}
& b'D_\mu^{-1}X'M^{(Z_1,Q)}\varphi \\
= \; & b'D_\mu^{-1}\Upsilon'M^{(Z_1,Q)}\varphi + b'D_\mu^{-1}\Phi'M^{(Z_1,Q)}\varphi + b'D_\mu^{-1}U'M^{(Z_1,Q)}\varphi \\
= \; & O_{a.s.}\left(\frac{\tau_n}{K_{1,n}^{\varrho_g}}\right) + O_{a.s.}\left(\frac{\tau_n\left(\kappa_n^{\max}\right)}{\left(\mu_n^{\min}\right) K_{1,n}^{\varrho_f+\varrho_g}}\right) + O_p\left(\frac{\tau_n}{\left(\mu_n^{\min}\right) K_{1,n}^{\varrho_g}}\right) \\
= \; & O_{a.s.}\left(\frac{\tau_n}{K_{1,n}^{\varrho_g}}\right).
\end{aligned}$$

Since the above argument holds for all $b \in \mathbb{R}^d$ such that $\|b\|_2 = 1$, we further deduce that

$$D_\mu^{-1}X'M^{(Z_1,Q)}\varphi = O_{a.s.}\left(\frac{\tau_n}{K_{1,n}^{\varrho_g}}\right). \; \square$$

Lemma OA-5: Under Assumptions 1, 2, 6, and 8; the following statements are true.

(a)

$$\frac{\varepsilon'M^Q\varepsilon}{n} - E\left[\frac{\varepsilon'M^Q\varepsilon}{n}\right] = O_p\left(\frac{1}{\sqrt{n}}\right),$$

and there exist positive constants $\underline{C} \leq \overline{C}$ such that

$$0 < \underline{C} \leq E\left[\frac{\varepsilon'M^Q\varepsilon}{n}\right] \leq \overline{C} < \infty$$

for all n sufficiently large.

(b)

$$\frac{U'M^{(Z_1,Q)}\varepsilon}{n} - E\left[\frac{U'M^Q\varepsilon}{n}\right] = O_p\left(\max\left\{\frac{1}{\sqrt{n}}, \frac{K_{1,n}}{n}\right\}\right),$$

where

$$E\left[\frac{U'M^Q\varepsilon}{n}\right] = O(1).$$

(c)

$$\frac{U'M^{(Z_1,Q)}\varepsilon}{\varepsilon'M^Q\varepsilon} - \rho = O_p\left(\max\left\{\frac{1}{\sqrt{n}}, \frac{K_{1,n}}{n}\right\}\right),$$

where

$$\rho = \lim_{n\to\infty}\rho_n = \lim_{n\to\infty}\frac{E\left[U'M^Q\varepsilon\right]/n}{E\left[\varepsilon'M^Q\varepsilon\right]/n}.$$

Proof of Lemma OA-5: To show part (a), note first that, making use of Assumptions 1, 2(i), and 6(ii); we have

$$
\begin{aligned}
& E\left(\frac{\varepsilon' M^Q \varepsilon - E\left[\varepsilon' M^Q \varepsilon\right]}{n}\right)^2 \\
= \; & \frac{E\left(\sum_{i=1}^{n}\sum_{t=1}^{T_i}\left(\frac{T_i-1}{T_i}\right)\left[\varepsilon_{(i,t)}^2 - E\left\{\varepsilon_{(i,t)}^2\right\}\right] - \sum_{i=1}^{n}\frac{1}{T_i}\sum_{r=1}^{T_i}\sum_{\substack{s=1\\ s\neq r}}^{T_i}\varepsilon_{(i,r)}\varepsilon_{(i,s)}\right)^2}{n^2} \\
= \; & \frac{1}{n^2}\sum_{i=1}^{n}\sum_{t=1}^{T_i}\sum_{j=1}^{n}\sum_{s=1}^{T_j}\left(\frac{T_i-1}{T_i}\right)\left(\frac{T_j-1}{T_j}\right)E\left(\left[\varepsilon_{(i,t)}^2 - E\left\{\varepsilon_{(i,t)}^2\right\}\right]\left[\varepsilon_{(j,s)}^2 - E\left\{\varepsilon_{(j,s)}^2\right\}\right]\right) \\
& -\frac{2}{n^2}\sum_{i=1}^{n}\sum_{t=1}^{T_i}\sum_{j=1}^{n}\sum_{r=1}^{T_j}\sum_{\substack{s=1\\ s\neq r}}^{T_j}\left(\frac{T_i-1}{T_i}\right)\frac{1}{T_j}E\left\{\left[\varepsilon_{(i,t)}^2 - E\left\{\varepsilon_{(i,t)}^2\right\}\right]\varepsilon_{(j,r)}\varepsilon_{(j,s)}\right\} \\
& +\frac{1}{n^2}\sum_{i=1}^{n}\sum_{t=1}^{T_i}\sum_{\substack{s=1\\ s\neq t}}^{T_i}\sum_{j=1}^{n}\sum_{r=1}^{T_j}\sum_{\substack{h=1\\ h\neq r}}^{T_j}\left(\frac{1}{T_i}\right)\left(\frac{1}{T_j}\right)E\left[\varepsilon_{(i,t)}\varepsilon_{(i,s)}\varepsilon_{(j,r)}\varepsilon_{(j,h)}\right] \\
= \; & \frac{1}{n^2}\sum_{i=1}^{n}\sum_{t=1}^{T_i}\sum_{j=1}^{n}\sum_{s=1}^{T_j}\left(\frac{T_i-1}{T_i}\right)\left(\frac{T_j-1}{T_j}\right)E\left(\left[\varepsilon_{(i,t)}^2 - E\left\{\varepsilon_{(i,t)}^2\right\}\right]\left[\varepsilon_{(j,s)}^2 - E\left\{\varepsilon_{(j,s)}^2\right\}\right]\right) \\
& +\frac{2}{n^2}\sum_{i=1}^{n}\sum_{t=1}^{T_i}\sum_{\substack{s=1\\ s\neq t}}^{T_i}\left(\frac{1}{T_i}\right)^2 E\left[\varepsilon_{(i,t)}^2\right]E\left[\varepsilon_{(i,s)}^2\right] \\
= \; & \frac{1}{n^2}\sum_{i=1}^{n}\sum_{t=1}^{T_i}\left(\frac{T_i-1}{T_i}\right)^2 E\left(\varepsilon_{(i,t)}^2 - E\left\{\varepsilon_{(i,t)}^2\right\}\right)^2 + \frac{2}{n^2}\sum_{i=1}^{n}\sum_{t=1}^{T_i}\sum_{\substack{s=1\\ s\neq t}}^{T_i}\left(\frac{1}{T_i}\right)^2 E\left[\varepsilon_{(i,t)}^2\right]E\left[\varepsilon_{(i,s)}^2\right] \\
\leq \; & \frac{1}{n^2}n\overline{T}\max_{1\leq(i,t)\leq m_n}E\left[\varepsilon_{(i,t)}^4\right] + \frac{2}{n^2}n\left(\max_{1\leq(i,t)\leq m_n}E\left[\varepsilon_{(i,t)}^2\right]\right)^2 \\
= \; & \frac{1}{n}\overline{T}\max_{1\leq(i,t)\leq m_n}E\left[\varepsilon_{(i,t)}^4\right] + \frac{2}{n}\left(\max_{1\leq(i,t)\leq m_n}E\left[\varepsilon_{(i,t)}^2\right]\right)^2 \\
= \; & O\left(\frac{1}{n}\right)
\end{aligned}
$$

It follows by Markov's inequality that

$$\frac{\varepsilon' M^Q \varepsilon - E\left[\varepsilon' M^Q \varepsilon\right]}{n} = O_p\left(\frac{1}{\sqrt{n}}\right). \tag{56}$$

Next, note that, by Assumption 6(i), $T_i \geq 3$ for all i, so that $\frac{T_i-1}{T_i} \geq \frac{2}{3}$ for all i. Hence, by

Assumption 2(ii), there exists a positive contant $\underline{C}$ such that $E\left[\varepsilon^2_{(i,t)}|\mathcal{F}^W_n\right] \geq \underline{C} > 0$ *a.s.*, so that

$$\begin{aligned}
\frac{E\left[\varepsilon' M^Q \varepsilon\right]}{n} &= \frac{1}{n} E_{W_n}\left\{\sum_{i=1}^{n}\sum_{t=1}^{T_i}\left[\left(\frac{T_i-1}{T_i}\right)\right] E\left[\varepsilon^2_{(i,t)}|\mathcal{F}^W_n\right]\right\} \\
&\geq \frac{2}{3n} E_{W_n}\left\{\sum_{i=1}^{n}\sum_{t=1}^{T_i} E\left[\varepsilon^2_{(i,t)}|\mathcal{F}^W_n\right]\right\} \\
&\geq \frac{2}{3n} E_{W_n}\left[\sum_{i=1}^{n}\sum_{t=1}^{T_i}\underline{C}\right] \\
&= \frac{2}{3n} E_{W_n}\left[\sum_{(i,t)=1}^{m_n}\underline{C}\right] \\
&= \frac{2}{3}\frac{m_n}{n}\underline{C} \\
&\geq \frac{2}{3}\underline{C} > 0
\end{aligned}$$

for all n sufficiently large. Furthermore, by Assumption 2(i), there also exists a positive contant C ($\geq \underline{C}$) such that

$$\left(\max_{1\leq (i,t)\leq m_n} E\left[\varepsilon^2_{(i,t)}|\mathcal{F}^W_n\right]\right) \leq C \ a.s.,$$

from which it follows that

$$\begin{aligned}
E\left[\frac{\varepsilon' M^Q \varepsilon}{n}\right] &= \frac{1}{n} E_{W_n}\left\{\sum_{i=1}^{n}\sum_{t=1}^{T_i}\left(\frac{T_i-1}{T_i}\right) E\left[\varepsilon^2_{(i,t)}|\mathcal{F}^W_n\right]\right\} \\
&\leq \frac{1}{n} E_{W_n}\left\{\sum_{i=1}^{n}\sum_{t=1}^{T_i} E\left[\varepsilon^2_{(i,t)}|\mathcal{F}^W_n\right]\right\} \\
&\leq \frac{m_n}{n} C = \overline{C} < \infty.
\end{aligned}$$

Next, to show part (b), let $b \in \mathbb{R}^d$ such that $\|b\|_2 = 1$; and since $M^{(Z_1,Q)} = M^Q - P^{Z_1^\perp}$, where $P^{Z_1^\perp} = M^Q Z_1\left(Z_1' M^Q Z_1\right)^{-1} Z_1' M^Q$, we can write

$$\frac{b'U'M^{(Z_1,Q)}\varepsilon}{n} = \frac{b'U'M^Q\varepsilon}{n} - \frac{b'U'P^{Z_1^\perp}\varepsilon}{n}.$$

Note first that by argument similar to that given in part (a) above, we can show that

$$\frac{b'U'M^Q\varepsilon}{n} - E\left[\frac{b'U'M^Q\varepsilon}{n}\right] = O_p\left(\frac{1}{\sqrt{n}}\right). \tag{57}$$

Next, note that, by application of the CS inequality and Assumptions 1, 2(i), and 5; we have

$$
\begin{aligned}
& E\left[\left|\frac{b'U'P^{Z_1^{\perp}}\varepsilon}{n}\right| |\mathcal{F}_n^W\right] \\
\leq & \sqrt{E\left[\frac{b'U'P^{Z_1^{\perp}}Ub}{n}|\mathcal{F}_n^W\right]}\sqrt{E\left[\frac{\varepsilon'P^{Z_1^{\perp}}\varepsilon}{n}|\mathcal{F}_n^W\right]} \\
= & \sqrt{\frac{1}{n}\sum_{(i,t)=1}^{m_n}\sum_{(j,s)=1}^{m_n}P^{Z_1^{\perp}}_{(i,t),(j,s)}b'E\left[U_{(i,t)}U'_{(j,s)}|\mathcal{F}_n^W\right]b}\sqrt{\frac{1}{n}\sum_{(i,t)=1}^{m_n}\sum_{(j,s)=1}^{m_n}P^{Z_1^{\perp}}_{(i,t),(j,s)}E\left[\varepsilon_{(i,t)}\varepsilon_{(j,s)}|\mathcal{F}_n^W\right]} \\
= & \sqrt{\frac{1}{n}\sum_{(i,t)=1}^{m_n}P^{Z_1^{\perp}}_{(i,t),(i,t)}b'E\left[U_{(i,t)}U'_{(i,t)}|\mathcal{F}_n^W\right]b}\sqrt{\frac{1}{n}\sum_{(i,t)=1}^{m_n}P^{Z_1^{\perp}}_{(i,t),(i,t)}E\left[\varepsilon^2_{(i,t)}|\mathcal{F}_n^W\right]} \\
\leq & \sqrt{\max_{1\leq(i,t)\leq m_n}E\left[\left\|U_{(i,t)}\right\|_2^2|\mathcal{F}_n^W\right]}\sqrt{\max_{1\leq(i,t)\leq m_n}E\left[\varepsilon^2_{(i,t)}|\mathcal{F}_n^W\right]}\frac{K_{1,n}}{n} = O_{a.s.}\left(\frac{K_{1,n}}{n}\right)
\end{aligned}
$$

Hence, there exists a constant $\overline{C} < \infty$ such that for all n sufficiently large

$$
E\left[\frac{n}{K_{1,n}}\left|\frac{b'U'P^{Z_1^{\perp}}\varepsilon}{n}\right|\right] = E_{W_n}\left(E\left[\frac{n}{K_{1,n}}\left|\frac{b'U'P^{Z_1^{\perp}}\varepsilon}{n}\right||\mathcal{F}_n^W\right]\right) \leq \overline{C}.
$$

Application of Markov's inequality then implies that for any $\epsilon > 0$,

$$
\Pr\left(\frac{n}{K_{1,n}}\left|\frac{b'U'P^{Z_1^{\perp}}\varepsilon}{n}\right| \geq \frac{\overline{C}}{\epsilon}\right) \leq \epsilon\frac{n}{K_{1,n}}\frac{E\left[\left|b'U'P^{Z_1^{\perp}}\varepsilon/n\right|\right]}{\overline{C}} \leq \epsilon
$$

for all n sufficiently large, which shows that

$$
\frac{b'U'P^{Z_1^{\perp}}\varepsilon}{n} = O_p\left(\frac{K_{1,n}}{n}\right). \tag{58}
$$

It follows from (57) and (58) that

$$
\frac{b'U'M^{(Z_1,Q)}\varepsilon}{n} - E\left[\frac{b'U'M^{Q}\varepsilon}{n}\right] = O_p\left(\max\left\{\frac{1}{\sqrt{n}}, \frac{K_{1,n}}{n}\right\}\right)
$$

In addition, by the CS inequality and Assumptions 2(i) and 6(ii),

$$\begin{aligned}
\left|\frac{E\left[b'U'M^{Q}\varepsilon\right]}{n}\right| &= \frac{1}{n}\left|\sum_{i=1}^{n}\sum_{t=1}^{T_i}\left(\frac{T_i-1}{T_i}\right)E\left[b'U_{(i,t)}\varepsilon_{(i,t)}\right]\right| \\
&\leq \frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T_i}E\left[\left|b'U_{(i,t)}\varepsilon_{(i,t)}\right|\right] \\
&\leq \frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T_i}\sqrt{b'E\left[U_{(i,t)}U_{(i,t)}'\right]b}\sqrt{E\left[\varepsilon_{(i,t)}^{2}\right]} \\
&\leq \overline{T}\sqrt{\max_{1\leq(i,t)\leq m_n}E\left\|U_{(i,t)}\right\|^{2}}\sqrt{\max_{1\leq(i,t)\leq m_n}E\left[\varepsilon_{(i,t)}^{2}\right]} \\
&= O(1)
\end{aligned}$$

Since the above argument holds for any $b\in\mathbb{R}^{d}$ such that $\|b\|_2=1$, we further deduce that

$$\frac{U'M^{(Z_1,Q)}\varepsilon}{n}-E\left[\frac{U'M^{Q}\varepsilon}{n}\right]=O_p\left(\max\left\{\frac{1}{\sqrt{n}},\frac{K_{1,n}}{n}\right\}\right).$$

and

$$E\left[\frac{U'M^{Q}\varepsilon}{n}\right]=O(1),$$

as required to show part (b).

Finally, to show part (c), we first write

$$\frac{U'M^{(Z_1,Q)}\varepsilon}{\varepsilon'M^{Q}\varepsilon}-\rho=\frac{U'M^{(Z_1,Q)}\varepsilon}{\varepsilon'M^{Q}\varepsilon}-\rho_n+\rho_n-\rho$$

By Assumption 8, the sequence $\{\rho_n\}$ has a limit defined to be ρ so that $\rho_n-\rho\to 0$ as $n\to\infty$. Next, note that

$$\begin{aligned}
&\frac{U'M^{(Z_1,Q)}\varepsilon}{\varepsilon'M^{Q}\varepsilon}-\rho_n \\
=\ &\frac{U'M^{(Z_1,Q)}\varepsilon}{\varepsilon'M^{Q}\varepsilon}-\frac{E\left[U'M^{Q}\varepsilon\right]}{E\left[\varepsilon'M^{Q}\varepsilon\right]} \\
=\ &\frac{U'M^{(Z_1,Q)}\varepsilon}{E\left[\varepsilon'M^{Q}\varepsilon\right]}-\frac{E\left[U'M^{Q}\varepsilon\right]}{E\left[\varepsilon'M^{Q}\varepsilon\right]}+\frac{U'M^{(Z_1,Q)}\varepsilon}{\varepsilon'M^{Q}\varepsilon}-\frac{U'M^{(Z_1,Q)}\varepsilon}{E\left[\varepsilon'M^{Q}\varepsilon\right]} \\
=\ &\frac{U'M^{(Z_1,Q)}\varepsilon-E\left[U'M^{Q}\varepsilon\right]}{E\left[\varepsilon'M^{Q}\varepsilon\right]}+\frac{U'M^{(Z_1,Q)}\varepsilon E\left[\varepsilon'M^{Q}\varepsilon\right]}{\left(\varepsilon'M^{Q}\varepsilon\right)E\left[\varepsilon'M^{Q}\varepsilon\right]}-\frac{\left(\varepsilon'M^{Q}\varepsilon\right)U'M^{(Z_1,Q)}\varepsilon}{\left(\varepsilon'M^{Q}\varepsilon\right)E\left[\varepsilon'M^{Q}\varepsilon\right]} \\
=\ &\frac{U'M^{(Z_1,Q)}\varepsilon-E\left[U'M^{Q}\varepsilon\right]}{E\left[\varepsilon'M^{Q}\varepsilon\right]}-\frac{U'M^{(Z_1,Q)}\varepsilon}{\varepsilon'M^{Q}\varepsilon}\frac{\left(\varepsilon'M^{Q}\varepsilon-E\left[\varepsilon'M^{Q}\varepsilon\right]\right)}{E\left[\varepsilon'M^{Q}\varepsilon\right]}
\end{aligned}$$

Now, applying the results of parts (a) and (b) above and the Slutsky's theorem, we obtain

$$\begin{aligned}
\frac{U'M^{(Z_1,Q)}\varepsilon - E\left[U'M^{Q}\varepsilon\right]}{E\left[\varepsilon'M^{Q}\varepsilon\right]} &= \frac{\left(U'M^{(Z_1,Q)}\varepsilon - E\left[U'M^{Q}\varepsilon\right]\right)/n}{E\left[\varepsilon'M^{Q}\varepsilon\right]/n} = O_p\left(\max\left\{\frac{1}{\sqrt{n}}, \frac{K_{1,n}}{n}\right\}\right), \\
\frac{\varepsilon'M^{Q}\varepsilon - E\left[\varepsilon'M^{Q}\varepsilon\right]}{E\left[\varepsilon'M^{Q}\varepsilon\right]} &= \frac{\left(\varepsilon'M^{Q}\varepsilon - E\left[\varepsilon'M^{Q}\varepsilon\right]\right)/n}{E\left[\varepsilon'M^{Q}\varepsilon\right]/n} = O_p\left(\frac{1}{\sqrt{n}}\right),
\end{aligned}$$

and

$$\begin{aligned}
\frac{U'M^{(Z_1,Q)}\varepsilon}{\varepsilon'M^{Q}\varepsilon} &= \frac{U'M^{(Z_1,Q)}\varepsilon/n}{\varepsilon'M^{Q}\varepsilon/n} \\
&= \frac{\left(U'M^{(Z_1,Q)}\varepsilon - E\left[U'M^{,Q}\varepsilon\right]\right)/n + E\left[U'M^{Q}\varepsilon\right]/n}{\left(\varepsilon'M^{Q}\varepsilon - E\left[\varepsilon'M^{Q}\varepsilon\right]\right)/n + E\left[\varepsilon'M^{Q}\varepsilon\right]/n} \\
&= \frac{E\left[U'M^{Q}\varepsilon\right]/n + o_p(1)}{E\left[\varepsilon'M^{Q}\varepsilon\right]/n + o_p(1)} = O_p(1).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{U'M^{(Z_1,Q)}\varepsilon}{\varepsilon'M^{Q}\varepsilon} - \rho_n \\
=\ &\frac{U'M^{(Z_1,Q)}\varepsilon}{\varepsilon'M^{Q}\varepsilon} - \frac{E\left[U'M^{Q}\varepsilon\right]}{E\left[\varepsilon'M^{Q}\varepsilon\right]} \\
=\ &\frac{\left(U'M^{(Z_1,Q)}\varepsilon - E\left[U'M^{Q}\varepsilon\right]\right)/n}{E\left[\varepsilon'M^{Q}\varepsilon\right]/n} - \frac{U'M^{(Z_1,Q)}\varepsilon}{\varepsilon'M^{Q}\varepsilon}\frac{\varepsilon'M^{Q}\varepsilon - E\left[\varepsilon'M^{Q}\varepsilon\right]/n}{E\left[\varepsilon'M^{Q}\varepsilon\right]/n} \\
=\ &O_p\left(\max\left\{\frac{1}{\sqrt{n}}, \frac{K_{1,n}}{n}\right\}\right) - O_p(1)\,O_p\left(\frac{1}{\sqrt{n}}\right) \\
=\ &O_p\left(\max\left\{\frac{1}{\sqrt{n}}, \frac{K_{1,n}}{n}\right\}\right). \ \square
\end{aligned}$$

Lemma OA-6:

Suppose that Assumptions 1-7 are satisfied. Then, the following statements are true.

(a)

$$\begin{aligned}
&D_\mu^{-1}X'A\left(y - X\delta_0\right) \\
=\ &\frac{\Gamma'M^{(Z_1,Q)}\varepsilon}{\sqrt{n}} + D_\mu^{-1}U'A\varepsilon + O_p\left(\frac{1}{K_{2,n}^{\rho_\gamma}}\right) + O_p\left(\frac{\kappa_n^{\max}}{\mu_n^{\min}}\frac{1}{K_{1,n}^{\varrho_f}}\right) + O_p\left(\frac{\tau_n}{K_{1,n}^{\varrho_g}}\right)
\end{aligned}$$

(b)

$$
\begin{aligned}
& D_\mu^{-1} X' M^{(Z_1,Q)} (y - X\delta_0) \\
= \; & \frac{\Gamma' M^{(Z_1,Q)} \varepsilon}{\sqrt{n}} + D_\mu^{-1} U' M^{(Z_1,Q)} \varepsilon + O_p\left(\frac{\kappa_n^{\max}}{\mu_n^{\min}} \frac{1}{K_{1,n}^{\varrho_f}} \right) + O_p\left(\frac{\tau_n}{K_{1,n}^{\varrho_g}} \right) \\
= \; & \frac{\Gamma' M^{(Z_1,Q)} \varepsilon}{\sqrt{n}} + D_\mu^{-1} U' M^{(Z_1,Q)} \varepsilon + o_p(1).
\end{aligned}
$$

(c)

$$
\begin{aligned}
\frac{(y - X\delta_0)' A (y - X\delta_0)}{\sqrt{K_{2,n}}} & = \frac{\varepsilon' A \varepsilon}{\sqrt{K_{2,n}}} + O_p\left(\frac{\tau_n}{\sqrt{K_{2,n}} K_{1,n}^{\varrho_g}} \right) \\
& = \frac{\varepsilon' A \varepsilon}{\sqrt{K_{2,n}}} [1 + o_p(1)],
\end{aligned}
$$

where $\varepsilon' A \varepsilon / \sqrt{K_{2,n}} = O_p(1)$.

(d)

$$
\frac{(y - X\delta_0)' M^{(Z_1,Q)} (y - X\delta_0)}{n} = \frac{\varepsilon' M^Q \varepsilon}{n} \left[1 + O_p\left(\frac{K_{1,n}}{n} \right) \right],
$$

where $\varepsilon' M^Q \varepsilon / n = O_p(1)$ and where $\varepsilon' M^Q \varepsilon / n > 0$ w.p.a.1.

(e)

$$
\widehat{\ell}(\delta_0) = \frac{\varepsilon' A \varepsilon}{\varepsilon' M^Q \varepsilon} \left[1 + O_p\left(\max\left\{ \frac{\tau_n}{\sqrt{K_{2,n}} K_{1,n}^{\varrho_g}}, \frac{K_{1,n}}{n} \right\} \right) \right],
$$

Proof of Lemma OA-6:

To show part (a), note that by parts (a) and (b) of Lemma S2-4, we have

$$
\begin{aligned}
& D_\mu^{-1} X' A (y - X\delta_0) \\
= \; & D_\mu^{-1} X' A (\varphi + Q\alpha + \varepsilon) \\
= \; & D_\mu^{-1} X' A \varepsilon + D_\mu^{-1} X' A \varphi \\
= \; & \frac{\Gamma' M^{(Z_1,Q)} \varepsilon}{\sqrt{n}} + D_\mu^{-1} U' A \varepsilon + O_p\left(\frac{1}{K_{2,n}^{\rho_\gamma}} \right) + O_p\left(\frac{K_{2,n}}{K_{2,n}^{\rho_\gamma} n} \right) \\
& + O_p\left(\frac{\kappa_n^{\max}}{\mu_n^{\min}} \frac{1}{K_{1,n}^{\varrho_f}} \right) + O_p\left(\frac{\tau_n}{K_{1,n}^{\varrho_g}} \right)
\end{aligned}
$$

To show part (b), write

$$
\begin{aligned}
& D_\mu^{-1} X' M^{(Z_1,Q)} (y - X\delta_0) \\
= \;& D_\mu^{-1} X' M^{(Z_1,Q)} (\varphi + Q\alpha + \varepsilon) \\
= \;& D_\mu^{-1} X' M^{(Z_1,Q)} \varphi + D_\mu^{-1} X' M^{(Z_1,Q)} \varepsilon
\end{aligned}
$$

Note first that by part (e) of Lemma OA-4, we have

$$
D_\mu^{-1} X' M^{(Z_1,Q)} \varphi = O_p\left(\frac{\tau_n}{K_{1,n}^{\varrho_g}}\right).
$$

Next, note that, applying part (b) of Lemma OA-4, we have

$$
\begin{aligned}
D_\mu^{-1} X' M^{(Z_1,Q)} \varepsilon &= D_\mu^{-1}\left(\frac{1}{\sqrt{n}} D_\mu \Gamma' + \frac{1}{\sqrt{n}} D_\kappa F' + \Xi' Q' + U'\right) M^{(Z_1,Q)} \varepsilon \\
&= \frac{\Gamma' M^{(Z_1,Q)} \varepsilon}{\sqrt{n}} + D_\mu^{-1} U' M^{(Z_1,Q)} \varepsilon + \frac{D_\mu^{-1} D_\kappa F' M^{(Z_1,Q)} \varepsilon}{\sqrt{n}} \\
&= \frac{\Gamma' M^{(Z_1,Q)} \varepsilon}{\sqrt{n}} + D_\mu^{-1} U' M^{(Z_1,Q)} \varepsilon + O_p\left(\frac{\kappa_n^{\max}}{\mu_n^{\min}} \frac{1}{K_{1,n}^{\varrho_f}}\right)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
D_\mu^{-1} X' M^{(Z_1,Q)} (y - X\delta_0) &= D_\mu^{-1} X' M^{(Z_1,Q)} \varepsilon + D_\mu^{-1} X' M^{(Z_1,Q)} \varphi \\
&= \frac{\Gamma' M^{(Z_1,Q)} \varepsilon}{\sqrt{n}} + D_\mu^{-1} U' M^{(Z_1,Q)} \varepsilon + O_p\left(\frac{\kappa_n^{\max}}{\mu_n^{\min}} \frac{1}{K_{1,n}^{\varrho_f}}\right) + O_p\left(\frac{\tau_n}{K_{1,n}^{\varrho_g}}\right) \\
&= \frac{\Gamma' M^{(Z_1,Q)} \varepsilon}{\sqrt{n}} + D_\mu^{-1} U' M^{(Z_1,Q)} \varepsilon + o_p(1).
\end{aligned}
$$

To show part (c), note that, by the symmetry of A and by making use of parts (a) and (b) of Lemma OA-3, we obtain

$$
\begin{aligned}
\frac{(y - X\delta_0)' A (y - X\delta_0)}{\sqrt{K_{2,n}}} &= \frac{(\varphi + Q\alpha + \varepsilon)' A (\varphi + Q\alpha + \varepsilon)}{\sqrt{K_{2,n}}} \\
&= \frac{(\varphi + \varepsilon)' A (\varphi + \varepsilon)}{\sqrt{K_{2,n}}} \\
&= \frac{\varepsilon' A \varepsilon}{\sqrt{K_{2,n}}} + 2\frac{\varphi' A \varepsilon}{\sqrt{K_{2,n}}} + \frac{\varphi' A \varphi}{\sqrt{K_{2,n}}} \\
&= \frac{\varepsilon' A \varepsilon}{\sqrt{K_{2,n}}} + O_p\left(\frac{\tau_n}{\sqrt{K_{2,n}} K_{1,n}^{\varrho_g}}\right) + O_p\left(\frac{\tau_n^2}{\sqrt{K_{2,n}} K_{1,n}^{2\varrho_g}}\right) \\
&= \frac{\varepsilon' A \varepsilon}{\sqrt{K_{2,n}}} \left[1 + o_p(1)\right],
\end{aligned}
$$

where $\varepsilon' A\varepsilon/\sqrt{K_{2,n}} = O_p(1)$ by part (c) of Lemma OA-3.

Turning our attention to part (d), note that, by the symmetry of $M^{(Z_1,Q)}$ and by making use of parts (c) and (d) of Lemma OA-4, we obtain

$$\begin{aligned}
\frac{(y - X\delta_0)' M^{(Z_1,Q)} (y - X\delta_0)}{n} &= \frac{(\varphi + Q\alpha + \varepsilon)' M^{(Z_1,Q)} (\varphi + Q\alpha + \varepsilon)}{n} \\
&= \frac{(\varphi + \varepsilon)' M^{(Z_1,Q)} (\varphi + \varepsilon)}{\sqrt{K_{2,n}}} \\
&= \frac{\varepsilon' M^{(Z_1,Q)} \varepsilon}{n} + 2\frac{\varphi' M^{(Z_1,Q)} \varepsilon}{n} + \frac{\varphi' M^{(Z_1,Q)} \varphi}{n} \\
&= \frac{\varepsilon' M^{(Z_1,Q)} \varepsilon}{n} + O_p\left(\frac{\tau_n}{nK_{1,n}^{\varrho_g}}\right) + O_p\left(\frac{\tau_n^2}{nK_{1,n}^{2\varrho_g}}\right) \qquad (59)
\end{aligned}$$

Moreover, note that $\varepsilon' M^{(Z_1,Q)}\varepsilon/n = \varepsilon' M^{Q}\varepsilon/\!\!/n - \varepsilon' P^{Z_1^{\perp}}\varepsilon/n$. By the results of part (a) of Lemma OA-5, we have that $\varepsilon' M^{Q}\varepsilon/n - E\left[\varepsilon' M^{Q}\varepsilon/n\right] = O_p\left(n^{-1/2}\right)$, and also that there exist positive constants $\underline{C} \leq \overline{C}$ such that, for all n sufficiently large, $0 < \underline{C} \leq E\left[\varepsilon' M^{Q}\varepsilon/n\right] \leq \overline{C} < \infty$. In addition, by argument similar to that used to obtain expression (58) in the proof of part (b) of Lemma OA-5, we can show that $\varepsilon' P^{Z_1^{\perp}}\varepsilon/n = O_p(K_{1,n}/n) = o_p(1)$. Applying these results to expression (59) above, we deduce that

$$\begin{aligned}
\frac{(y - X\delta_0)' M^{(Z_1,Q)} (y - X\delta_0)}{n} &= \frac{\varepsilon' M^{Q}\varepsilon}{n}\left[1 + O_p\left(\max\left\{\frac{K_{1,n}}{n}, \frac{\tau_n}{nK_{1,n}^{\varrho_g}}\right\}\right)\right] \\
&= \frac{\varepsilon' M^{Q}\varepsilon}{n}\left[1 + O_p\left(\frac{K_{1,n}}{n}\right)\right],
\end{aligned}$$

where $\varepsilon' M^{Q}\varepsilon/n = O_p(1)$ and where $\varepsilon' M^{Q}\varepsilon/n > 0$ w.p.a.1. □

Lemma OA-7: Suppose that Assumptions 1-7 are satisfied. Let $\widehat{\delta}_n$ be any estimator that satisfies the following conditions as $n \to \infty$

(i) If $K_{2,n}/\left(\mu_n^{\min}\right)^2 = O(1)$, then
$$D_\mu\left(\widehat{\delta}_n - \delta_0\right) = O_p(1)$$

(ii) If $\left(\mu_n^{\min}\right)^2/K_{2,n} = o(1)$ but $\sqrt{K_{2,n}}/\left(\mu_n^{\min}\right)^2 \to 0$, then
$$\frac{\left(\mu_n^{\min}\right) D_\mu\left(\widehat{\delta}_n - \delta_0\right)}{\sqrt{K_{2,n}}} = O_p(1)$$

Under these conditions, the following statements are true.

(a) Under case (i),
$$D_\mu^{-1} X' A\left(y - X\widehat{\delta}_n\right) = O_p(1)$$

while, under case (ii),

$$D_{\mu}^{-1}X'A\left(y-X\widehat{\delta}_n\right)=O_p\left(\frac{\sqrt{K_{2,n}}}{(\mu_n^{\min})}\right).$$

(b) Under both case (i) and case (ii),

$$D_{\mu}^{-1}X'M^{(Z_1,Q)}\left(y-X\widehat{\delta}_n\right)=O_p\left(\frac{n}{(\mu_n^{\min})}\right)$$

(c) Under both case (i) and case (ii),

$$\frac{\left(y-X\widehat{\delta}_n\right)'A\left(y-X\widehat{\delta}_n\right)}{\sqrt{K_{2,n}}}=\frac{\varepsilon'A\varepsilon}{\sqrt{K_{2,n}}}+o_p\left(1\right),$$

where $\varepsilon'A\varepsilon/\sqrt{K_{2,n}}=O_p\left(1\right)$.

(d) Under both case (i) and case (ii),

$$\frac{\left(y-X\widehat{\delta}_n\right)'M^{(Z_1,Q)}\left(y-X\widehat{\delta}_n\right)}{n}=\frac{\varepsilon'M^{Q}\varepsilon}{n}\left[1+o_p\left(1\right)\right],$$

where $\varepsilon'M^{Q}\varepsilon/n=O_p\left(1\right)$ and where $\varepsilon'M^{Q}\varepsilon/n>0$ w.p.a.1.

(e) Under both case (i) and case (ii),

$$\widehat{\ell}\left(\widehat{\delta}_n\right)=\frac{\varepsilon'A\varepsilon}{\varepsilon'M^{Q}\varepsilon}+o_p\left(\frac{\sqrt{K_{2,n}}}{n}\right)=O_p\left(\frac{\sqrt{K_{2,n}}}{n}\right),$$

where

$$\widehat{\ell}\left(\widehat{\delta}_n\right)=\frac{\left(y-X\widehat{\delta}_n\right)'A\left(y-X\widehat{\delta}_n\right)}{\left(y-X\widehat{\delta}_n\right)'M^{(Z_1,Q)}\left(y-X\widehat{\delta}_n\right)}.$$

Proof of Lemma OA-7:

To show part (a), write

$$\begin{aligned}D_{\mu}^{-1}X'A\left(y-X\widehat{\delta}_n\right) &= D_{\mu}^{-1}X'A\left(y-X\delta_0\right)+D_{\mu}^{-1}X'AX\left(\widehat{\delta}_n-\delta_0\right)\\ &= D_{\mu}^{-1}X'A\left(y-X\delta_0\right)+D_{\mu}^{-1}X'AXD_{\mu}^{-1}D_{\mu}\left(\widehat{\delta}_n-\delta_0\right)\end{aligned}$$

First, note that, by part (a) of Lemma OA-6, we have

$$\begin{aligned}&D_{\mu}^{-1}X'A\left(y-X\delta_0\right)\\ =\;&\frac{\Gamma'M^{(Z_1,Q)}\varepsilon}{\sqrt{n}}+D_{\mu}^{-1}U'A\varepsilon+O_p\left(\frac{1}{K_{2,n}^{\rho_\gamma}}\right)+O_p\left(\frac{\kappa_n^{\max}}{\mu_n^{\min}}\frac{1}{K_{1,n}^{\varrho_f}}\right)+O_p\left(\frac{\tau_n}{K_{1,n}^{\varrho_g}}\right)\end{aligned}$$

Now, consider case (i) where $\sqrt{K_{2,n}}/\left(\mu_n^{\min}\right) = O(1)$. In this case, we have by part (c) of Lemma OA-2 and expression (8) that

$$\frac{\Gamma' M^{(Z_1,Q)}\varepsilon}{\sqrt{n}} = O_p(1) \text{ and } D_\mu^{-1} U' A \varepsilon = O_p\left(\frac{\sqrt{K_{2,n}}}{\left(\mu_n^{\min}\right)}\right),$$

from which it follows that, in this case, $D_\mu^{-1} X' A (y - X\delta_0) = O_p(1)$. In addition, by part (b) of Lemma S2-2, we have in this case

$$D_\mu^{-1} X' A X D_\mu^{-1} D_\mu \left(\widehat{\delta}_n - \delta_0\right) = O_p(1),$$

so that, in this case,

$$\begin{aligned} D_\mu^{-1} X' A \left(y - X\widehat{\delta}_n\right) &= D_\mu^{-1} X' A (y - X\delta_0) + D_\mu^{-1} X' A X D_\mu^{-1} D_\mu \left(\widehat{\delta}_n - \delta_0\right) \\ &= O_p(1) + O_p(1) = O_p(1) \end{aligned}$$

Next, consider case (ii) where $\left(\mu_n^{\min}\right)/\sqrt{K_{2,n}} = o(1)$ but $\sqrt{K_{2,n}}/\left(\mu_n^{\min}\right)^2 \to 0$. Here, again by part (a) of Lemma OA-6, part (c) of Lemma OA-2, and expression (8), we have

$$D_\mu^{-1} X' A (y - X\delta_0) = O_p\left(\frac{\sqrt{K_{2,n}}}{\left(\mu_n^{\min}\right)}\right)$$

Furthermore, in this case, we also have, by using part (b) of Lemma S2-2

$$\begin{aligned} & D_\mu^{-1} X' A X D_\mu^{-1} D_\mu \left(\widehat{\delta}_n - \delta_0\right) \\ = \ & \frac{\sqrt{K_{2,n}}}{\left(\mu_n^{\min}\right)} D_\mu^{-1} X' A X D_\mu^{-1} \frac{\mu_n^{\min} D_\mu \left(\widehat{\delta}_n - \delta_0\right)}{\sqrt{K_{2,n}}} = O_p\left(\frac{\sqrt{K_{2,n}}}{\left(\mu_n^{\min}\right)}\right) \end{aligned}$$

from which it follows that in case (ii)

$$\begin{aligned} D_\mu^{-1} X' A \left(y - X\widehat{\delta}_n\right) &= D_\mu^{-1} X' A (y - X\delta_0) + D_\mu^{-1} X' A X D_\mu^{-1} D_\mu \left(\widehat{\delta}_n - \delta_0\right) \\ &= O_p\left(\frac{\sqrt{K_{2,n}}}{\left(\mu_n^{\min}\right)}\right) + O_p\left(\frac{\sqrt{K_{2,n}}}{\left(\mu_n^{\min}\right)}\right) \\ &= O_p\left(\frac{\sqrt{K_{2,n}}}{\left(\mu_n^{\min}\right)}\right). \end{aligned}$$

Next, consider part (b). In this case, note that

$$
\begin{aligned}
& D_\mu^{-1} X' M^{(Z_1,Q)} \left(y - X\widehat{\delta}_n\right) \\
= \; & D_\mu^{-1} X' M^{(Z_1,Q)} \left(y - X\delta_0\right) + D_\mu^{-1} X' M^{(Z_1,Q)} X \left(\widehat{\delta}_n - \delta_0\right) \\
= \; & D_\mu^{-1} X' M^{(Z_1,Q)} \left(y - X\delta_0\right) + D_\mu^{-1} X' M^{(Z_1,Q)} X D_\mu^{-1} D_\mu \left(\widehat{\delta}_n - \delta_0\right)
\end{aligned}
$$

Note that, by part (b) of Lemma OA-6, we have

$$
\begin{aligned}
& D_\mu^{-1} X' M^{(Z_1,Q)} \left(y - X\delta_0\right) \\
= \; & \frac{\Gamma' M^{(Z_1,Q)} \varepsilon}{\sqrt{n}} + D_\mu^{-1} U' M^{(Z_1,Q)} \varepsilon + O_p\left(\frac{\kappa_n^{\max}}{\mu_n^{\min}} \frac{1}{K_{1,n}^{\varrho_f}}\right) + O_p\left(\frac{\tau_n}{K_{1,n}^{\varrho_g}}\right).
\end{aligned}
$$

Now, consider case (i) where $\sqrt{K_{2,n}} / \left(\mu_n^{\min}\right) = O(1)$. In this case, we have by part (c) of Lemma OA-2 and expression (9) that

$$
\frac{\Gamma' M^{(Z_1,Q)} \varepsilon}{\sqrt{n}} = O_p(1) \text{ and } D_\mu^{-1} U' M^{(Z_1,Q)} \varepsilon = O_p\left(\frac{n}{\left(\mu_n^{\min}\right)}\right),
$$

from which it follows that, in this case,

$$
D_\mu^{-1} X' M^{(Z_1,Q)} \left(y - X\delta_0\right) = O_p\left(\frac{n}{\left(\mu_n^{\min}\right)}\right).
$$

In addition, by part (a) of Lemma S2-2, we have in this case

$$
D_\mu^{-1} X' M^{(Z_1,Q)} X D_\mu^{-1} D_\mu \left(\widehat{\delta}_n - \delta_0\right) = O_p\left(\frac{n}{\left(\mu_n^{\min}\right)^2}\right),
$$

so that, in this case,

$$
\begin{aligned}
& D_\mu^{-1} X' M^{(Z_1,Q)} \left(y - X\widehat{\delta}_n\right) \\
= \; & D_\mu^{-1} X' M^{(Z_1,Q)} \left(y - X\delta_0\right) + D_\mu^{-1} X' M^{(Z_1,Q)} X D_\mu^{-1} D_\mu \left(\widehat{\delta}_n - \delta_0\right) \\
= \; & O_p\left(\frac{n}{\left(\mu_n^{\min}\right)}\right) + O_p\left(\frac{n}{\left(\mu_n^{\min}\right)^2}\right) = O_p\left(\frac{n}{\left(\mu_n^{\min}\right)}\right)
\end{aligned}
$$

Next, consider case (ii) where $\left(\mu_n^{\min}\right) / \sqrt{K_{2,n}} = o(1)$ but $\sqrt{K_{2,n}} / \left(\mu_n^{\min}\right)^2 \to 0$. Here, again by part (c) of Lemma OA-2 and expression (9), we have

$$
D_\mu^{-1} X' M^{(Z_1,Q)} \left(y - X\delta_0\right) = O_p\left(\frac{n}{\left(\mu_n^{\min}\right)}\right)
$$

Furthermore, using part (a) of Lemma S2-2, we have

$$\begin{aligned}
D_\mu^{-1}X'M^{(Z_1,Q)}XD_\mu^{-1}D_\mu\left(\widehat{\delta}_n-\delta_0\right) &= \frac{\sqrt{K_{2,n}}}{\left(\mu_n^{\min}\right)}D_\mu^{-1}X'M^{(Z_1,Q)}XD_\mu^{-1}\frac{\left(\mu_n^{\min}\right)D_\mu\left(\widehat{\delta}_n-\delta_0\right)}{\sqrt{K_{2,n}}} \\
&= O_p\left(\frac{\sqrt{K_{2,n}}n}{\left(\mu_n^{\min}\right)^3}\right)
\end{aligned}$$

from which it follows that in case (ii)

$$\begin{aligned}
& D_\mu^{-1}X'M^{(Z_1,Q)}\left(y-X\widehat{\delta}_n\right) \\
=\; & D_\mu^{-1}X'M^{(Z_1,Q)}\left(y-X\delta_0\right)+D_\mu^{-1}X'M^{(Z_1,Q)}XD_\mu^{-1}D_\mu\left(\widehat{\delta}_n-\delta_0\right) \\
=\; & O_p\left(\frac{n}{\left(\mu_n^{\min}\right)}\right)+O_p\left(\frac{\sqrt{K_{2,n}}n}{\left(\mu_n^{\min}\right)^3}\right)=O_p\left(\frac{n}{\left(\mu_n^{\min}\right)}\right)
\end{aligned}$$

Combining the results for cases (i) and (ii), we have

$$D_\mu^{-1}X'M^{(Z_1,Q)}\left(y-X\widehat{\delta}_n\right)=O_p\left(\frac{n}{\left(\mu_n^{\min}\right)}\right)$$

Consider now part (c). Write

$$\begin{aligned}
& \frac{\left(y-X\widehat{\delta}_n\right)'A\left(y-X\widehat{\delta}_n\right)}{\sqrt{K_{2,n}}} \\
=\; & \frac{\left[y-X\delta_0-X\left(\widehat{\delta}_n-\delta_0\right)\right]'A\left[y-X\delta_0-X\left(\widehat{\delta}_n-\delta_0\right)\right]}{\sqrt{K_{2,n}}} \\
=\; & \frac{\left(y-X\delta_0\right)'A\left(y-X\delta_0\right)}{\sqrt{K_{2,n}}}-2\frac{\left(y-X\delta_0\right)'AX\left(\widehat{\delta}_n-\delta_0\right)}{\sqrt{K_{2,n}}} \\
& +\frac{\left(\widehat{\delta}_n-\delta_0\right)'X'AX\left(\widehat{\delta}_n-\delta_0\right)}{\sqrt{K_{2,n}}} \\
=\; & \frac{\left(y-X\delta_0\right)'A\left(y-X\delta_0\right)}{\sqrt{K_{2,n}}}-2\left(y-X\delta_0\right)'AXD_\mu^{-1}\frac{D_\mu\left(\widehat{\delta}_n-\delta_0\right)}{\sqrt{K_{2,n}}} \\
& +\left(\widehat{\delta}_n-\delta_0\right)'D_\mu\frac{D_\mu^{-1}X'AXD_\mu^{-1}}{\sqrt{K_{2,n}}}D_\mu\left(\widehat{\delta}_n-\delta_0\right)
\end{aligned}$$

Note first that, by part (c) of Lemma OA-6, we have

$$\frac{\left(y-X\delta_0\right)'A\left(y-X\delta_0\right)}{\sqrt{K_{2,n}}}=\frac{\varepsilon'A\varepsilon}{\sqrt{K_{2,n}}}+O_p\left(\frac{\tau_n}{\sqrt{K_{2,n}}K_{1,n}^{\varrho_g}}\right).$$

Moreover, by part (a) of Lemma OA-6, we have

$$\begin{aligned}
& (y - X\delta_0)' AXD_\mu^{-1} \\
= \ & \frac{\varepsilon' M^{(Z_1,Q)}\Gamma}{\sqrt{n}} + \varepsilon' AUD_\mu^{-1} + O_p\left(\frac{1}{K_{2,n}^{\varrho_\gamma}}\right) + O_p\left(\frac{\kappa_n^{\max}}{\mu_n^{\min}} \frac{1}{K_{1,n}^{\varrho_f}}\right) + O_p\left(\frac{\tau_n}{K_{1,n}^{\varrho_g}}\right)
\end{aligned}$$

Now, consider case (i) where $\sqrt{K_{2,n}}/\left(\mu_n^{\min}\right) = O(1)$. In this case, we have by part (c) of Lemma OA-2 and expression (8) that

$$\frac{\varepsilon' M^{(Z_1,Q)}\Gamma}{\sqrt{n}} = O_p(1) \text{ and } \varepsilon' AUD_\mu^{-1} = O_p\left(\frac{\sqrt{K_{2,n}}}{(\mu_n^{\min})}\right),$$

so that $(y - X\delta_0)' AXD_\mu^{-1} = O_p(1)$ and, thus,

$$(y - X\delta_0)' AXD_\mu^{-1} \frac{D_\mu\left(\widehat{\delta}_n - \delta_0\right)}{\sqrt{K_{2,n}}} = O_p\left(\frac{1}{\sqrt{K_{2,n}}}\right) = o_p(1)$$

In addition, by part (b) of Lemma S2-2, we have in this case

$$\begin{aligned}
& \left(\widehat{\delta}_n - \delta_0\right)' D_\mu \frac{D_\mu^{-1} X' AXD_\mu^{-1}}{\sqrt{K_{2,n}}} D_\mu \left(\widehat{\delta}_n - \delta_0\right) \\
= \ & \frac{1}{\sqrt{K_{2,n}}} \left(\widehat{\delta}_n - \delta_0\right)' D_\mu D_\mu^{-1} X' AXD_\mu^{-1} D_\mu \left(\widehat{\delta}_n - \delta_0\right) \\
= \ & O_p\left(\frac{1}{\sqrt{K_{2,n}}}\right) = o_p(1),
\end{aligned}$$

from which it follows that

$$\begin{aligned}
& \frac{\left(y - X\widehat{\delta}_n\right)' A \left(y - X\widehat{\delta}_n\right)}{\sqrt{K_{2,n}}} \\
= \ & \frac{\varepsilon' A\varepsilon}{\sqrt{K_{2,n}}} + O_p\left(\frac{\tau_n}{\sqrt{K_{2,n}} K_{1,n}^{\varrho_g}}\right) + O_p\left(\frac{1}{\sqrt{K_{2,n}}}\right) + O_p\left(\frac{1}{\sqrt{K_{2,n}}}\right) \\
= \ & \frac{\varepsilon' A\varepsilon}{\sqrt{K_{2,n}}} + o_p(1).
\end{aligned}$$

Next, consider case (ii) where $\left(\mu_n^{\min}\right)/\sqrt{K_{2,n}} = o(1)$ but $\sqrt{K_{2,n}}/\left(\mu_n^{\min}\right)^2 \to 0$. Here, again by part (c) of Lemma OA-6, part (c) of Lemma OA-2, and expression (8), we have

$$(y - X\delta_0)' AXD_\mu^{-1} = O_p\left(\frac{\sqrt{K_{2,n}}}{(\mu_n^{\min})}\right)$$

so that

$$
\begin{aligned}
& (y - X\delta_0)' AXD_\mu^{-1} \frac{D_\mu \left(\widehat{\delta}_n - \delta_0\right)}{\sqrt{K_{2,n}}} \\
= \;& \frac{1}{(\mu_n^{\min})} (y - X\delta_0)' AXD_\mu^{-1} \frac{(\mu_n^{\min}) D_\mu \left(\widehat{\delta}_n - \delta_0\right)}{\sqrt{K_{2,n}}} \\
= \;& O_p\left(\frac{1}{(\mu_n^{\min})}\right) O_p\left(\frac{\sqrt{K_{2,n}}}{(\mu_n^{\min})}\right) O_p(1) = O_p\left(\frac{\sqrt{K_{2,n}}}{(\mu_n^{\min})^2}\right) = o_p(1).
\end{aligned}
$$

Furthermore, in this case,

$$
\begin{aligned}
& \left(\widehat{\delta}_n - \delta_0\right)' D_\mu \frac{D_\mu^{-1} X' AXD_\mu^{-1}}{\sqrt{K_{2,n}}} D_\mu \left(\widehat{\delta}_n - \delta_0\right) \\
= \;& \frac{\sqrt{K_{2,n}}}{(\mu_n^{\min})^2} \left[\frac{(\mu_n^{\min}) \left(\widehat{\delta}_n - \delta_0\right)' D_\mu}{\sqrt{K_{2,n}}} D_\mu^{-1} X' AXD_\mu^{-1} \frac{(\mu_n^{\min}) D_\mu \left(\widehat{\delta}_n - \delta_0\right)}{\sqrt{K_{2,n}}} \right] \\
= \;& O_p\left(\frac{\sqrt{K_{2,n}}}{(\mu_n^{\min})^2}\right) = o_p(1)
\end{aligned}
$$

from which it follows that in case (ii)

$$
\begin{aligned}
& \frac{\left(y - X\widehat{\delta}_n\right)' A \left(y - X\widehat{\delta}_n\right)}{\sqrt{K_{2,n}}} \\
= \;& \frac{\varepsilon' A\varepsilon}{\sqrt{K_{2,n}}} + O_p\left(\frac{\tau_n}{\sqrt{K_{2,n}} K_{1,n}^{\varrho_g}}\right) + O_p\left(\frac{\sqrt{K_{2,n}}}{(\mu_n^{\min})^2}\right) + O_p\left(\frac{\sqrt{K_{2,n}}}{(\mu_n^{\min})^2}\right) \\
= \;& \frac{\varepsilon' A\varepsilon}{\sqrt{K_{2,n}}} + o_p(1).
\end{aligned}
$$

Combining the results for cases (i) and (ii), we have

$$
\frac{\left(y - X\widehat{\delta}_n\right)' A \left(y - X\widehat{\delta}_n\right)}{\sqrt{K_{2,n}}} = \frac{\varepsilon' A\varepsilon}{\sqrt{K_{2,n}}} + o_p(1).
$$

where $\varepsilon' A\varepsilon / \sqrt{K_{2,n}} = O_p(1)$ by part (c) of Lemma OA-3.

Turning our attention to part (d), note that

$$\begin{aligned}
& \frac{\left(y-X\widehat{\delta}_n\right)' M^{(Z_1,Q)} \left(y-X\widehat{\delta}_n\right)}{n} \\
= & \frac{\left[y-X\delta_0-X\left(\widehat{\delta}_n-\delta_0\right)\right]' M^{(Z_1,Q)} \left[y-X\delta_0-X\left(\widehat{\delta}_n-\delta_0\right)\right]}{n} \\
= & \frac{\left(y-X\delta_0\right)' M^{(Z_1,Q)} \left(y-X\delta_0\right)}{n} - 2\frac{\left(y-X\delta_0\right)' M^{(Z_1,Q)} X\left(\widehat{\delta}_n-\delta_0\right)}{n} \\
& + \frac{\left(\widehat{\delta}_n-\delta_0\right)' X' M^{(Z_1,Q)} X\left(\widehat{\delta}_n-\delta_0\right)}{n} \\
= & \frac{\left(y-X\delta_0\right)' M^{(Z_1,Q)} \left(y-X\delta_0\right)}{n} - 2\frac{\left(y-X\delta_0\right)' M^{(Z_1,Q)} XD_\mu^{-1}}{n} D_\mu \left(\widehat{\delta}_n-\delta_0\right) \\
& + \left(\widehat{\delta}_n-\delta_0\right)' D_\mu \frac{D_\mu^{-1} X' M^{(Z_1,Q)} XD_\mu^{-1}}{n} D_\mu \left(\widehat{\delta}_n-\delta_0\right).
\end{aligned}$$

Note first that, by part (d) of Lemma OA-6

$$\frac{\left(y-X\delta_0\right)' M^{(Z_1,Q)} \left(y-X\delta_0\right)}{n} = \frac{\varepsilon' M^Q \varepsilon}{n} + O_p\left(\frac{K_{1,n}}{n}\right),$$

Moreover, by part (b) of Lemma OA-6, we have

$$\begin{aligned}
& \left(y-X\delta_0\right)' M^{(Z_1,Q)} XD_\mu^{-1} \\
= & \frac{\varepsilon' M^{(Z_1,Q)} \Gamma}{\sqrt{n}} + \varepsilon' M^{(Z_1,Q)} UD_\mu^{-1} + O_p\left(\frac{\kappa_n^{\max}}{\mu_n^{\min}} \frac{1}{K_{1,n}^{\varrho_f}}\right) + O_p\left(\frac{\tau_n}{K_{1,n}^{\varrho_g}}\right)
\end{aligned}$$

Now, consider case (i) where $\sqrt{K_{2,n}}/\left(\mu_n^{\min}\right) = O(1)$. In this case, we have by part (c) of Lemma OA-2 and expression (9) that

$$\frac{\varepsilon' M^{(Z_1,Q)} \Gamma}{\sqrt{n}} = O_p(1) \text{ and } \varepsilon' M^{(Z_1,Q)} UD_\mu^{-1} = O_p\left(\frac{n}{\left(\mu_n^{\min}\right)}\right),$$

so that $\left(y-X\delta_0\right)' M^{(Z_1,Q)} XD_\mu^{-1} = O_p\left(n/\left(\mu_n^{\min}\right)\right)$ and, thus,

$$\begin{aligned}
\frac{\left(y-X\delta_0\right)' M^{(Z_1,Q)} XD_\mu^{-1}}{n} D_\mu \left(\widehat{\delta}_n-\delta_0\right) &= O_p\left(\frac{1}{\left(\mu_n^{\min}\right)}\right) O_p(1) \\
&= O_p\left(\frac{1}{\left(\mu_n^{\min}\right)}\right)
\end{aligned}$$

In addition, by part (a) of Lemma S2-2, we have in this case

$$\begin{aligned}
& \left(\widehat{\delta}_n-\delta_0\right)' D_\mu \frac{D_\mu^{-1} X' M^{(Z_1,Q)} X D_\mu^{-1}}{n} D_\mu \left(\widehat{\delta}_n-\delta_0\right) \\
= \; & O_p(1)\, O_p\left(\frac{1}{\left(\mu_n^{\min}\right)^2}\right) O_p(1) = O_p\left(\frac{1}{\left(\mu_n^{\min}\right)^2}\right) = o_p(1),
\end{aligned}$$

from which it follows that

$$\begin{aligned}
& \frac{\left(y - X\widehat{\delta}_n\right)' M^{(Z_1,Q)} \left(y - X\widehat{\delta}_n\right)}{n} \\
= \; & \frac{\varepsilon' M^Q \varepsilon}{n} + O_p\left(\frac{K_{1,n}}{n}\right) + O_p\left(\frac{1}{\left(\mu_n^{\min}\right)}\right) + O_p\left(\frac{1}{\left(\mu_n^{\min}\right)^2}\right) \\
= \; & \frac{\varepsilon' M^Q \varepsilon}{n} + o_p(1).
\end{aligned}$$

Next, consider case (ii) where $\left(\mu_n^{\min}\right)/\sqrt{K_{2,n}} = o(1)$ but $\sqrt{K_{2,n}}/\left(\mu_n^{\min}\right)^2 \to 0$. Here, again by part (b) of Lemma OA-6, part (c) of Lemma OA-2, and expression (9); we have

$$\frac{\left(y - X\delta_0\right)' M^{(Z_1,Q)} X D_\mu^{-1}}{n} = O_p\left(\frac{1}{\left(\mu_n^{\min}\right)}\right) = o_p(1)$$

so that

$$\begin{aligned}
& \frac{\left(y - X\delta_0\right)' M^{(Z_1,Q)} X D_\mu^{-1}}{n} D_\mu \left(\widehat{\delta}_n-\delta_0\right) \\
= \; & \frac{\sqrt{K_{2,n}}}{\left(\mu_n^{\min}\right)} \frac{\left(y - X\delta_0\right)' M^{(Z_1,Q)} X D_\mu^{-1}}{n} \frac{\left(\mu_n^{\min}\right) D_\mu \left(\widehat{\delta}_n-\delta_0\right)}{\sqrt{K_{2,n}}} \\
= \; & O_p\left(\frac{\sqrt{K_{2,n}}}{\left(\mu_n^{\min}\right)}\right) O_p\left(\frac{1}{\left(\mu_n^{\min}\right)}\right) O_p(1) = O_p\left(\frac{\sqrt{K_{2,n}}}{\left(\mu_n^{\min}\right)^2}\right) = o_p(1).
\end{aligned}$$

Furthermore, in this case,

$$\begin{aligned}
& \left(\widehat{\delta}_n-\delta_0\right)' D_\mu \frac{D_\mu^{-1} X' M^{(Z_1,Q)} X D_\mu^{-1}}{n} D_\mu \left(\widehat{\delta}_n-\delta_0\right) \\
= \; & \frac{K_{2,n}}{\left(\mu_n^{\min}\right)^2} \frac{\left(\mu_n^{\min}\right)\left(\widehat{\delta}_n-\delta_0\right)' D_\mu}{\sqrt{K_{2,n}}} \frac{D_\mu^{-1} X' M^{(Z_1,Q)} X D_\mu^{-1}}{n} \frac{\left(\mu_n^{\min}\right) D_\mu \left(\widehat{\delta}_n-\delta_0\right)}{\sqrt{K_{2,n}}} \\
= \; & O_p\left(\frac{K_{2,n}}{\left(\mu_n^{\min}\right)^2}\right) O_p(1)\, O_p\left(\frac{1}{\left(\mu_n^{\min}\right)^2}\right) O_p(1) = O_p\left(\frac{K_{2,n}}{\left(\mu_n^{\min}\right)^4}\right) = o_p(1)
\end{aligned}$$

from which it follows that in case (ii)

$$\begin{aligned}\frac{\left(y-X\widehat{\delta}_n\right)' M^{(Z_1,Q)}\left(y-X\widehat{\delta}_n\right)}{n} &= \frac{\varepsilon' M^Q \varepsilon}{n} + O_p\left(\frac{K_{1,n}}{n}\right) + O_p\left(\frac{\sqrt{K_{2,n}}}{(\mu_n^{\min})^2}\right) + O_p\left(\frac{K_{2,n}}{(\mu_n^{\min})^4}\right) \\ &= \frac{\varepsilon' M^Q \varepsilon}{n} + o_p(1)\end{aligned}$$

Combining the results for cases (i) and (ii), we have

$$\frac{\left(y-X\widehat{\delta}_n\right)' M^{(Z_1,Q)}\left(y-X\widehat{\delta}_n\right)}{n} = \frac{\varepsilon' M^Q \varepsilon}{n} + o_p(1).$$

where $\varepsilon' M^Q \varepsilon/n = O_p(1)$ and $\varepsilon' M^Q \varepsilon/n > 0$ w.p.a.1. were both shown in part (d) of Lemma OA-6.

Finally, to show part (e), we apply the results of parts (c) and (d) above to obtain

$$\begin{aligned}&\widehat{\ell}\left(\widehat{\delta}_n\right) \\ =\ & \frac{\left(y-X\widehat{\delta}_n\right)' A\left(y-X\widehat{\delta}_n\right)}{\left(y-X\widehat{\delta}_n\right)' M^{(Z_1,Q)}\left(y-X\widehat{\delta}_n\right)} \\ =\ & \frac{\sqrt{K_{2,n}}}{n}\frac{\left(y-X\widehat{\delta}_n\right)' A\left(y-X\widehat{\delta}_n\right)/\sqrt{K_{2,n}}}{\left(y-X\widehat{\delta}_n\right)' M^{(Z_1,Q)}\left(y-X\widehat{\delta}_n\right)/n} \\ =\ & \frac{\sqrt{K_{2,n}}}{n}\frac{\varepsilon' A\varepsilon/\sqrt{K_{2,n}} + o_p(1)}{\varepsilon' M^Q\varepsilon/n + o_p(1)} \\ =\ & \frac{\sqrt{K_{2,n}}}{n}\left[\frac{\varepsilon' A\varepsilon/\sqrt{K_{2,n}}}{\varepsilon' M^Q\varepsilon/n} + o_p(1)\right] \quad \left(\text{since } \frac{\varepsilon' M^Q\varepsilon}{n} > 0 \text{ w.p.a.1}\right) \\ =\ & \frac{\varepsilon' A\varepsilon}{\varepsilon' M^Q\varepsilon} + o_p\left(\frac{\sqrt{K_{2,n}}}{n}\right) = O_p\left(\frac{\sqrt{K_{2,n}}}{n}\right)\end{aligned}$$

given that $\varepsilon' A\varepsilon/\sqrt{K_{2,n}} = O_p(1)$ and $\varepsilon' M^Q\varepsilon/n > 0$ w.p.a.1. □

Lemma OA-8: (Decoupling Inequality) For natural number $n \geq G$, let $\{X_i\}_{i=1}^n$ be n independent random variables taking on values in a measurable space $(S, \mathfrak{S})$, and let $\left\{X_i^{(k)}\right\}_{i=1}^n$ $k = 1, ..., G$ be G independent copies of this sequence. Let B be a separable Banach space and for each $(i_1, .., i_{g_*}) \in I_n^{g_*}$, where

$$I_n^{g_*} = \{(i_1, .., i_{g_*}) : i_j \in \mathbb{N},\ 1 \leq i_j \leq n,\ i_j \neq i_k \text{ if } j \neq k\},$$

let $h_{i_1 \ldots i_G} : S^G \longmapsto B$ be a measurable functions such that $E\left(\left\|h_{i_1 \ldots i_G}\left(X_{i_1}, ..., X_{i_G}\right)\right\|\right) < \infty$. Let $\Phi : [0, \infty) \longrightarrow [0, \infty)$ be a convex nondecreasing function such that $E\Phi\left(\left\|h_{i_1 \ldots i_G}\left(X_{i_1}, ..., X_{i_G}\right)\right\|\right) <$

∞ for all $(i_1,..,i_G) \in I_n^G$. Then,

$$E\Phi\left(\left\|\sum_{I_n^G} h_{i_1...i_G}\left(X_{i_1},...,X_{i_G}\right)\right\|\right) \leq E\Phi\left(C_G\left\|\sum_{I_n^G} h_{i_1...i_G}\left(X_{i_1},...,X_{i_G}\right)\right\|\right) \tag{60}$$

where $C_G = 2^G\left[G^G - 1\right]\left[(G-1)^{(G-1)} - 1\right] \times \cdots \times 3$.

Lemma OA-8 gives the inequality result stated in the first half of Theorem 3.1.1 of de la Peña and Giné (1999). Theorem 3.1.1 also gives a reverse inequality under some additional symmetry conditions on the kernel $h_{i_1...i_G}$ which we will not give here, since we will not be using the reverse inequality any of the results stated below. Proof of a more general decoupling inequality which contains the inequality given in expression (60) as a special case is provided in de la Peña (1992). See Theorem 2 of de la Peña (1992).

Lemma OA-9:

Let $D(\varepsilon \circ \varepsilon) = diag\left(\varepsilon_{(1,1)}^2, ..., \varepsilon_{(n,T_n)}^2\right)$. Under Assumptions 2, 5, 6, and 7; the following statements hold.

(a)
$$\frac{\Gamma' P^{\perp} D(\varepsilon \circ \varepsilon) P^{\perp}\Gamma}{n} = O_p(1)$$

(b)
$$\frac{\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D(\varepsilon \circ \varepsilon) M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\Gamma}{n} = O_p\left(\frac{K_{2,n}^2}{n^2}\right) = o_p(1)$$

(c)
$$\frac{\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D(\varepsilon \circ \varepsilon) P^{\perp}\Gamma}{n} = O_p\left(\frac{K_{2,n}}{n}\right) = o_p(1)$$

(d)
$$\frac{\Gamma' P^{\perp} D(\varepsilon \circ \varepsilon) M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\Gamma}{n} = O_p\left(\frac{K_{2,n}}{n}\right) = o_p(1)$$

(e)
$$\frac{\Gamma' P^{\perp} D(\varepsilon \circ \varepsilon) P^{\perp}\Gamma}{n} - \frac{\Gamma' P^{\perp} D_{\sigma^2} P^{\perp}\Gamma}{n} = O_p\left(\sqrt{\frac{K_{2,n}}{n}}\right),$$

where $D_{\sigma^2} = diag\left(E\left[\varepsilon_{(1,1)}^2|\mathcal{F}_n^W\right], ..., E\left[\varepsilon_{(n,T_n)}^2|\mathcal{F}_n^W\right]\right)$.

(f)
$$\frac{\Gamma' A D(\varepsilon \circ \varepsilon) A\Gamma}{n} - \frac{\Gamma' M^{(Z_1,Q)} D_{\sigma^2} M^{(Z_1,Q)}\Gamma}{n} = O_p\left(\max\left\{\frac{1}{K_{2,n}^{\varrho_\gamma}}, \sqrt{\frac{K_{2,n}}{n}}\right\}\right) = o_p(1).$$

Proof of Lemma OA-9:

To show part (a), let $a, b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$; and, by the CS inequality, we obtain

$$\left|\frac{a'\Gamma' P^{\perp} D(\varepsilon\circ\varepsilon) P^{\perp}\Gamma b}{n}\right| \leq \sqrt{\frac{a'\Gamma' P^{\perp} D(\varepsilon\circ\varepsilon) P^{\perp}\Gamma a}{n}}\sqrt{\frac{b'\Gamma' P^{\perp} D(\varepsilon\circ\varepsilon) P^{\perp}\Gamma b}{n}}.$$

Next, note that, since $a'\Gamma' P^{\perp} D(\varepsilon\circ\varepsilon) P^{\perp}\Gamma a/n \geq 0$, we have that

$$\begin{aligned}
E\left[\left|\frac{a'\Gamma' P^{\perp} D(\varepsilon\circ\varepsilon) P^{\perp}\Gamma a}{n}\right| |\mathcal{F}_n^W\right] &= \frac{1}{n}E\left[a'\Gamma' P^{\perp} D(\varepsilon\circ\varepsilon) P^{\perp}\Gamma a|\mathcal{F}_n^W\right] \\
&= \frac{1}{n}a'\Gamma' P^{\perp} D\left(E\left[\varepsilon\circ\varepsilon|\mathcal{F}_n^W\right]\right) P^{\perp}\Gamma a \\
&\leq \left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right]\right)\frac{a'\Gamma' P^{\perp}\Gamma a}{n} \\
&\leq \left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right]\right)\frac{a'\Gamma'\Gamma a}{n} \\
&= O_{a.s.}(1)
\end{aligned}$$

given Assumptions 2(i) and 3(iii). Hence, there exists a constant $\overline{C} < \infty$ such that for all n sufficiently large

$$\begin{aligned}
E\left[\left|\frac{a'\Gamma' P^{\perp} D(\varepsilon\circ\varepsilon) P^{\perp}\Gamma a}{n}\right|\right] &= E\left[\frac{a'\Gamma' P^{\perp} D(\varepsilon\circ\varepsilon) P^{\perp}\Gamma a}{n}\right] \\
&= E_{W_n}\left(\frac{1}{n}E\left[a'\Gamma' P^{\perp} D(\varepsilon\circ\varepsilon) P^{\perp}\Gamma a|\mathcal{F}_n^W\right]\right) \leq \overline{C}.
\end{aligned}$$

It follows from the Markov's inequality that for any $\epsilon > 0$,

$$\begin{aligned}
\Pr\left(\left|\frac{a'\Gamma' P^{\perp} D(\varepsilon\circ\varepsilon) P^{\perp}\Gamma a}{n}\right| \geq \frac{\overline{C}}{\epsilon}\right) &\leq \frac{\epsilon}{\overline{C}}E\left[\left|\frac{a'\Gamma' P^{\perp} D(\varepsilon\circ\varepsilon) P^{\perp}\Gamma a}{n}\right|\right] \\
&\leq \epsilon
\end{aligned}$$

for all n sufficiently large, which shows that

$$\frac{a'\Gamma' P^{\perp} D(\varepsilon\circ\varepsilon) P^{\perp}\Gamma a}{n} = O_p(1).$$

In the same way, we can also show that

$$\frac{b'\Gamma' P^{\perp} D(\varepsilon\circ\varepsilon) P^{\perp}\Gamma b}{n} = O_p(1),$$

from which it follows immediately that

$$\begin{aligned}
\left|\frac{a'\Gamma' P^{\perp} D(\varepsilon\circ\varepsilon) P^{\perp}\Gamma b}{n}\right| &\leq \sqrt{\frac{a'\Gamma' P^{\perp} D(\varepsilon\circ\varepsilon) P^{\perp}\Gamma a}{n}}\sqrt{\frac{b'\Gamma' P^{\perp} D(\varepsilon\circ\varepsilon) P^{\perp}\Gamma b}{n}} \\
&= O_p(1).
\end{aligned}$$

Since the above argument holds for all $a, b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$, we further deduce that

$$\frac{\Gamma' P^{\perp} D\left(\varepsilon \circ \varepsilon\right) P^{\perp} \Gamma}{n} = O_p\left(1\right).$$

For part (b), again let $a, b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$; and, by the CS inequality, we obtain

$$\begin{aligned}
& \left| \frac{a'\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D\left(\varepsilon \circ \varepsilon\right) M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} \Gamma b}{n} \right| \\
\leq & \sqrt{\frac{a'\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D\left(\varepsilon \circ \varepsilon\right) M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} \Gamma a}{n}} \\
& \times \sqrt{\frac{b'\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D\left(\varepsilon \circ \varepsilon\right) M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} \Gamma b}{n}}.
\end{aligned}$$

Next, note that, since $a'\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D\left(\varepsilon \circ \varepsilon\right) M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} \Gamma a / n \geq 0$, we have, by Assumptions 2(i) and 3(iii) as well as part (b) of Lemma OA-1,

$$\begin{aligned}
& E\left[\left| \frac{a'\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D\left(\varepsilon \circ \varepsilon\right) M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} \Gamma a}{n} \right| |\mathcal{F}_n^W\right] \\
= & \frac{1}{n} E\left[a'\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D\left(\varepsilon \circ \varepsilon\right) M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} \Gamma a |\mathcal{F}_n^W\right] \\
= & \frac{1}{n} a'\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D\left(E\left[\varepsilon \circ \varepsilon |\mathcal{F}_n^W\right]\right) M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} \Gamma a \\
\leq & \left(\max_{1 \leq (i,t) \leq m_n} E\left[\varepsilon_{(i,t)}^2 |\mathcal{F}_n^W\right]\right) \left(\max_{1 \leq (i,t) \leq m_n} \left|\widehat{\vartheta}_{(i,t)}\right|^2\right) \frac{a'\Gamma'\Gamma a}{n} \\
= & O_{a.s.}\left(\frac{K_{2,n}^2}{n^2}\right) = o_{a.s.}\left(1\right).
\end{aligned}$$

Hence, there exists a constant $\overline{C} < \infty$ such that for all n sufficiently large

$$\begin{aligned}
& E\left[\left| \frac{n^2}{K_{2,n}^2} \frac{a'\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D\left(\varepsilon \circ \varepsilon\right) M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} \Gamma a}{n} \right|\right] \\
= & E\left[\frac{n^2}{K_{2,n}^2} \frac{a'\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D\left(\varepsilon \circ \varepsilon\right) M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} \Gamma a}{n}\right] \\
= & E_{W_n}\left(\frac{n^2}{K_{2,n}^2} E\left[\frac{a'\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D\left(\varepsilon \circ \varepsilon\right) M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} \Gamma a}{n} |\mathcal{F}_n^W\right]\right) \leq \overline{C}.
\end{aligned}$$

It follows from the Markov's inequality that for any $\epsilon > 0$,

$$\begin{aligned}
&\Pr\left(\left|\frac{n^2}{K_{2,n}^2}\frac{a'\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D\left(\varepsilon\circ\varepsilon\right) M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\Gamma a}{n}\right| \geq \frac{\overline{C}}{\epsilon}\right) \\
\leq\ & \frac{\epsilon}{\overline{C}} E\left[\left|\frac{n^2}{K_{2,n}^2}\frac{a'\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D\left(\varepsilon\circ\varepsilon\right) M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\Gamma a}{n}\right|\right] \leq \epsilon
\end{aligned}$$

for all n sufficiently large, which shows that

$$\frac{a'\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D\left(\varepsilon\circ\varepsilon\right) M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\Gamma a}{n} = O_p\left(\frac{K_{2,n}^2}{n^2}\right).$$

In the same way, we can also show that

$$\frac{b'\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D\left(\varepsilon\circ\varepsilon\right) M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\Gamma b}{n} = O_p\left(\frac{K_{2,n}^2}{n^2}\right),$$

from which it follows immediately that

$$\begin{aligned}
&\left|\frac{a'\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D\left(\varepsilon\circ\varepsilon\right) M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\Gamma b}{n}\right| \\
\leq\ & \sqrt{\frac{a'\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D\left(\varepsilon\circ\varepsilon\right) M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\Gamma a}{n}} \\
& \times\sqrt{\frac{b'\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D\left(\varepsilon\circ\varepsilon\right) M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\Gamma b}{n}} \\
=\ & O_p\left(\frac{K_{2,n}^2}{n^2}\right).
\end{aligned}$$

Since the above argument holds for all $a, b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$, we further deduce that

$$\frac{\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D\left(\varepsilon\circ\varepsilon\right) M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\Gamma}{n} = O_p\left(\frac{K_{2,n}^2}{n^2}\right) = o_p\left(1\right).$$

To show part (c), let $a, b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$; and, making use of the CS inequality as well as the results of parts (a) and (b) of this lemma, we obtain

$$
\begin{aligned}
& \left|\frac{a'\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D\left(\varepsilon \circ \varepsilon\right) P^{\perp}\Gamma b}{n}\right| \\
\leq & \sqrt{\frac{a'\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D\left(\varepsilon \circ \varepsilon\right) M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} \Gamma a}{n}} \sqrt{\frac{b'\Gamma' P^{\perp} D\left(\varepsilon \circ \varepsilon\right) P^{\perp}\Gamma b}{n}} \\
= & O_p\left(\frac{K_{2,n}}{n}\right) O_p\left(1\right) \\
= & O_p\left(\frac{K_{2,n}}{n}\right)
\end{aligned}
$$

Since the above argument holds for all $a, b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$, we further deduce that

$$
\frac{\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D\left(\varepsilon \circ \varepsilon\right) P^{\perp}\Gamma}{n} = O_p\left(\frac{K_{2,n}}{n}\right).
$$

Part (d) can be shown in essentially the same way as part (c). Hence, to avoid redundancy, we omit the proof.

Turning our attention to part (e), let $a, b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$, and note that we can write

$$
\begin{aligned}
& \frac{a'\Gamma' P^{\perp} D\left(\varepsilon \circ \varepsilon\right) P^{\perp}\Gamma b}{n} - \frac{a'\Gamma' P^{\perp} D_{\sigma^2} P^{\perp}\Gamma b}{n} \\
= & \frac{1}{n}\sum_{(i,t)=1}^{m_n}\sum_{(j,s)=1}^{m_n}\sum_{(k,v)=1}^{m_n} P^{\perp}_{(i,t),(j,s)} P^{\perp}_{(i,t),(k,v)} a'\gamma_{(j,s)}\gamma'_{(k,v)} b \left\{\varepsilon^2_{(i,t)} - E\left[\varepsilon^2_{(i,t)} | \mathcal{F}^W_n\right]\right\}
\end{aligned}
$$

Now, making use of the CS inequality as well as Assumptions 2(i), 3(iii) and 5; we obtain

$$\begin{aligned}
& E\left[\left(\frac{1}{n}\sum_{(i,t)=1}^{m_n}\sum_{(j,s)=1}^{m_n}\sum_{(k,v)=1}^{m_n} P^{\perp}_{(i,t),(j,s)}P^{\perp}_{(i,t),(k,v)}a'\gamma_{(j,s)}\gamma'_{(k,v)}b\left\{\varepsilon^2_{(i,t)} - E\left[\varepsilon^2_{(i,t)}|\mathcal{F}^W_n\right]\right\}\right)^2 |\mathcal{F}^W_n\right] \\
= & \frac{1}{n^2}\sum_{(i,t)=1}^{m_n}\sum_{(j,s)=1}^{m_n}\sum_{(k,v)=1}^{m_n}\sum_{(\ell,h)=1}^{m_n}\sum_{(r,c)=1}^{m_n} P^{\perp}_{(i,t),(j,s)}P^{\perp}_{(i,t),(k,v)}P^{\perp}_{(i,t),(\ell,h)}P^{\perp}_{(i,t),(r,c)} \\
& \times\left(a'\gamma_{(j,s)}\right)\left(a'\gamma_{(\ell,h)}\right)\left(\gamma'_{(k,v)}b\right)\left(\gamma'_{(r,c)}b\right)E\left\{\left(\varepsilon^2_{(i,t)} - E\left[\varepsilon^2_{(i,t)}|\mathcal{F}^W_n\right]\right)^2 |\mathcal{F}^W_n\right\} \\
= & \frac{1}{n^2}\sum_{(i,t)=1}^{m_n} E\left\{\left(\varepsilon^2_{(i,t)} - E\left[\varepsilon^2_{(i,t)}|\mathcal{F}^W_n\right]\right)^2 |\mathcal{F}^W_n\right\}\left(e'_{(i,t)}P^{\perp}\Gamma a\right)^2\left(e'_{(i,t)}P^{\perp}\Gamma b\right)^2 \\
\leq & \left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon^4_{(i,t)}|\mathcal{F}^W_n\right]\right)\frac{1}{n^2}\sum_{(i,t)=1}^{m_n}\left(a'\Gamma'P^{\perp}e_{(i,t)}e'_{(i,t)}P^{\perp}\Gamma a\right)\left(b'\Gamma'P^{\perp}e_{(i,t)}e'_{(i,t)}P^{\perp}\Gamma b\right) \\
\leq & \left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon^4_{(i,t)}|\mathcal{F}^W_n\right]\right)\frac{1}{n}\left(a'\Gamma'P^{\perp}\sum_{(i,t)=1}^{m_n} e_{(i,t)}e'_{(i,t)}P^{\perp}\Gamma a\right)e'_{(i,t)}P^{\perp}e_{(i,t)}\frac{b'\Gamma'\Gamma b}{n} \\
\leq & \left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon^4_{(i,t)}|\mathcal{F}^W_n\right]\right)\left(\max_{1\leq(i,t)\leq m_n} P^{\perp}_{(i,t),(i,t)}\right)\frac{a'\Gamma'P^{\perp}\Gamma a}{n}\frac{b'\Gamma'\Gamma b}{n} \\
= & O_{a.s.}\left(\frac{K_{2,n}}{n}\right) = o_{a.s.}(1).
\end{aligned}$$

Hence, there exists a constant $\overline{C} < \infty$ such that for all n sufficiently large

$$\begin{aligned}
& E\left[\frac{n}{K_{2,n}}\left(\frac{a'\Gamma'P^{\perp}D(\varepsilon\circ\varepsilon)P^{\perp}\Gamma b}{n} - \frac{a'\Gamma'P^{\perp}D_{\sigma^2}P^{\perp}\Gamma b}{n}\right)^2\right] \\
= & E_{W_n}\left(\frac{n}{K_{2,n}}E\left[\left(\frac{a'\Gamma'P^{\perp}D(\varepsilon\circ\varepsilon)P^{\perp}\Gamma b}{n} - \frac{a'\Gamma'P^{\perp}D_{\sigma^2}P^{\perp}\Gamma b}{n}\right)^2 |\mathcal{F}^W_n\right]\right) \leq \overline{C}.
\end{aligned}$$

It follows from the Markov's inequality that for any $\epsilon > 0$,

$$\begin{aligned}
& \Pr\left(\sqrt{\frac{n}{K_{2,n}}}\left|\frac{a'\Gamma'P^{\perp}D(\varepsilon\circ\varepsilon)P^{\perp}\Gamma b}{n} - \frac{a'\Gamma'P^{\perp}D_{\sigma^2}P^{\perp}\Gamma b}{n}\right| \geq \sqrt{\frac{\overline{C}}{\epsilon}}\right) \\
= & \Pr\left(\frac{n}{K_{2,n}}\left(\frac{a'\Gamma'P^{\perp}D(\varepsilon\circ\varepsilon)P^{\perp}\Gamma b}{n} - \frac{a'\Gamma'P^{\perp}D_{\sigma^2}P^{\perp}\Gamma b}{n}\right)^2 \geq \frac{\overline{C}}{\epsilon}\right) \\
\leq & \frac{\epsilon}{\overline{C}}E\left[\frac{n}{K_{2,n}}\left(\frac{a'\Gamma'P^{\perp}D(\varepsilon\circ\varepsilon)P^{\perp}\Gamma b}{n} - \frac{a'\Gamma'P^{\perp}D_{\sigma^2}P^{\perp}\Gamma b}{n}\right)^2\right] \leq \epsilon
\end{aligned}$$

for all n sufficiently large, which shows that

$$\frac{a'\Gamma' P^{\perp} D\left(\varepsilon \circ \varepsilon\right) P^{\perp}\Gamma b}{n} - \frac{a'\Gamma' P^{\perp} D_{\sigma^2} P^{\perp}\Gamma b}{n} = O_p\left(\sqrt{\frac{K_{2,n}}{n}}\right).$$

Since the above result holds for all $a, b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$, we further deduce that

$$\begin{aligned}
& \frac{\Gamma' P^{\perp} D\left(\varepsilon \circ \varepsilon\right) P^{\perp}\Gamma}{n} - \frac{\Gamma' P^{\perp} D_{\sigma^2} P^{\perp}\Gamma}{n} \\
= & \frac{1}{n}\sum_{(i,t)=1}^{m_n}\sum_{(j,s)=1}^{m_n}\sum_{(k,v)=1}^{m_n} P^{\perp}_{(i,t),(j,s)} P^{\perp}_{(i,t),(k,v)} \gamma_{(j,s)}\gamma'_{(k,v)} \left\{\varepsilon^2_{(i,t)} - E\left[\varepsilon^2_{(i,t)} | \mathcal{F}^W_n\right]\right\} \\
= & O_p\left(\sqrt{\frac{K_{2,n}}{n}}\right).
\end{aligned}$$

Finally, consider part (f). Using the fact that $A = P^{\perp} - M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}$ and applying the results of parts (b)-(e) above, we obtain

$$\begin{aligned}
& \frac{\Gamma' A D\left(\varepsilon \circ \varepsilon\right) A\Gamma}{n} - \frac{\Gamma' M^{(Z_1,Q)} D_{\sigma^2} M^{(Z_1,Q)}\Gamma}{n} \\
= & \frac{\Gamma' P^{\perp} D\left(\varepsilon \circ \varepsilon\right) P^{\perp}\Gamma}{n} - \frac{\Gamma' M^{(Z_1,Q)} D_{\sigma^2} M^{(Z_1,Q)}\Gamma}{n} - \frac{\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D\left(\varepsilon \circ \varepsilon\right) P^{\perp}\Gamma}{n} \\
& - \frac{\Gamma' P^{\perp} D\left(\varepsilon \circ \varepsilon\right) M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\Gamma}{n} + \frac{\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D\left(\varepsilon \circ \varepsilon\right) M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\Gamma}{n} \\
= & \frac{\Gamma' P^{\perp} D_{\sigma^2} P^{\perp}\Gamma}{n} - \frac{\Gamma' M^{(Z_1,Q)} D_{\sigma^2} M^{(Z_1,Q)}\Gamma}{n} + \frac{\Gamma' P^{\perp} D\left(\varepsilon \circ \varepsilon\right) P^{\perp}\Gamma}{n} - \frac{\Gamma' P^{\perp} D_{\sigma^2} P^{\perp}\Gamma}{n} \\
& - \frac{\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D\left(\varepsilon \circ \varepsilon\right) P^{\perp}\Gamma}{n} - \frac{\Gamma' P^{\perp} D\left(\varepsilon \circ \varepsilon\right) M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\Gamma}{n} \\
& + \frac{\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D\left(\varepsilon \circ \varepsilon\right) M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\Gamma}{n} \\
= & \frac{\Gamma' P^{\perp} D_{\sigma^2} P^{\perp}\Gamma}{n} - \frac{\Gamma' M^{(Z_1,Q)} D_{\sigma^2} M^{(Z_1,Q)}\Gamma}{n} + O_p\left(\sqrt{\frac{K_{2,n}}{n}}\right) + O_p\left(\frac{K_{2,n}}{n}\right) \\
& + O_p\left(\frac{K_{2,n}}{n}\right) + O_p\left(\frac{K_{2,n}^2}{n^2}\right) \\
= & \frac{\Gamma' P^{\perp} D_{\sigma^2} P^{\perp}\Gamma}{n} - \frac{\Gamma' M^{(Z_1,Q)} D_{\sigma^2} M^{(Z_1,Q)}\Gamma}{n} + O_p\left(\sqrt{\frac{K_{2,n}}{n}}\right) \qquad (61)
\end{aligned}$$

Next, write

$$
\begin{aligned}
& \frac{\Gamma' P^{\perp} D_{\sigma^2} P^{\perp} \Gamma}{n} - \frac{\Gamma' M^{(Z_1,Q)} D_{\sigma^2} M^{(Z_1,Q)} \Gamma}{n} \\
= \ & \frac{\Gamma' M^{(Z_1,Q)} D_{\sigma^2} M^{(Z_1,Q)} \Gamma}{n} - \frac{\Gamma' M^{(Z_1,Q)} D_{\sigma^2} M^{(Z_1,Q)} \Gamma}{n} + \frac{\Gamma' \left[P^{\perp} - M^{(Z_1,Q)}\right] D_{\sigma^2} M^{(Z_1,Q)} \Gamma}{n} \\
& + \frac{\Gamma' M^{(Z_1,Q)} D_{\sigma^2} \left[P^{\perp} - M^{(Z_1,Q)}\right] \Gamma}{n} + \frac{\Gamma' \left[P^{\perp} - M^{(Z_1,Q)}\right] D_{\sigma^2} \left[P^{\perp} - M^{(Z_1,Q)}\right] \Gamma}{n} \\
= \ & \frac{\Gamma' \left[P^{\perp} - M^{(Z_1,Q)}\right] D_{\sigma^2} M^{(Z_1,Q)} \Gamma}{n} + \frac{\Gamma' M^{(Z_1,Q)} D_{\sigma^2} \left[P^{\perp} - M^{(Z_1,Q)}\right] \Gamma}{n} \\
& + \frac{\Gamma' \left[P^{\perp} - M^{(Z_1,Q)}\right] D_{\sigma^2} \left[P^{\perp} - M^{(Z_1,Q)}\right] \Gamma}{n}.
\end{aligned}
$$

Let $a, b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$; and, applying the triangle and CS inequalities, we have

$$
\begin{aligned}
& \left| \frac{a'\Gamma' P^{\perp} D_{\sigma^2} P^{\perp} \Gamma b}{n} - \frac{a'\Gamma' M^{(Z_1,Q)} D_{\sigma^2} M^{(Z_1,Q)} \Gamma b}{n} \right| \\
\leq \ & \frac{\left| a'\Gamma' \left[P^{\perp} - M^{(Z_1,Q)}\right] D_{\sigma^2} M^{(Z_1,Q)} \Gamma b \right|}{n} + \frac{\left| a'\Gamma' M^{(Z_1,Q)} D_{\sigma^2} \left[P^{\perp} - M^{(Z_1,Q)}\right] \Gamma b \right|}{n} \\
& + \frac{\left| a'\Gamma' \left[P^{\perp} - M^{(Z_1,Q)}\right] D_{\sigma^2} \left[P^{\perp} - M^{(Z_1,Q)}\right] \Gamma b \right|}{n} \\
\leq \ & \sqrt{\frac{a'\Gamma' \left[P^{\perp} - M^{(Z_1,Q)}\right] D_{\sigma^2} \left[P^{\perp} - M^{(Z_1,Q)}\right] \Gamma a}{n}} \sqrt{\frac{b'\Gamma' M^{(Z_1,Q)} D_{\sigma^2} M^{(Z_1,Q)} \Gamma b}{n}} \\
& + \sqrt{\frac{a'\Gamma' M^{(Z_1,Q)} D_{\sigma^2} M^{(Z_1,Q)} \Gamma a}{n}} \sqrt{\frac{b'\Gamma' \left[P^{\perp} - M^{(Z_1,Q)}\right] D_{\sigma^2} \left[P^{\perp} - M^{(Z_1,Q)}\right] \Gamma b}{n}} \\
& + \left\{ \sqrt{\frac{a'\Gamma' \left[P^{\perp} - M^{(Z_1,Q)}\right] D_{\sigma^2} \left[P^{\perp} - M^{(Z_1,Q)}\right] \Gamma a}{n}} \right. \\
& \qquad \left. \times \sqrt{\frac{b'\Gamma' \left[P^{\perp} - M^{(Z_1,Q)}\right] D_{\sigma^2} \left[P^{\perp} - M^{(Z_1,Q)}\right] \Gamma b}{n}} \right\}
\end{aligned} \tag{62}
$$

Next, note that, using Assumptions 2(i), 3(iii), 5, and 7(iii)

$$
\begin{aligned}
\frac{a'\Gamma' M^{(Z_1,Q)} D_{\sigma^2} M^{(Z_1,Q)} \Gamma a}{n} & \leq \left(\max_{1 \leq (i,t) \leq m_n} E\left[\varepsilon^2_{(i,t)} | \mathcal{F}^W_n \right] \right) \frac{a'\Gamma' M^{(Z_1,Q)} \Gamma a}{n} \\
& \leq \left(\max_{1 \leq (i,t) \leq m_n} E\left[\varepsilon^2_{(i,t)} | \mathcal{F}^W_n \right] \right) \frac{a'\Gamma' \Gamma a}{n} = O_{a.s.}(1),
\end{aligned}
$$

and

$$\begin{aligned}
& \frac{a'\Gamma'\left[P^{\perp}-M^{(Z_1,Q)}\right]D_{\sigma^2}\left[P^{\perp}-M^{(Z_1,Q)}\right]\Gamma a}{n} \\
= \ & \frac{a'\left(\Gamma-Z_2\Pi_{K_{2,n}}\right)'\left[P^{\perp}-M^{(Z_1,Q)}\right]D_{\sigma^2}\left[P^{\perp}-M^{(Z_1,Q)}\right]\left(\Gamma-Z_2\Pi_{K_{2,n}}\right)a}{n} \\
\leq \ & \left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon^2_{(i,t)}|\mathcal{F}^W_n\right]\right)\frac{a'\left(\Gamma-Z_2\Pi_{K_{2,n}}\right)'\left(\Gamma-Z_2\Pi_{K_{2,n}}\right)a}{n} \\
\leq \ & \left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon^2_{(i,t)}|\mathcal{F}^W_n\right]\right)\frac{m_n}{n}\left\|\gamma\left(\cdot\right)-\Pi'_{K_{2,n}}Z_2\left(\cdot\right)\right\|^2_{\infty,d} \\
= \ & O_{a.s.}\left(\frac{1}{K^{2\varrho_\gamma}_{2,n}}\right)
\end{aligned}$$

Similarly, of course, we also have

$$\begin{aligned}
\frac{b'\Gamma'M^{(Z_1,Q)}D_{\sigma^2}M^{(Z_1,Q)}\Gamma b}{n} &= O_{a.s.}\left(1\right) \text{ and} \\
\frac{b'\Gamma'\left[P^{\perp}-M^{(Z_1,Q)}\right]D_{\sigma^2}\left[P^{\perp}-M^{(Z_1,Q)}\right]\Gamma b}{n} &= O_{a.s.}\left(\frac{1}{K^{2\varrho_\gamma}_{2,n}}\right)
\end{aligned}$$

It follows from (62) that

$$\begin{aligned}
& \left|\frac{a'\Gamma'P^{\perp}D_{\sigma^2}P^{\perp}\Gamma b}{n}-\frac{a'\Gamma'M^{(Z_1,Q)}D_{\sigma^2}M^{(Z_1,Q)}\Gamma b}{n}\right| \\
= \ & O_{a.s.}\left(\frac{1}{K^{\varrho_\gamma}_{2,n}}\right)O_{a.s.}\left(1\right)+O_{a.s.}\left(1\right)O_{a.s.}\left(\frac{1}{K^{\varrho_\gamma}_{2,n}}\right)+O_{a.s.}\left(\frac{1}{K^{\varrho_\gamma}_{2,n}}\right)O_{a.s.}\left(\frac{1}{K^{\varrho_\gamma}_{2,n}}\right) \\
= \ & O_{a.s.}\left(\frac{1}{K^{\varrho_\gamma}_{2,n}}\right)=o_{a.s.}\left(1\right)
\end{aligned}$$

Since the above argument holds for all $a,b\in\mathbb{R}^d$ such that $\|a\|_2=\|b\|_2=1$, we further deduce that

$$\frac{\Gamma'P^{\perp}D_{\sigma^2}P^{\perp}\Gamma}{n}-\frac{\Gamma'M^{(Z_1,Q)}D_{\sigma^2}M^{(Z_1,Q)}\Gamma}{n}=O_{a.s.}\left(\frac{1}{K^{\varrho_\gamma}_{2,n}}\right)=o_{a.s.}\left(1\right).$$

Combining this with (61), we have

$$\begin{aligned}
& \frac{\Gamma' AD(\varepsilon\circ\varepsilon)A\Gamma}{n} - \frac{\Gamma' M^{(Z_1,Q)}D_{\sigma^2}M^{(Z_1,Q)}\Gamma}{n} \\
= \; & \frac{\Gamma' P^{\perp}D_{\sigma^2}P^{\perp}\Gamma}{n} - \frac{\Gamma' M^{(Z_1,Q)}D_{\sigma^2}M^{(Z_1,Q)}\Gamma}{n} + O_p\left(\sqrt{\frac{K_{2,n}}{n}}\right) \\
= \; & O_{a.s.}\left(\frac{1}{K_{2,n}^{\varrho_\gamma}}\right) + O_p\left(\sqrt{\frac{K_{2,n}}{n}}\right) \\
= \; & O_p\left(\max\left\{\frac{1}{K_{2,n}^{\varrho_\gamma}}, \sqrt{\frac{K_{2,n}}{n}}\right\}\right) = o_p(1),
\end{aligned}$$

as required. $\square$

Lemma OA-10:

Suppose that Assumptions 1-7 are satisfied. Then,

(a)

$$\begin{aligned}
& D_\mu^{-1}U'AD(\varepsilon\circ\varepsilon)AUD_\mu^{-1} \\
& - \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^2 E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right] D_\mu^{-1}E\left[U_{(j,s)}U_{(j,s)}'|\mathcal{F}_n^W\right] D_\mu^{-1} \\
= \; & O_p\left(\frac{\sqrt{K_{2,n}}}{(\mu_n^{\min})^2}\right) = o_p(1)
\end{aligned}$$

(b)

$$\frac{\Gamma' AD(\varepsilon\circ\varepsilon)AUD_\mu^{-1}}{\sqrt{n}} = o_p(1)$$

(c)

$$\frac{D_\mu^{-1}D_\kappa F' AD(\varepsilon\circ\varepsilon)AUD_\mu^{-1}}{\sqrt{n}} = o_p(1)$$

(d)

$$\frac{D_\mu^{-1}D_\kappa F' AD(\varepsilon\circ\varepsilon)AFD_\kappa D_\mu^{-1}}{n} = o_p(1)$$

(e)

$$\frac{D_\mu^{-1}D_\kappa F' AD(\varepsilon\circ\varepsilon)A\Gamma}{n} = o_p(1)$$

Proof of Lemma OA-10:

To show part (a), let $a, b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$, and define $u_{a,(i,t)} = a' D_\mu^{-1} U_{(i,t)}$ and $u_{b,(j,s)} = b' D_\mu^{-1} U_{(j,s)}$. Note that

$$\begin{aligned}
& a' D_\mu^{-1} U' A D (\varepsilon \circ \varepsilon) A U D_\mu^{-1} b \\
& - \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^2 E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W\right] a' D_\mu^{-1} E\left[U_{(j,s)} U_{(j,s)}' | \mathcal{F}_n^W\right] D_\mu^{-1} b \\
= & \sum_{(j,s),(k,v)=1}^{m_n} \sum_{(i,t)\neq\{(j,s),(k,v)\}} A_{(i,t),(j,s)} A_{(i,t),(k,v)} \varepsilon_{(i,t)}^2 u_{a,(j,s)} u_{b,(k,v)} \\
& - \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^2 E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W\right] E\left[u_{a,(j,s)} u_{b,(k,v)} | \mathcal{F}_n^W\right] \\
= & \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^2 \left(\varepsilon_{(i,t)}^2 u_{a,(j,s)} u_{b,(j,s)} - E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W\right] E\left[u_{a,(j,s)} u_{b,(j,s)} | \mathcal{F}_n^W\right]\right) \\
& + \sum_{\substack{(j,s),(k,v)=1 \\ (j,s)\neq(k,v)}}^{m_n} \sum_{(i,t)\neq\{(j,s),(k,v)\}} A_{(i,t),(j,s)} A_{(i,t),(k,v)} \varepsilon_{(i,t)}^2 u_{a,(j,s)} u_{b,(k,v)} \qquad (63)
\end{aligned}$$

Focusing on the first term in expression (63) above, we apply the CS inequality, parts (b) and

(c) of Lemma S2-1, and Assumptions 2(i) and 3(ii) to obtain

$$
\begin{aligned}
& E\left[\left(\sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^2 \left(\varepsilon_{(i,t)}^2 u_{a,(j,s)} u_{b,(j,s)} - E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W\right] E\left[u_{a,(j,s)} u_{b,(j,s)} | \mathcal{F}_n^W\right]\right)\right)^2 | \mathcal{F}_n^W\right] \\
= & \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^4 E\left\{\left(\varepsilon_{(i,t)}^2 u_{a,(j,s)} u_{b,(j,s)} - E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W\right] E\left[u_{a,(j,s)} u_{b,(j,s)} | \mathcal{F}_n^W\right]\right)^2 | \mathcal{F}_n^W\right\} \\
& + \sum_{\substack{(i,t),(k,v)=1 \\ (i,t)\neq(k,v)}}^{m_n} \sum_{(j,s)\neq\{(i,t),(k,v)\}} A_{(i,t),(j,s)}^2 A_{(k,v),(j,s)}^2 \left\{E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W\right] E\left[\varepsilon_{(k,v)}^2 | \mathcal{F}_n^W\right] E\left[u_{a,(j,s)}^2 u_{b,(j,s)}^2 | \mathcal{F}_n^W\right]\right. \\
& \left. - E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W\right] E\left[\varepsilon_{(k,v)}^2 | \mathcal{F}_n^W\right] \left(E\left[u_{a,(j,s)} u_{b,(j,s)} | \mathcal{F}_n^W\right]\right)^2\right\} \\
& + \sum_{\substack{(i,t),(k,v)=1 \\ (i,t)\neq(k,v)}}^{m_n} \sum_{(j,s)\neq\{(i,t),(k,v)\}} \left\{A_{(i,t),(j,s)}^2 A_{(i,t),(k,v)}^2\right. \\
& \quad \times \left(E\left[\varepsilon_{(i,t)}^4 | \mathcal{F}_n^W\right] E\left[u_{a,(j,s)} u_{b,(j,s)} | \mathcal{F}_n^W\right] E\left[u_{a,(k,v)} u_{b,(k,v)} | \mathcal{F}_n^W\right]\right. \\
& \quad \left.\left. - \left(E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W\right]\right)^2 E\left[u_{a,(j,s)} u_{b,(j,s)} | \mathcal{F}_n^W\right] E\left[u_{a,(k,v)} u_{b,(k,v)} | \mathcal{F}_n^W\right]\right)\right\} \\
\leq & \; 2\left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon_{(i,t)}^4 | \mathcal{F}_n^W\right]\right)\left(\max_{1\leq(i,t)\leq m_n} E\left[\left\|U_{(i,t)}\right\|_2^4 | \mathcal{F}_n^W\right]\right) \frac{1}{\left(\mu_n^{\min}\right)^4} \left\{\sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^4\right. \\
& \left. + \sum_{\substack{(i,t),(k,v),(j,s)=1 \\ (i,t)\neq(k,v), \\ (j,s)\neq(i,t),(j,s)\neq(k,v)}}^{m_n} A_{(i,t),(j,s)}^2 A_{(k,v),(j,s)}^2 + \sum_{\substack{(i,t),(k,v),(j,s)=1 \\ (i,t)\neq(k,v), \\ (j,s)\neq(i,t),(j,s)\neq(k,v)}}^{m_n} A_{(i,t),(j,s)}^2 A_{(i,t),(k,v)}^2\right\} \\
= & \; O_{a.s.}\left(\frac{K_{2,n}^3}{\left(\mu_n^{\min}\right)^4 n^2}\right) + O_{a.s.}\left(\frac{K_{2,n}^2}{\left(\mu_n^{\min}\right)^4 n}\right) + O_{a.s.}\left(\frac{K_{2,n}^2}{\left(\mu_n^{\min}\right)^4 n}\right) \\
= & \; O_{a.s.}\left(\frac{K_{2,n}^2}{\left(\mu_n^{\min}\right)^4 n}\right)
\end{aligned}
$$

Hence, there exists a constant $\overline{C} < \infty$ such that for all n sufficiently large

$$
\begin{aligned}
& E\left[\frac{\left(\mu_n^{\min}\right)^4 n}{K_{2,n}^2}\left(\sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^2\left(\varepsilon_{(i,t)}^2 u_{a,(j,s)} u_{b,(j,s)} - E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W\right] E\left[u_{a,(j,s)} u_{b,(j,s)} | \mathcal{F}_n^W\right]\right)\right)^2\right] \\
= & \frac{\left(\mu_n^{\min}\right)^4 n}{K_{2,n}^2} \times \\
& E_{W_n}\left(E\left[\left(\sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^2\left(\varepsilon_{(i,t)}^2 u_{a,(j,s)} u_{b,(j,s)} - E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W\right] E\left[u_{a,(j,s)} u_{b,(j,s)} | \mathcal{F}_n^W\right]\right)\right)^2 | \mathcal{F}_n^W\right]\right) \\
\leq & \overline{C}.
\end{aligned}
$$

It follows from the Markov's inequality that for any $\epsilon > 0$,

$$
\begin{aligned}
& \Pr\left(\frac{\left(\mu_n^{\min}\right)^2 n^{\frac{1}{2}}}{K_{2,n}}\left|\sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^2\left(\varepsilon_{(i,t)}^2 u_{a,(j,s)} u_{b,(j,s)} - E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W\right] E\left[u_{a,(j,s)} u_{b,(j,s)} | \mathcal{F}_n^W\right]\right)\right| \geq \sqrt{\frac{\overline{C}}{\epsilon}}\right) \\
= & \Pr\left\{\frac{\left(\mu_n^{\min}\right)^4 n}{K_{2,n}^2}\left(\sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^2\left(\varepsilon_{(i,t)}^2 u_{a,(j,s)} u_{b,(j,s)} - E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W\right] E\left[u_{a,(j,s)} u_{b,(j,s)} | \mathcal{F}_n^W\right]\right)\right)^2 \geq \frac{\overline{C}}{\epsilon}\right\} \\
\leq & \epsilon \frac{E\left[\frac{\left(\mu_n^{\min}\right)^4 n}{K_{2,n}^2}\left(\sum_{(i,t)\neq(j,s)} A_{(i,t),(j,s)}^2\left\{\varepsilon_{(i,t)}^2 u_{a,(j,s)} u_{b,(j,s)} - E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W\right] E\left[u_{a,(j,s)} u_{b,(j,s)} | \mathcal{F}_n^W\right]\right\}\right)^2\right]}{\overline{C}} \\
\leq & \epsilon
\end{aligned}
$$

for all n sufficiently large, which shows that

$$
\begin{aligned}
& \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^2\left(\varepsilon_{(i,t)}^2 u_{a,(j,s)} u_{b,(j,s)} - E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W\right] E\left[u_{a,(j,s)} u_{b,(j,s)} | \mathcal{F}_n^W\right]\right) \\
= & O_p\left(\frac{K_{2,n}}{\left(\mu_n^{\min}\right)^2 \sqrt{n}}\right). \qquad (64)
\end{aligned}
$$

Turning our attention to the second term in expression (63), we first define $\varsigma_{(i,t)} = \varepsilon_{(i,t)}^2 - E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W\right]$. Note that by the decoupling inequality given in Lemma OA-8, there exist finite constants C_2 and C_3, whose explicit forms are given in Lemma OA-8, and independent copies $\left\{\left(u_{a,(i,t)}^{(g)}, u_{b,(i,t)}^{(g)}, \varsigma_{(i,t)}^{(g)}\right)\right\}_{(i,t)=1}^{m_n}$ (for $g = 1,2,3$) of the sequence $\left\{\left(u_{a,(i,t)}, u_{b,(i,t)}, \varsigma_{(i,t)}\right)\right\}_{(i,t)=1}^{m_n}$ such that

$$
\begin{aligned}
& E\left[\left(\sum_{\substack{(j,s),(k,v)=1 \\ (j,s)\neq(k,v)}}^{m_n}\sum_{(i,t)\neq\{(j,s),(k,v)\}} A_{(i,t),(j,s)}A_{(i,t),(k,v)}\varepsilon_{(i,t)}^2 u_{a,(j,s)}u_{b,(k,v)}\right)^2 |\mathcal{F}_n^W\right] \\
\leq\ & 2E\left[\left(\sum_{\substack{(j,s),(k,v)=1 \\ (j,s)\neq(k,v)}}^{m_n}\sum_{(i,t)\neq\{(j,s),(k,v)\}} A_{(i,t),(j,s)}A_{(i,t),(k,v)}E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right] u_{a,(j,s)}u_{b,(k,v)}\right)^2 |\mathcal{F}_n^W\right] \\
& +2E\left[\left(\sum_{\substack{(j,s),(k,v)=1 \\ (j,s)\neq(k,v)}}^{m_n}\sum_{(i,t)\neq\{(j,s),(k,v)\}} A_{(i,t),(j,s)}A_{(i,t),(k,v)}u_{a,(j,s)}u_{b,(k,v)}\left[\varepsilon_{(i,t)}^2 - E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right]\right]\right)^2 |\mathcal{F}_n^W\right] \\
\leq\ & 2C_2E\left[\left(\sum_{\substack{(j,s),(k,v)=1 \\ (j,s)\neq(k,v)}}^{m_n}\sum_{(i,t)\neq\{(j,s),(k,v)\}} A_{(i,t),(j,s)}A_{(i,t),(k,v)}E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right] u_{a,(j,s)}^{(1)}u_{b,(k,v)}^{(2)}\right)^2 |\mathcal{F}_n^W\right] \\
& +2C_3E\left[\left(\sum_{\substack{(j,s),(k,v)=1 \\ (j,s)\neq(k,v)}}^{m_n}\sum_{(i,t)\neq\{(j,s),(k,v)\}} A_{(i,t),(j,s)}A_{(i,t),(k,v)}u_{a,(j,s)}^{(1)}u_{b,(k,v)}^{(2)}\varsigma_{(i,t)}^{(3)}\right)^2 |\mathcal{F}_n^W\right] \\
=\ & 2C_2\sum_{\substack{(j,s),(k,v)=1 \\ (j,s)\neq(k,v)}}^{m_n}\sum_{(i,t)\neq\{(j,s),(k,v)\}}\sum_{(\ell,h)\neq\{(i,t),(j,s),(k,v)\}}\Big\{A_{(i,t),(j,s)}A_{(i,t),(k,v)}A_{(\ell,h),(j,s)}A_{(\ell,h),(k,v)} \\
& \qquad\times E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right]E\left[\varepsilon_{(\ell,h)}^2|\mathcal{F}_n^W\right]E\left[\left(u_{a,(j,s)}^{(1)}\right)^2|\mathcal{F}_n^W\right]E\left[\left(u_{b,(k,v)}^{(1)}\right)^2|\mathcal{F}_n^W\right]\Big\} \\
& +2C_2\sum_{\substack{(j,s),(k,v),(i,t)=1 \\ (j,s)\neq(k,v) \\ (i,t)\neq\{(j,s),(k,v)\}}}^{m_n} A_{(i,t),(j,s)}^2A_{(i,t),(k,v)}^2\left(E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right]\right)^2E\left[\left(u_{a,(j,s)}^{(1)}\right)^2|\mathcal{F}_n^W\right]E\left[\left(u_{b,(k,v)}^{(2)}\right)^2|\mathcal{F}_n^W\right] \\
& +2C_3\sum_{\substack{(j,s),(k,v),(i,t)=1 \\ (j,s)\neq(k,v) \\ (i,t)\neq\{(j,s),(k,v)\}}}^{m_n} A_{(i,t),(j,s)}^2A_{(i,t),(k,v)}^2\Big\{E\left[\left(u_{a,(j,s)}^{(1)}\right)^2|\mathcal{F}_n^W\right]E\left[\left(u_{b,(k,v)}^{(2)}\right)^2|\mathcal{F}_n^W\right] \\
& \qquad\times E\left[\left(\varsigma_{(i,t)}^{(3)}\right)^2|\mathcal{F}_n^W\right]\Big\} \qquad (65)
\end{aligned}
$$

Define $D_{\sigma^2} = diag\left(\sigma_{(1,2)}^2\left(W_n\right),....,\sigma_{(n,T_n)}^2\left(W_n\right)\right)$ where $\sigma_{(i,t)}^2 = E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right]$ for $(i,t) = 1,...,m_n$. By applying Assumptions 2(i), 3(ii), and 5 as well as parts (a) and (b) of Lemma OA-1, we can estimate the (almost sure) order of magnitude of the first term on the right-hand side of (65) as follows

$$
\begin{aligned}
& \sum_{\substack{(j,s),(k,v)=1 \\ (j,s)\neq(k,v)}}^{m_n} \sum_{(i,t)\neq\{(j,s),(k,v)\}} \sum_{(\ell,h)\neq\{(i,t),(j,s),(k,v)\}} \left\{A_{(i,t),(j,s)}A_{(i,t),(k,v)}A_{(\ell,h),(j,s)}A_{(\ell,h),(k,v)}\right. \\
& \qquad \left. \times E\left[\varepsilon^2_{(i,t)}|\mathcal{F}^W_n\right] E\left[\varepsilon^2_{(\ell,h)}|\mathcal{F}^W_n\right] E\left[\left(u^{(1)}_{a,(j,s)}\right)^2|\mathcal{F}^W_n\right] E\left[\left(u^{(2)}_{b,(k,v)}\right)^2|\mathcal{F}^W_n\right]\right\} \\
= & \sum_{(j,s)=1}^{m_n} E\left[\left(u^{(1)}_{a,(j,s)}\right)^2|\mathcal{F}^W_n\right] \sum_{\substack{(k,v)=1 \\ (k,v)\neq(j,s)}}^{m_n} e'_{(j,s)}A \sum_{(i,t)\neq\{(j,s),(k,v)\}} e_{(i,t)}E\left[\varepsilon^2_{(i,t)}|\mathcal{F}^W_n\right] e'_{(i,t)}Ae_{(k,v)} \\
& \times E\left[\left(u^{(2)}_{b,(k,v)}\right)^2|\mathcal{F}^W_n\right] e'_{(k,v)} \sum_{(\ell,h)\neq\{(i,t),(j,s),(k,v)\}} Ae_{(\ell,h)}E\left[\varepsilon^2_{(\ell,h)}|\mathcal{F}^W_n\right] e'_{(\ell,h)}Ae_{(j,s)} \\
= & \sum_{(j,s)=1}^{m_n} E\left[\left(u^{(1)}_{a,(j,s)}\right)^2|\mathcal{F}^W_n\right] \sum_{\substack{(k,v)=1 \\ (k,v)\neq(j,s)}}^{m_n} e'_{(j,s)}A \sum_{(i,t)\neq\{(j,s),(k,v)\}} e_{(i,t)}E\left[\varepsilon^2_{(i,t)}|\mathcal{F}^W_n\right] e'_{(i,t)}Ae_{(k,v)} \\
& \times E\left[\left(u^{(2)}_{b,(k,v)}\right)^2|\mathcal{F}^W_n\right] \left[e'_{(k,v)}AD_{\sigma^2}Ae_{(j,s)} - e'_{(k,v)}Ae_{(i,t)}E\left[\varepsilon^2_{(i,t)}|\mathcal{F}^W_n\right] e'_{(i,t)}Ae_{(j,s)}\right] \\
= & \sum_{(j,s)=1}^{m_n} E\left[\left(u^{(1)}_{a,(j,s)}\right)^2|\mathcal{F}^W_n\right] \sum_{\substack{(k,v)=1 \\ (k,v)\neq(j,s)}}^{m_n} E\left[\left(u^{(2)}_{b,(k,v)}\right)^2|\mathcal{F}^W_n\right] \left[e'_{(k,v)}AD_{\sigma^2}Ae_{(j,s)}\right]^2 \\
& - \sum_{(j,s)=1}^{m_n} E\left[\left(u^{(1)}_{a,(j,s)}\right)^2|\mathcal{F}^W_n\right] \\
& \times \sum_{\substack{(k,v),(i,t)=1 \\ (k,v)\neq(j,s) \\ (i,t)\neq\{(j,s),(k,v)\}}}^{m_n} \left[e'_{(j,s)}Ae_{(i,t)}E\left[\varepsilon^2_{(i,t)}|\mathcal{F}^W_n\right] e'_{(i,t)}Ae_{(k,v)}\right]^2 E\left[\left(u^{(2)}_{b,(k,v)}\right)^2|\mathcal{F}^W_n\right] \\
\leq & \sum_{(j,s)=1}^{m_n} E\left[\left(u^{(1)}_{a,(j,s)}\right)^2|\mathcal{F}^W_n\right] \sum_{(k,v)=1}^{m_n} E\left[\left(u^{(2)}_{b,(k,v)}\right)^2|\mathcal{F}^W_n\right] e'_{(j,s)}AD_{\sigma^2}Ae_{(k,v)}e'_{(k,v)}AD_{\sigma^2}Ae_{(j,s)} \\
\leq & \left(\max_{1\leq(i,t)\leq m_n} E\left[\left\|U_{(i,t)}\right\|_2^2|\mathcal{F}^W_n\right]\right)^2 \frac{1}{(\mu^{\min}_n)^4} \sum_{(j,s)=1}^{m_n} e'_{(j,s)}AD_{\sigma^2}A^2D_{\sigma^2}Ae_{(j,s)}
\end{aligned}
$$

$$\begin{aligned}
&\leq \left(\max_{1\leq(i,t)\leq m_n} E\left[\left\|U_{(i,t)}\right\|_2^2|\mathcal{F}_n^W\right]\right)^2\left(1+\max_{1\leq(i,t)\leq m_n}\left|\widehat{\vartheta}_{(i,t)}\right|^2\right)\left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right]\right)^2 \\
&\quad\times\frac{1}{(\mu_n^{\min})^4}\sum_{(j,s)=1}^{m_n} e_{(j,s)}'A^2e_{(j,s)} \\
&\leq \left(\max_{1\leq(i,t)\leq m_n} E\left[\left\|U_{(i,t)}\right\|_2^2|\mathcal{F}_n^W\right]\right)^2\left(1+\max_{1\leq(i,t)\leq m_n}\left|\widehat{\vartheta}_{(i,t)}\right|^2\right)\left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right]\right)^2 \\
&\quad\times\frac{1}{(\mu_n^{\min})^4}\left[tr\left\{P^{\perp}\right\}+tr\left\{D_{\widehat{\vartheta}}^2\right\}\right] \\
&= O_{a.s.}\left(\frac{K_{2,n}}{(\mu_n^{\min})^4}\right)+O_{a.s.}\left(\frac{K_{2,n}^2}{(\mu_n^{\min})^4 n}\right)=O_{a.s.}\left(\frac{K_{2,n}}{(\mu_n^{\min})^4}\right).
\end{aligned}$$

By applying Assumptions 2(i) and 3(ii) as well as part (c) of Lemma S2-1, we can also derive the (almost sure) order of magnitude for the second and the third terms of (65) as follows

$$\begin{aligned}
&\sum_{(i,t)=1}^{m_n}\sum_{\substack{(j,s),(k,v)=1\\(j,s)\neq(k,v),(i,t)\\(k,v)\neq(i,t)}}^{m_n} A_{(i,t),(j,s)}^2A_{(i,t),(k,v)}^2\left(E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right]\right)^2E\left[\left(u_{a,(j,s)}^{(1)}\right)^2|\mathcal{F}_n^W\right]E\left[\left(u_{b,(k,v)}^{(2)}\right)^2|\mathcal{F}_n^W\right] \\
&\leq \left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right]\right)^2\left(\max_{1\leq(i,t)\leq m_n} E\left[\left\|U_{(i,t)}\right\|_2^2|\mathcal{F}_n^W\right]\right)^2\frac{1}{(\mu_n^{\min})^4} \\
&\quad\times\sum_{(i,t)=1}^{m_n}\sum_{\substack{(j,s),(k,v)=1\\(j,s)\neq\{(k,v),(i,t)\},(k,v)\neq(i,t)}}^{m_n} A_{(i,t),(j,s)}^2A_{(i,t),(k,v)}^2 \\
&= O_{a.s.}\left(\frac{K_{2,n}^2}{(\mu_n^{\min})^4 n}\right)=o_{a.s.}(1)
\end{aligned}$$

and

$$
\begin{aligned}
&\sum_{(i,t)=1}^{m_n} \sum_{\substack{(j,s),(k,v)=1 \\ (j,s)\neq(k,v),(i,t) \\ (k,v)\neq(i,t)}}^{m_n} \left\{ A^2_{(i,t),(j,s)} A^2_{(i,t),(k,v)} E\left[\left(u^{(1)}_{a,(j,s)}\right)^2 |\mathcal{F}^W_n\right] \right. \\
&\qquad\qquad \left. \times E\left[\left(u^{(2)}_{b,(k,v)}\right)^2 |\mathcal{F}^W_n\right] E\left[\left(\varsigma^{(3)}_{(i,t)}\right)^2 |\mathcal{F}^W_n\right] \right\} \\
\leq\ & \left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon^4_{(i,t)} |\mathcal{F}^W_n\right] \right) \left(\max_{1\leq(i,t)\leq m_n} E\left[\left\|U_{(i,t)}\right\|_2^2 |\mathcal{F}^W_n\right] \right)^2 \frac{1}{\left(\mu_n^{\min}\right)^4} \\
& \times \sum_{(i,t)=1}^{m_n} \sum_{\substack{(j,s),(k,v)=1 \\ (j,s)\neq\{(k,v),(i,t)\},(k,v)\neq(i,t)}}^{m_n} A^2_{(i,t),(j,s)} A^2_{(i,t),(k,v)} \\
=\ & O_{a.s.}\left(\frac{K^2_{2,n}}{\left(\mu_n^{\min}\right)^4 n} \right) = o_{a.s.}(1).
\end{aligned}
$$

These results imply that

$$
\begin{aligned}
& E\left[\left(\sum_{(i,t)=1}^{m_n} \sum_{\substack{(j,s),(k,v)=1 \\ (j,s)\neq\{(k,v),(i,t)\},(k,v)\neq(i,t)}}^{m_n} A_{(i,t),(j,s)} A_{(i,t),(k,v)} \varepsilon^2_{(i,t)} u_{a,(j,s)} u_{b,(k,v)} \right)^2 |\mathcal{F}^W_n \right] \\
=\ & O_{a.s.}\left(\frac{K_{2,n}}{\left(\mu_n^{\min}\right)^4} \right)
\end{aligned}
$$

Hence, there exists a constant $\overline{C} < \infty$ such that for all n sufficiently large

$$
\begin{aligned}
& E\left[\frac{\left(\mu_n^{\min}\right)^4}{K^2_{2,n}} \left(\sum_{(i,t)=1}^{m_n} \sum_{\substack{(j,s),(k,v)=1 \\ (j,s)\neq\{(k,v),(i,t)\},(k,v)\neq(i,t)}}^{m_n} A_{(i,t),(j,s)} A_{(i,t),(k,v)} \varepsilon^2_{(i,t)} u_{a,(j,s)} u_{b,(k,v)} \right)^2 \right] \\
=\ & E_{W_n}\left(E\left[\frac{\left(\mu_n^{\min}\right)^4}{K^2_{2,n}} \left(\sum_{(i,t)=1}^{m_n} \sum_{\substack{(j,s),(k,v)=1 \\ (j,s)\neq\{(k,v),(i,t)\},(k,v)\neq(i,t)}}^{m_n} A_{(i,t),(j,s)} A_{(i,t),(k,v)} \varepsilon^2_{(i,t)} u_{a,(j,s)} u_{b,(k,v)} \right)^2 |\mathcal{F}^W_n \right] \right) \\
\leq\ & \overline{C}.
\end{aligned}
$$

It follows from the Markov's inequality that for any $\epsilon > 0$,

$$
\begin{aligned}
& \Pr\left(\frac{\left(\mu_n^{\min}\right)^2}{K_{2,n}}\left|\sum_{(i,t)=1}^{m_n}\sum_{\substack{(j,s),(k,v)=1\\(j,s)\neq\{(k,v),(i,t)\},(k,v)\neq(i,t)}}^{m_n} A_{(i,t),(j,s)}A_{(i,t),(k,v)}\varepsilon_{(i,t)}^2 u_{a,(j,s)}u_{b,(k,v)}\right| \geq \sqrt{\frac{\overline{C}}{\epsilon}}\right) \\
= & \Pr\left\{\frac{\left(\mu_n^{\min}\right)^4}{K_{2,n}^2}\left(\sum_{(i,t)=1}^{m_n}\sum_{\substack{(j,s),(k,v)=1\\(j,s)\neq\{(k,v),(i,t)\},(k,v)\neq(i,t)}}^{m_n} A_{(i,t),(j,s)}A_{(i,t),(k,v)}\varepsilon_{(i,t)}^2 u_{a,(j,s)}u_{b,(k,v)}\right)^2 \geq \frac{\overline{C}}{\epsilon}\right\} \\
\leq & \frac{\epsilon}{\overline{C}}E\left[\frac{\left(\mu_n^{\min}\right)^4}{K_{2,n}^2}\left(\sum_{(i,t)=1}^{m_n}\sum_{\substack{(j,s),(k,v)=1\\(j,s)\neq\{(k,v),(i,t)\},(k,v)\neq(i,t)}}^{m_n} A_{(i,t),(j,s)}A_{(i,t),(k,v)}\varepsilon_{(i,t)}^2 u_{a,(j,s)}u_{b,(k,v)}\right)^2\right] \leq \epsilon
\end{aligned}
$$

for all n sufficiently large, which shows that

$$
\sum_{(i,t)=1}^{m_n}\sum_{\substack{(j,s),(k,v)=1\\(j,s)\neq\{(k,v),(i,t)\},(k,v)\neq(i,t)}}^{m_n} A_{(i,t),(j,s)}A_{(i,t),(k,v)}\varepsilon_{(i,t)}^2 u_{a,(j,s)}u_{b,(k,v)} = O_p\left(\frac{\sqrt{K_{2,n}}}{\left(\mu_n^{\min}\right)^2}\right) \tag{66}
$$

Now, (64) and (66) together imply that

$$
\begin{aligned}
& a'D_\mu^{-1}U'AD\left(\varepsilon\circ\varepsilon\right)AUD_\mu^{-1}b \\
& -\sum_{\substack{(i,t),(j,s)=1\\(i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^2 E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right]a'D_\mu^{-1}E\left[U_{(j,s)}U_{(j,s)}'|\mathcal{F}_n^W\right]D_\mu^{-1}b \\
= & \sum_{\substack{(i,t),(j,s)=1\\(i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^2\left(\varepsilon_{(i,t)}^2 u_{a,(j,s)}u_{b,(j,s)} - E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right]E\left[u_{a,(j,s)}u_{b,(j,s)}|\mathcal{F}_n^W\right]\right) \\
& +\sum_{\substack{(j,s),(k,v)=1\\(j,s)\neq(k,v)}}^{m_n}\sum_{(i,t)\neq\{(j,s),(k,v)\}} A_{(i,t),(j,s)}A_{(i,t),(k,v)}\varepsilon_{(i,t)}^2 u_{a,(j,s)}u_{b,(k,v)} \\
= & O_p\left(\frac{K_{2,n}}{\left(\mu_n^{\min}\right)^2\sqrt{n}}\right) + O_p\left(\frac{\sqrt{K_{2,n}}}{\left(\mu_n^{\min}\right)^2}\right) \\
= & O_p\left(\frac{\sqrt{K_{2,n}}}{\left(\mu_n^{\min}\right)^2}\right).
\end{aligned}
$$

Finally, since the above argument holds for all a, $b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$, we obtain the desired result

$$
\begin{aligned}
& D_\mu^{-1}U'AD\left(\varepsilon\circ\varepsilon\right)AUD_\mu^{-1} - \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^2 E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right] D_\mu^{-1}E\left[U_{(j,s)}U_{(j,s)}'|\mathcal{F}_n^W\right] D_\mu^{-1} \\
= \; & O_p\left(\frac{\sqrt{K_{2,n}}}{\left(\mu_n^{\min}\right)^2}\right).
\end{aligned}
$$

To show part (b), again let a, $b\in\mathbb{R}^d$ such that $\|a\|_2=\|b\|_2=1$ and define $u_{b,(k,v)}=b'D_\mu^{-1}U_{(k,v)}$ and $u_b=UD_\mu^{-1}b$. We can apply Loève's c_r inequality to obtain

$$
\begin{aligned}
& E\left[\left(\frac{a'\Gamma'AD\left(\varepsilon\circ\varepsilon\right)AUD_\mu^{-1}b}{\sqrt{n}}\right)^2|\mathcal{F}_n^W\right] \\
= \; & \frac{1}{n}E\left[\left(\sum_{(j,s),(k,v)=1}^{m_n}\sum_{(i,t)\neq\{(j,s),(k,v)\}} A_{(i,t),(j,s)}A_{(i,t),(k,v)}a'\gamma_{(j,s)}\varepsilon_{(i,t)}^2 u_{b,(k,v)}\right)^2|\mathcal{F}_n^W\right] \\
\leq \; & \frac{2}{n}E\left[\left(\sum_{(j,s),(k,v)=1}^{m_n}\sum_{(i,t)\neq\{(j,s),(k,v)\}} A_{(i,t),(j,s)}A_{(i,t),(k,v)}a'\gamma_{(j,s)}E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right]u_{b,(k,v)}\right)^2|\mathcal{F}_n^W\right] \\
& +\frac{2}{n}E\left[\left(\sum_{(j,s),(k,v)=1}^{m_n}\sum_{(i,t)\neq\{(j,s),(k,v)\}} A_{(i,t),(j,s)}A_{(i,t),(k,v)}a'\gamma_{(j,s)}\left\{\varepsilon_{(i,t)}^2 - E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right]\right\}u_{b,(k,v)}\right)^2|\mathcal{F}_n^W\right] \\
= \; & \frac{2}{n}E\left[\left(a'\Gamma'AD_{\sigma^2}Au_b\right)^2|\mathcal{F}_n^W\right] \\
& +\frac{2}{n}E\left[\left(\sum_{(j,s),(k,v)=1}^{m_n}\sum_{(i,t)\neq\{(j,s),(k,v)\}} A_{(i,t),(j,s)}A_{(i,t),(k,v)}a'\gamma_{(j,s)}\varsigma_{(i,t)}u_{b,(k,v)}\right)^2|\mathcal{F}_n^W\right] \qquad (67)
\end{aligned}
$$

where $\varsigma_{(i,t)}=\varepsilon_{(i,t)}^2-E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right]$. Focusing on the first term of expression (67) above, we can apply Assumptions 2(i), 3(iii), and 3(ii) as well as part (b) of Lemma OA-1 to obtain

$$
\begin{aligned}
& \frac{2}{n} E\left[\left(a' \Gamma' A D_{\sigma^2} A u_b\right)^2 | \mathcal{F}_n^W\right] \\
= & \frac{2}{n} a' \Gamma' A D_{\sigma^2} A E\left[u_b u_b' | \mathcal{F}_n^W\right] A D_{\sigma^2} A \Gamma a \\
\leq & \frac{1}{\left(\mu_n^{\min}\right)^2}\left(\max_{1 \leq (i,t) \leq m_n} E\left[\left\|U_{(i,t)}\right\|_2^2 | \mathcal{F}_n^W\right]\right) \frac{a' \Gamma' A D_{\sigma^2} A^2 D_{\sigma^2} A \Gamma a}{n} \\
\leq & \frac{1}{\left(\mu_n^{\min}\right)^2}\left(\max_{1 \leq (i,t) \leq m_n} E\left[\left\|U_{(i,t)}\right\|_2^2 | \mathcal{F}_n^W\right]\right)\left(1+\max_{1 \leq (i,t) \leq m_n}\left|\widehat{\vartheta}_{(i,t)}\right|^2\right)^2 \\
& \times\left(\max_{1 \leq (i,t) \leq m_n} E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W\right]\right)^2 \frac{a' \Gamma' \Gamma a}{n} \\
= & O_{a.s.}\left(\frac{1}{\left(\mu_n^{\min}\right)^2}\right)=o_{a.s.}(1) .
\end{aligned}
$$

Turning our attention now to the second term of (67), note that we can apply the inequality $|XY| \leq (1/2) X^2 + (1/2) Y^2$, Assumptions 1, 2(i), and 3(iii) as well as part (c) of Lemma S2-1 to get

$$
\begin{aligned}
& \frac{2}{n} E\left[\left(\sum_{(j,s),(k,v)=1}^{m_n} \sum_{(i,t)\neq\{(j,s),(k,v)\}} A_{(i,t),(j,s)} A_{(i,t),(k,v)} a' \gamma_{(j,s)} \varsigma_{(i,t)} u_{b,(k,v)}\right)^2 |\mathcal{F}_n^W\right] \\
\leq\; & \frac{2}{n} \sum_{(j,s),(k,v),(\ell,h)=1}^{m_n} \sum_{(i,t)\neq\{(j,s),(k,v),(\ell,h)\}} \Big| A_{(i,t),(j,s)} A_{(i,t),(\ell,h)} A^2_{(i,t),(k,v)} \\
& \qquad \times a'\gamma_{(j,s)} a'\gamma_{(\ell,h)} E\left[\varsigma^2_{(i,t)} |\mathcal{F}_n^W\right] E\left[u^2_{b,(k,v)} |\mathcal{F}_n^W\right]\Big| \\
& + \frac{2}{n} \sum_{(j,s),(k,v),(\ell,h)=1}^{m_n} \sum_{(i,t)\neq\{(j,s),(k,v),(\ell,h)\}} \Big| A_{(i,t),(j,s)} A_{(i,t),(k,v)} A_{(k,v),(\ell,h)} A_{(k,v),(i,t)} \\
& \qquad \times a'\gamma_{(j,s)} a'\gamma_{(\ell,h)} E\left[\varsigma_{(i,t)} u_{b,(i,t)} |\mathcal{F}_n^W\right] E\left[\varsigma_{(k,v)} u_{b,(k,v)} |\mathcal{F}_n^W\right]\Big| \\
\leq\; & \left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon^4_{(i,t)} |\mathcal{F}_n^W\right]\right)^2 \frac{a'\Gamma'\Gamma a}{n} \frac{1}{\left(\mu_n^{\min}\right)^2} \sum_{(i,t)=1}^{m_n} \sum_{\substack{(k,v),(\ell,h)=1 \\ (k,v)\neq(i,t) \\ (\ell,h)\neq(i,t)}}^{m_n} A^2_{(i,t),(\ell,h)} A^2_{(i,t),(k,v)} \\
& + \left(\max_{1\leq(i,t)\leq m_n} E\left[\left\|U_{(i,t)}\right\|_2^2 |\mathcal{F}_n^W\right]\right)^2 \frac{a'\Gamma'\Gamma a}{n} \frac{1}{\left(\mu_n^{\min}\right)^2} \sum_{(i,t)=1}^{m_n} \sum_{\substack{(k,v),(j,s)=1 \\ (k,v)\neq(i,t) \\ (j,s)\neq(i,t)}}^{m_n} A^2_{(i,t),(j,s)} A^2_{(i,t),(k,v)} \\
& + \left(\max_{1\leq(i,t)\leq m_n} E\left[\left\|U_{(i,t)}\right\|_2^2 |\mathcal{F}_n^W\right]\right) \left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon^4_{(i,t)} |\mathcal{F}_n^W\right]\right) \frac{a'\Gamma'\Gamma a}{n\left(\mu_n^{\min}\right)^2} \\
& \quad \times \sum_{(k,v)=1}^{m_n} \sum_{\substack{(i,t),(\ell,h)=1 \\ (i,t)\neq(k,v),(\ell,h)\neq(k,v)}}^{m_n} A^2_{(k,v),(\ell,h)} A^2_{(i,t),(k,v)} \\
& + \left(\max_{1\leq(i,t)\leq m_n} E\left[\left\|U_{(i,t)}\right\|_2^2 |\mathcal{F}_n^W\right]\right) \left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon^4_{(i,t)} |\mathcal{F}_n^W\right]\right) \frac{a'\Gamma'\Gamma a}{n\left(\mu_n^{\min}\right)^2} \\
& \quad \times \sum_{(i,t)=1}^{m_n} \sum_{\substack{(k,v),(j,s)=1 \\ (k,v)\neq(i,t),(j,s)\neq(i,t)}}^{m_n} A^2_{(i,t),(j,s)} A^2_{(k,v),(i,t)} \\
=\; & O_{a.s.}\left(\frac{K^2_{2,n}}{\left(\mu_n^{\min}\right)^2 n}\right)
\end{aligned}
$$

It follows from these results that

$$
\begin{aligned}
& E\left[\left(\frac{a'\Gamma' AD\left(\varepsilon\circ\varepsilon\right)AUD_{\mu}^{-1}b}{\sqrt{n}}\right)^{2}|\mathcal{F}_{n}^{W}\right] \\
\leq\; & \frac{2}{n}E\left[\left(a'\Gamma' AD_{\sigma^{2}}Au_{b}\right)^{2}|\mathcal{F}_{n}^{W}\right] \\
& +\frac{2}{n}E\left[\left(\sum_{(j,s),(k,v)=1}^{m_{n}}\sum_{(i,t)\neq\{(j,s),(k,v)\}}A_{(i,t),(j,s)}A_{(i,t),(k,v)}a'\gamma_{(j,s)}\varsigma_{(i,t)}u_{b,(k,v)}\right)^{2}|\mathcal{F}_{n}^{W}\right] \\
=\; & O_{a.s.}\left(\frac{1}{\left(\mu_{n}^{\min}\right)^{2}}\right)+O_{a.s.}\left(\frac{K_{2,n}^{2}}{\left(\mu_{n}^{\min}\right)^{2}n}\right)=o_{a.s.}\left(1\right)
\end{aligned}
$$

Now, by the conditional version of the Markov's inequality, we deduce that, for any $\epsilon>0$,

$$
\Pr\left(\left|\frac{a'\Gamma' AD\left(\varepsilon\circ\varepsilon\right)AUD_{\mu}^{-1}b}{\sqrt{n}}\right|\geq\epsilon\mid\mathcal{F}_{n}^{W}\right)\rightarrow 0\quad a.s.
$$

Since

$$
\sup_{n}E\left[\left|\Pr\left(\left|\frac{a'\Gamma' AD\left(\varepsilon\circ\varepsilon\right)AUD_{\mu}^{-1}b}{\sqrt{n}}\right|\geq\epsilon\mid\mathcal{F}_{n}^{W}\right)\right|^{2}\right]<\infty,
$$

it then follows by a version of the dominated convergence theorem, as given by Theorem 25.12 of Billingsley (1986), that as $n\rightarrow\infty$

$$
\begin{aligned}
& \Pr\left(\left|\frac{a'\Gamma' AD\left(\varepsilon\circ\varepsilon\right)AUD_{\mu}^{-1}b}{\sqrt{n}}\right|\geq\epsilon\right) \\
=\; & E\left[\Pr\left(\left|\frac{a'\Gamma' AD\left(\varepsilon\circ\varepsilon\right)AUD_{\mu}^{-1}b}{\sqrt{n}}\right|\geq\epsilon\mid\mathcal{F}_{n}^{W}\right)\right]\rightarrow 0,
\end{aligned}
$$

Finally, since the above result holds for all $a,b\in\mathbb{R}^{d}$ such that $\left\|a\right\|_{2}=\left\|b\right\|_{2}=1$, we further deduce that

$$
\frac{\Gamma' AD\left(\varepsilon\circ\varepsilon\right)AUD_{\mu}^{-1}}{\sqrt{n}}\overset{p}{\rightarrow}0,\text{ as }n\rightarrow\infty
$$

as required for part (b).

To show part (c), let a, $b\in\mathbb{R}^{d}$ such that $\left\|a\right\|_{2}=\left\|b\right\|_{2}=1$, and define $u_{b,(k,v)}=b'D_{\mu}^{-1}U_{(k,v)}$,

$u_b = UD_\mu^{-1}b$, and $D_{\sigma^2} = diag\left(\sigma^2_{(1,1)}, ..., \sigma^2_{(n,T_n)}\right)$. Applying Loève's c_r inequality, we obtain

$$
\begin{aligned}
& E\left[\left(\frac{a'D_\mu^{-1}D_\kappa F'AD\left(\varepsilon\circ\varepsilon\right)AUD_\mu^{-1}b}{\sqrt{n}}\right)^2 |\mathcal{F}_n^W\right] \\
= & \frac{1}{n}E\left[\left(\sum_{(j,s),(k,v)=1}^{m_n}\sum_{(i,t)\neq\{(j,s),(k,v)\}}A_{(i,t),(j,s)}A_{(i,t),(k,v)}a'D_\mu^{-1}D_\kappa f_{(j,s)}\varepsilon^2_{(i,t)}u_{b,(k,v)}\right)^2 |\mathcal{F}_n^W\right] \\
\leq & \frac{2}{n}E\left[\left(\sum_{(j,s),(k,v)=1}^{m_n}\sum_{(i,t)\neq\{(j,s),(k,v)\}}A_{(i,t),(j,s)}A_{(i,t),(k,v)}a'D_\mu^{-1}D_\kappa f_{(j,s)}E\left[\varepsilon^2_{(i,t)}|\mathcal{F}_n^W\right]u_{b,(k,v)}\right)^2 |\mathcal{F}_n^W\right] \\
& +\frac{2}{n}E\left[\left(\sum_{\substack{(i,t),(j,s),(k,v)=1\\(i,t)\neq(j,s)\\(i,t)\neq(k,v)}}^{m_n}A_{(i,t),(j,s)}A_{(i,t),(k,v)}a'D_\mu^{-1}D_\kappa f_{(j,s)}\left\{\varepsilon^2_{(i,t)}-E\left[\varepsilon^2_{(i,t)}|\mathcal{F}_n^W\right]\right\}u_{b,(k,v)}\right)^2 |\mathcal{F}_n^W\right] \\
= & \frac{2}{n}E\left[\left(a'D_\mu^{-1}D_\kappa F'AD_{\sigma^2}Au_b\right)^2|\mathcal{F}_n^W\right] \\
& +\frac{2}{n}E\left[\left(\sum_{(j,s),(k,v)=1}^{m_n}\sum_{(i,t)\neq\{(j,s),(k,v)\}}A_{(i,t),(j,s)}A_{(i,t),(k,v)}a'D_\mu^{-1}D_\kappa f_{(j,s)}\varsigma_{(i,t)}u_{b,(k,v)}\right)^2 |\mathcal{F}_n^W\right] \qquad (68)
\end{aligned}
$$

where $\varsigma_{(i,t)} = \varepsilon^2_{(i,t)} - E\left[\varepsilon^2_{(i,t)}|\mathcal{F}_n^W\right]$. Focusing on the first term of expression (68) above, we can apply Assumptions 1, 2(i), 3(ii), 4(i), 5, and 7(ii) as well as part (b) of Lemma OA-1 to obtain

$$
\begin{aligned}
& \frac{2}{n}E\left[\left(a'D_\mu^{-1}D_\kappa F'AD_{\sigma^2}Au_b\right)^2|\mathcal{F}_n^W\right] \\
= & \frac{2}{n}a'D_\mu^{-1}D_\kappa F'AD_{\sigma^2}AE\left[u_bu_b'|\mathcal{F}_n^W\right]AD_{\sigma^2}AFD_\kappa D_\mu^{-1}a \\
\leq & \frac{1}{\left(\mu_n^{\min}\right)^2}\left(\max_{1\leq(i,t)\leq m_n}E\left[\left\|U_{(i,t)}\right\|_2^2|\mathcal{F}_n^W\right]\right) \\
& \times\frac{a'D_\mu^{-1}D_\kappa\left(F-Z_1\Theta_{K_{1,n}}\right)'AD_{\sigma^2}A^2D_{\sigma^2}A\left(F-Z_1\Theta_{K_{1,n}}\right)D_\kappa D_\mu^{-1}a}{n} \\
\leq & \frac{1}{\left(\mu_n^{\min}\right)^2}\left(\max_{1\leq(i,t)\leq m_n}E\left[\left\|U_{(i,t)}\right\|_2^2|\mathcal{F}_n^W\right]\right)\left(1+\max_{1\leq(i,t)\leq m_n}\left|\widehat{\vartheta}_{(i,t)}\right|^2\right)^2 \\
& \times\left(\max_{1\leq(i,t)\leq m_n}E\left[\varepsilon^2_{(i,t)}|\mathcal{F}_n^W\right]\right)^2\frac{\left(\kappa_n^{\max}\right)^2}{\left(\mu_n^{\min}\right)^2}\frac{m_nd}{n}\left\|f\left(\cdot\right)-\Theta'_{K_{1,n}}Z_1\left(\cdot\right)\right\|_{\infty,d}^2 \\
= & O_{a.s.}\left(\frac{\left(\kappa_n^{\max}\right)^2}{\left(\mu_n^{\min}\right)^4}\frac{1}{K_{1,n}^{2\varrho_f}}\right).
\end{aligned}
$$

Turning our attention now to the second term of (68), we first define

$$\begin{aligned}
D_{\varsigma} &= diag\left(E\left[\varsigma_{(1,1)}^{2}|\mathcal{F}_{n}^{W}\right],....,E\left[\varsigma_{(n,T_n)}^{2}|\mathcal{F}_{n}^{W}\right]\right), \\
D_{b} &= diag\left(E\left[u_{b,(1,1)}^{2}|\mathcal{F}_{n}^{W}\right],....,E\left[u_{b,(n,T_n)}^{2}|\mathcal{F}_{n}^{W}\right]\right), \text{ and} \\
D_{\varsigma b} &= diag\left(E\left[\varsigma_{(1,1)}u_{b,(1,1)}|\mathcal{F}_{n}^{W}\right],....,E\left[\varsigma_{(n,T_n)}u_{b,(n,T_n)}|\mathcal{F}_{n}^{W}\right]\right).
\end{aligned}$$

Next, let $e_{(i,t)}$ denote an $m_n \times 1$ elementary vector whose $(i,t)^{th}$ component is 1 and all other components are 0; and note that we can apply the inequality $|XY| \leq (1/2)\,X^2 + (1/2)\,Y^2$, part (b) of Lemma OA-1, part (b) of Lemma S2-1, as well as Assumptions 1, 2(i), 3(ii), 4(i), 5, and 7(ii) to obtain

$$\begin{aligned}
& \frac{2}{n}E\left[\left(\sum_{(j,s),(k,v)=1}^{m_n}\sum_{(i,t)\neq\{(j,s),(k,v)\}}A_{(i,t),(j,s)}A_{(i,t),(k,v)}a'D_\mu^{-1}D_\kappa f_{(j,s)}\varsigma_{(i,t)}u_{b,(k,v)}\right)^2|\mathcal{F}_n^W\right] \\
\leq\ & \frac{2}{n}\sum_{(j,s),(k,v),(\ell,h)=1}^{m_n}\sum_{(i,t)\neq\{(j,s),(k,v),(\ell,h)\}}A_{(i,t),(j,s)}A_{(i,t),(\ell,h)}A_{(i,t),(k,v)}^2a'D_\mu^{-1}D_\kappa f_{(j,s)} \\
& \qquad\times a'D_\mu^{-1}D_\kappa f_{(\ell,h)}E\left[\varsigma_{(i,t)}^2|\mathcal{F}_n^W\right]E\left[u_{b,(k,v)}^2|\mathcal{F}_n^W\right] \\
& +\frac{2}{n}\sum_{(j,s),(k,v),(\ell,h)=1}^{m_n}\sum_{(i,t)\neq\{(j,s),(k,v),(\ell,h)\}}A_{(i,t),(j,s)}A_{(i,t),(k,v)}A_{(k,v),(\ell,h)}A_{(k,v),(i,t)} \\
& \qquad\times a'D_\mu^{-1}D_\kappa f_{(j,s)}a'D_\mu^{-1}D_\kappa f_{(\ell,h)}E\left[\varsigma_{(i,t)}u_{b,(i,t)}|\mathcal{F}_n^W\right]E\left[\varsigma_{(k,v)}u_{b,(k,v)}|\mathcal{F}_n^W\right] \\
=\ & \frac{2}{n}a'D_\mu^{-1}D_\kappa F'A\sum_{(i,t)\neq\{(j,s),(k,v),(\ell,h)\}}e_{(i,t)}E\left[\varsigma_{(i,t)}^2|\mathcal{F}_n^W\right]e'_{(i,t)}AD_bAe_{(i,t)}e'_{(i,t)}AFD_\kappa D_\mu^{-1}a \\
& +\frac{2}{n}a'D_\mu^{-1}D_\kappa F'A\sum_{(i,t)\neq\{(j,s),(k,v),(\ell,h)\}}e_{(i,t)}E\left[\varsigma_{(i,t)}u_{b,(i,t)}|\mathcal{F}_n^W\right]e'_{(i,t)}\left(A\circ A\right)D_{\varsigma b}AFD_\kappa D_\mu^{-1}a \\
\leq\ & \frac{1}{n}a'D_\mu^{-1}D_\kappa F'AD_\varsigma AD_bA^2D_bAD_\varsigma AFD_\kappa D_\mu^{-1}a \\
& +\frac{1}{n}a'D_\mu^{-1}D_\kappa F'A^2FD_\kappa D_\mu^{-1}a+\frac{1}{n}a'D_\mu^{-1}D_\kappa F'AD_{\varsigma b}^2AFD_\kappa D_\mu^{-1}a \\
& +\frac{1}{n}a'D_\mu^{-1}D_\kappa F'AD_{\varsigma b}\left(A\circ A\right)^2D_{\varsigma b}AFD_\kappa D_\mu^{-1}a \\
\leq\ & \left(1+\max_{1\leq(i,t)\leq m_n}\left|\widehat{\vartheta}_{(i,t)}\right|^2\right)^3\left(\max_{1\leq(i,t)\leq m_n}E\left[\left\|U_{(i,t)}\right\|_2^2|\mathcal{F}_n^W\right]\right)^2\left(\max_{1\leq(i,t)\leq m_n}E\left[\varepsilon_{(i,t)}^4|\mathcal{F}_n^W\right]\right)^2 \\
& \times\frac{(\kappa_n^{\max})^2}{(\mu_n^{\min})^6}\frac{m_nd}{n}\left\|f(\cdot)-\Theta'_{K_{1,n}}Z_1(\cdot)\right\|_{\infty,d}^2 \\
& +\left(1+\max_{1\leq(i,t)\leq m_n}\left|\widehat{\vartheta}_{(i,t)}\right|^2\right)\frac{(\kappa_n^{\max})^2}{(\mu_n^{\min})^2}\frac{m_nd}{n}\left\|f(\cdot)-\Theta'_{K_{1,n}}Z_1(\cdot)\right\|_{\infty,d}^2 \\
& +\left(\max_{1\leq(i,t)\leq m_n}E\left[\varepsilon_{(i,t)}^4|\mathcal{F}_n^W\right]\right)\left(\max_{1\leq(i,t)\leq m_n}E\left[\left\|U_{(i,t)}\right\|_2^2|\mathcal{F}_n^W\right]\right)\left(1+\max_{1\leq(i,t)\leq m_n}\left|\widehat{\vartheta}_{(i,t)}\right|^2\right) \\
& \times\frac{(\kappa_n^{\max})^2}{(\mu_n^{\min})^4}\frac{m_nd}{n}\left\|f(\cdot)-\Theta'_{K_{1,n}}Z_1(\cdot)\right\|_{\infty,d}^2 \\
& +\left(\max_{1\leq(i,t)\leq m_n}E\left[\varepsilon_{(i,t)}^4|\mathcal{F}_n^W\right]\right)\left(\max_{1\leq(i,t)\leq m_n}E\left[\left\|U_{(i,t)}\right\|_2^2|\mathcal{F}_n^W\right]\right)\left(1+\max_{1\leq(i,t)\leq m_n}\left|\widehat{\vartheta}_{(i,t)}\right|^2\right) \\
& \times\frac{(\kappa_n^{\max})^2}{(\mu_n^{\min})^4}\frac{m_nd}{n}\left\|f(\cdot)-\Theta'_{K_{1,n}}Z_1(\cdot)\right\|_{\infty,d}^2\sum_{\substack{(i,t),(j,s)=1\\(i,t)\neq(j,s)}}^{m_n}A_{(i,t),(j,s)}^4
\end{aligned}$$

$$
\begin{aligned}
&= O_{a.s.}\left(\frac{(\kappa_n^{\max})^2}{(\mu_n^{\min})^6 K_{1,n}^{2\varrho_f}}\right) + O_{a.s.}\left(\frac{(\kappa_n^{\max})^2}{(\mu_n^{\min})^2 K_{1,n}^{2\varrho_f}}\right) + O_{a.s.}\left(\frac{(\kappa_n^{\max})^2}{(\mu_n^{\min})^4 K_{1,n}^{2\varrho_f}}\right) \\
&\quad + O_{a.s.}\left(\frac{(\kappa_n^{\max})^2 K_{2,n}^3}{(\mu_n^{\min})^4 K_{1,n}^{2\varrho_f} n^2}\right) \\
&= O_{a.s.}\left(\frac{(\kappa_n^{\max})^2}{(\mu_n^{\min})^2 K_{1,n}^{2\varrho_f}} \max\left\{1, \frac{K_{2,n}^3}{(\mu_n^{\min})^2 n^2}\right\}\right) = O_{a.s.}\left(\frac{(\kappa_n^{\max})^2}{(\mu_n^{\min})^2 K_{1,n}^{2\varrho_f}}\right) = o_{a.s.}(1).
\end{aligned}
$$

It follows from these results that

$$
\begin{aligned}
&E\left[\left(\frac{a' D_\mu^{-1} D_\kappa F' A D(\varepsilon \circ \varepsilon) A U D_\mu^{-1} b}{\sqrt{n}}\right)^2 |\mathcal{F}_n^W\right] \\
\leq\ & \frac{2}{n} E\left[\left(a' D_\mu^{-1} D_\kappa F' A D_{\sigma^2} A u_b\right)^2 |\mathcal{F}_n^W\right] \\
&+ \frac{2}{n} E\left[\left(\sum_{(j,s),(k,v)=1}^{m_n} \sum_{(i,t)\neq\{(j,s),(k,v)\}} A_{(i,t),(j,s)} A_{(i,t),(k,v)} a' D_\mu^{-1} D_\kappa f_{(j,s)} \varsigma_{(i,t)} u_{b,(k,v)}\right)^2 |\mathcal{F}_n^W\right] \\
=\ & O_{a.s.}\left(\frac{(\kappa_n^{\max})^2}{(\mu_n^{\min})^4} \frac{1}{K_{1,n}^{2\varrho_f}}\right) + O_{a.s.}\left(\frac{(\kappa_n^{\max})^2}{(\mu_n^{\min})^2 K_{1,n}^{2\varrho_f}}\right) = O_{a.s.}\left(\frac{(\kappa_n^{\max})^2}{(\mu_n^{\min})^2 K_{1,n}^{2\varrho_f}}\right) = o_{a.s.}(1).
\end{aligned}
$$

Now, by the conditional version of the Markov's inequality, we deduce that, for any $\epsilon > 0$,

$$
\Pr\left(\left|\frac{a' D_\mu^{-1} D_\kappa F' A D(\varepsilon \circ \varepsilon) A U D_\mu^{-1} b}{\sqrt{n}}\right| \geq \epsilon \mid \mathcal{F}_n^W\right) \to 0 \quad a.s.
$$

Since

$$
\sup_n E\left[\left|\Pr\left(\left|\frac{a' D_\mu^{-1} D_\kappa F' A D(\varepsilon \circ \varepsilon) A U D_\mu^{-1} b}{\sqrt{n}}\right| \geq \epsilon \mid \mathcal{F}_n^W\right)\right|^2\right] < \infty,
$$

it then follows by a version of the dominated convergence theorem, as given by Theorem 25.12 of Billingsley (1986), that as $n \to \infty$

$$
\begin{aligned}
&\Pr\left(\left|\frac{a' D_\mu^{-1} D_\kappa F' A D(\varepsilon \circ \varepsilon) A U D_\mu^{-1} b}{\sqrt{n}}\right| \geq \epsilon\right) \\
=\ & E\left[\Pr\left(\left|\frac{a' D_\mu^{-1} D_\kappa F' A D(\varepsilon \circ \varepsilon) A U D_\mu^{-1} b}{\sqrt{n}}\right| \geq \epsilon \mid \mathcal{F}_n^W\right)\right] \to 0,
\end{aligned}
$$

Finally, since the above result holds for all $a, b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$, we further deduce that

$$
\frac{D_\mu^{-1} D_\kappa F' A D(\varepsilon \circ \varepsilon) A U D_\mu^{-1}}{\sqrt{n}} \xrightarrow{p} 0, \text{ as } n \to \infty
$$

as required for part (c).

To show part (d), note that, for any a, $b \in \mathbb{R}^d$ such that $\|a\| = \|b\| = 1$, we have by the CS inequality

$$\begin{aligned} & \left| \frac{a' D_\mu^{-1} D_\kappa F' A D \left(\varepsilon \circ \varepsilon \right) A F D_\kappa D_\mu^{-1} b}{n} \right| \\ \leq & \sqrt{\frac{a' D_\mu^{-1} D_\kappa F' A D \left(\varepsilon \circ \varepsilon \right) A F D_\kappa D_\mu^{-1} a}{n}} \\ & \times \sqrt{\frac{b' D_\mu^{-1} D_\kappa F' A D \left(\varepsilon \circ \varepsilon \right) A F D_\kappa D_\mu^{-1} b}{n}}. \end{aligned}$$

Next, define $D_{\sigma^2} = diag\left(E\left[\varepsilon_{(1,1)}^2 | \mathcal{F}_n^W \right], ..., E\left[\varepsilon_{(n,T_n)}^2 | \mathcal{F}_n^W \right] \right)$. Note that, by applying Assumptions 2(i), 3(ii), 4(i), 5, and 7(ii) as well as part (b) of Lemma OA-1; we obtain

$$\begin{aligned} & E\left[\frac{a' D_\mu^{-1} D_\kappa F' A D \left(\varepsilon \circ \varepsilon \right) A F D_\kappa D_\mu^{-1} a}{n} | \mathcal{F}_n^W \right] \\ = & \frac{a' D_\mu^{-1} D_\kappa F' A D_{\sigma^2} A F D_\kappa D_\mu^{-1} a}{n} \\ \leq & \left(\max_{1 \leq (i,t) \leq m_n} E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^W \right] \right) \left(1 + \max_{1 \leq (i,t) \leq m_n} \left| \widehat{\vartheta}_{(i,t)} \right|^2 \right) \\ & \times \frac{\left(\kappa_n^{\max} \right)^2}{\left(\mu_n^{\min} \right)^2} \frac{m_n d}{n} \left\| f\left(\cdot \right) - \Theta_{K_{1,n}}' Z_1\left(\cdot \right) \right\|_{\infty,d}^2 \\ = & O_{a.s.}\left(\frac{\left(\kappa_n^{\max} \right)^2}{\left(\mu_n^{\min} \right)^2 K_{1,n}^{2\varrho_f}} \right) = o_{a.s.}\left(1 \right). \end{aligned}$$

It follows by applying the conditional version of Markov's inequality and Theorem 25.12 of Billingsley (1986) that

$$\frac{a' D_\mu^{-1} D_\kappa F' A D \left(\varepsilon \circ \varepsilon \right) A F D_\kappa D_\mu^{-1} a}{n} = o_p\left(1 \right)$$

Similarly, we also have

$$\frac{b' D_\mu^{-1} D_\kappa F' A D \left(\varepsilon \circ \varepsilon \right) A F D_\kappa D_\mu^{-1} b}{n} = o_p\left(1 \right).$$

Combining these results, we obtain

$$
\begin{aligned}
&\left|\frac{a'D_\mu^{-1}D_\kappa F'AD\left(\varepsilon\circ\varepsilon\right)AFD_\kappa D_\mu^{-1}b}{n}\right| \\
\leq\ & \sqrt{\frac{a'D_\mu^{-1}D_\kappa F'AD\left(\varepsilon\circ\varepsilon\right)AFD_\kappa D_\mu^{-1}a}{n}} \\
&\times\sqrt{\frac{b'D_\mu^{-1}D_\kappa F'AD\left(\varepsilon\circ\varepsilon\right)AFD_\kappa D_\mu^{-1}b}{n}} \\
=\ & o_p\left(1\right).
\end{aligned}
$$

Moreover, since the above result holds for all $a, b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$, we further deduce that

$$
\frac{D_\mu^{-1}D_\kappa F'AD\left(\varepsilon\circ\varepsilon\right)AFD_\kappa D_\mu^{-1}}{n} = o_p\left(1\right)
$$

as required for part (d).

Finally, consider part (e). For any $a, b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$, we can apply the CS inequality

$$
\begin{aligned}
&\left|\frac{a'D_\mu^{-1}D_\kappa F'AD\left(\varepsilon\circ\varepsilon\right)A\Gamma b}{n}\right| \\
=\ & \left|\frac{a'D_\mu^{-1}D_\kappa F'AD\left(\varepsilon\right)^2 A\Gamma b}{n}\right| \\
\leq\ & \sqrt{\frac{a'D_\mu^{-1}D_\kappa F'AD\left(\varepsilon\circ\varepsilon\right)AFD_\kappa D_\mu^{-1}a}{n}}\sqrt{\frac{b'\Gamma' AD\left(\varepsilon\circ\varepsilon\right)A\Gamma b}{n}}
\end{aligned}
$$

Now, from the result of part (d) above, we see that

$$
\frac{a'D_\mu^{-1}D_\kappa F'AD\left(\varepsilon\circ\varepsilon\right)AFD_\kappa D_\mu^{-1}a}{n} = o_p\left(1\right)
$$

In addition, let $D_{\sigma^2} = diag\left(E\left[\varepsilon_{(1,1)}^2|\mathcal{F}_n^W\right], ..., E\left[\varepsilon_{(n,T_n)}^2|\mathcal{F}_n^W\right]\right)$ as before. Note that, by applying Assumptions 2(i), 3(iii), 5, and 6 as well as part (b) of Lemma OA-1; we get

$$
\begin{aligned}
E\left[\frac{b'\Gamma' AD\left(\varepsilon\circ\varepsilon\right)A\Gamma b}{n}|\mathcal{F}_n^W\right] &= \frac{b'\Gamma' AD_{\sigma^2}A\Gamma b}{n} \\
&\leq \left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right]\right)\frac{b'\Gamma' A^2\Gamma b}{n} \\
&\leq \left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^W\right]\right)\left(1+\max_{1\leq(i,t)\leq m_n}\left|\widehat{\vartheta}_{(i,t)}\right|^2\right)\frac{b'\Gamma'\Gamma b}{n} \\
&= O_{a.s.}\left(1\right)
\end{aligned}
$$

from which it follows by applying a conditional version of Markov's inequality and Theorem 25.12

of Billingsley (1986) that

$$\frac{b'\Gamma' AD\left(\varepsilon\circ\varepsilon\right)A\Gamma b}{n}=O_p\left(1\right).$$

Using these results, we further deduce that

$$\begin{aligned}
&\left|\frac{a'D_{\mu}^{-1}D_{\kappa}F'AD\left(\varepsilon\circ\varepsilon\right)A\Gamma b}{n}\right| \\
\leq\ & \sqrt{\frac{a'D_{\mu}^{-1}D_{\kappa}F'AD\left(\varepsilon\circ\varepsilon\right)AFD_{\kappa}D_{\mu}^{-1}a}{n}}\sqrt{\frac{b'\Gamma' AD\left(\varepsilon\circ\varepsilon\right)A\Gamma b}{n}} \\
=\ & o_p\left(1\right).
\end{aligned}$$

Finally, since the above result holds for all $a, b\in\mathbb{R}^d$ such that $\left\|a\right\|_2=\left\|b\right\|_2=1$, we conclude

$$\frac{D_{\mu}^{-1}D_{\kappa}F'AD\left(\varepsilon\circ\varepsilon\right)A\Gamma}{n}=o_p\left(1\right).\ \square$$

Lemma OA-11:

Under Assumptions 5 and 6, the following statements are true.

(a)

$$\begin{aligned}
tr\left\{A^4\right\} &= tr\left\{\left(P^{\perp}-M^{(Z,Q)}D_{\widehat{\vartheta}}M^{(Z,Q)}\right)^4\right\} \\
&= O_{a.s.}\left(K_{2,n}\right).
\end{aligned}$$

(b) $\left|S_n\right|=O_{a.s.}\left(K_{2,n}\right)$, where

$$\begin{aligned}
S_n = \sum_{1\leq(i,t)<(j,s)<(k,v)<(\ell,h)\leq m_n} \big(&A_{(i,t),(j,s)}A_{(j,s),(k,v)}A_{(i,t),(\ell,h)}A_{(k,v),(\ell,h)} \\
&+A_{(i,t),(j,s)}A_{(i,t),(k,v)}A_{(j,s),(\ell,h)}A_{(k,v),(\ell,h)} \\
&+A_{(i,t),(k,v)}A_{(j,s),(k,v)}A_{(i,t),(\ell,h)}A_{(j,s),(\ell,h)}\big)
\end{aligned}$$

(c) $\left|\sum_{1\leq(i,t)<(j,s)<(k,v)<(\ell,h)\leq m_n}A_{(i,t),(k,v)}A_{(j,s),(k,v)}A_{(i,t),(\ell,h)}A_{(j,s),(\ell,h)}\right|=O_{a.s.}\left(K_{2,n}\right).$

Proof of Lemma OA-11:

To show part (a), note first that $P^{\perp}M^{(Z,Q)} = 0 = M^{(Z,Q)}P^{\perp}$. Now, write

$$
\begin{aligned}
& \left(P^{\perp} - M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\right)^4 \\
= & \left[\left(P^{\perp} - M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\right)\left(P^{\perp} - M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\right)\right]^2 \\
= & \left(P^{\perp} + M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\right)^2 \\
= & \left(P^{\perp} + M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\right)\left(P^{\perp} + M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\right) \\
= & P^{\perp} + M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& tr\left\{\left(P^{\perp} - M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\right)^4\right\} \\
= & tr\left\{P^{\perp}\right\} + tr\left\{M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\right\} \\
= & K_{2,n} + tr\left\{M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\right\}
\end{aligned}
$$

In addition, note that

$$
\begin{aligned}
0 & \leq tr\left\{M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\right\} \\
& \leq tr\left\{M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}}^2 M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\right\} \\
& \leq \left(\max_{(i,t)\in\Lambda_1} \left|\widehat{\vartheta}_{(i,t)}\right|^2\right) tr\left\{M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\right\} \\
& \leq \left(\max_{(i,t)\in\Lambda_1} \left|\widehat{\vartheta}_{(i,t)}\right|^2\right) tr\left\{M^{(Z,Q)} D_{\widehat{\vartheta}}^2 M^{(Z,Q)}\right\} \\
& = \left(\max_{(i,t)\in\Lambda_1} \left|\widehat{\vartheta}_{(i,t)}\right|^2\right) tr\left\{D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}}\right\} \\
& \leq \left(\max_{(i,t)\in\Lambda_1} \left|\widehat{\vartheta}_{(i,t)}\right|^2\right) tr\left\{D_{\widehat{\vartheta}}^2\right\}
\end{aligned}
$$

It follows from applying parts (a) and (b) of Lemma OA-1 that

$$
\begin{aligned}
tr\left\{A^4\right\} & = tr\left\{\left(P^{\perp} - M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\right)^4\right\} \\
& = K_{2,n} + tr\left\{M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\right\} \\
& = K_{2,n} + \left(\max_{(i,t)\in\Lambda_1} \left|\widehat{\vartheta}_{(i,t)}\right|^2\right) tr\left\{D_{\widehat{\vartheta}}^2\right\} \\
& = O_{a.s.}\left(K_{2,n}\right).
\end{aligned}
$$

Next, to show part (b), decompose $A = P^{\perp} - M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}$ as follows

$$A = L + L',$$

where L be the lower triangular matrix such that $L_{(i,t),(j,s)} = A_{(i,t),(j,s)} \mathbb{I}\{(i,t) > (j,s)\}$, i.e., L is lower triangular matrix whose lower triangular elements correspond to the lower triangular elements of A. It follows that

$$\begin{aligned}
& A^4 \\
= \;& (L + L')^4 = \left(L^2 + LL' + L'L + (L')^2\right)^2 \\
= \;& L^4 + L^3 L' + L^2 L' L + L^2 (L')^2 + LL'L^2 + LL'LL' + L (L')^2 L + L (L')^3 \\
& + L'L^3 + L'L^2 L' + L'LL'L + L'L (L')^2 + (L')^2 L^2 + (L')^2 LL' + (L')^3 L + (L')^4
\end{aligned}$$

Using the fact that $tr\{AB\} = tr\{BA\}$ and $tr\{A\} = tr\{A'\}$, we obtain

$$tr\left\{A^4\right\} = 2tr\left\{L^4\right\} + 8tr\left\{L^3 L'\right\} + 4tr\left\{L^2 (L')^2\right\} + 2tr\left\{LL'LL'\right\}$$

We compute each of the terms on the right-hand side above as follows.

$$\begin{aligned}
& tr\left\{L^4\right\} \\
= \;& \sum_{(i,t)} \sum_{(j,s)} \sum_{(k,\upsilon)} \sum_{(\ell,h)} \left[A_{(i,t),(j,s)} \mathbb{I}\{(i,t) > (j,s)\} A_{(j,s),(k,\upsilon)} \mathbb{I}\{(j,s) > (k,\upsilon)\}\right. \\
& \qquad \left. \times A_{(k,\upsilon),(\ell,h)} \mathbb{I}\{(k,\upsilon) > (\ell,h)\} A_{(\ell,h),(i,t)} \mathbb{I}\{(\ell,h) > (i,t)\}\right] \\
= \;& 0
\end{aligned}$$

$$\begin{aligned}
& tr\left\{L^3 L'\right\} \\
= \;& \sum_{(i,t)} \sum_{(j,s)} \sum_{(k,\upsilon)} \sum_{(\ell,h)} \left[A_{(i,t),(j,s)} \mathbb{I}\{(i,t) > (j,s)\} A_{(j,s),(k,\upsilon)} \mathbb{I}\{(j,s) > (k,\upsilon)\}\right. \\
& \qquad \left. \times A_{(k,\upsilon),(\ell,h)} \mathbb{I}\{(k,\upsilon) > (\ell,h)\} A_{(\ell,h),(i,t)} \mathbb{I}\{(i,t) > (\ell,h)\}\right] \\
= \;& \sum_{1 \leq (\ell,h) < (k,\upsilon) < (j,s) < (i,t) \leq m_n} A_{(i,t),(j,s)} A_{(j,s),(k,\upsilon)} A_{(k,\upsilon),(\ell,h)} A_{(\ell,h),(i,t)} \\
= \;& \sum_{1 \leq (i,t) < (j,s) < (k,\upsilon) < (\ell,h) \leq m_n} A_{(\ell,h),(k,\upsilon)} A_{(k,\upsilon),(j,s)} A_{(j,s),(i,t)} A_{(i,t),(\ell,h)} \\
= \;& \sum_{1 \leq (i,t) < (j,s) < (k,\upsilon) < (\ell,h) \leq m_n} A_{(i,t),(j,s)} A_{(j,s),(k,\upsilon)} A_{(\ell,h),(i,t)} A_{(k,\upsilon),(\ell,h)} \text{ (by symmetry of } A)
\end{aligned}$$

$$
\begin{aligned}
& tr\left\{L^2\left(L'\right)^2\right\} \\
= & \sum_{(i,t)}\sum_{(j,s)}\sum_{(k,\upsilon)}\sum_{(\ell,h)}\left[A_{(i,t),(j,s)}\mathbb{I}\left\{(i,t)>(j,s)\right\}A_{(j,s),(k,\upsilon)}\mathbb{I}\left\{(j,s)>(k,\upsilon)\right\}\right. \\
& \qquad\qquad\qquad \left.\times A_{(k,\upsilon),(\ell,h)}\mathbb{I}\left\{(\ell,h)>(k,\upsilon)\right\}A_{(\ell,h),(i,t)}\mathbb{I}\left\{(i,t)>(\ell,h)\right\}\right] \\
= & \sum_{1\le(k,\upsilon)<(\ell,h)=(j,s)<(i,t)\le m_n} A_{(i,t),(j,s)}A_{(j,s),(k,\upsilon)}A_{(k,\upsilon),(\ell,h)}A_{(\ell,h),(i,t)} \\
& +\sum_{1\le(k,\upsilon)<(\ell,h)<(j,s)<(i,t)\le m_n} A_{(i,t),(j,s)}A_{(j,s),(k,\upsilon)}A_{(k,\upsilon),(\ell,h)}A_{(\ell,h),(i,t)} \\
& +\sum_{1\le(k,\upsilon)<(j,s)<(\ell,h)<(i,t)\le m_n} A_{(i,t),(j,s)}A_{(j,s),(k,\upsilon)}A_{(k,\upsilon),(\ell,h)}A_{(\ell,h),(i,t)} \\
= & \sum_{1\le(k,\upsilon)<(j,s)<(i,t)\le m_n} A_{(i,t),(j,s)}A_{(j,s),(k,\upsilon)}A_{(k,\upsilon),(j,s)}A_{(j,s),(i,t)} \\
& +\sum_{1\le(k,\upsilon)<(\ell,h)<(j,s)<(i,t)\le m_n} \left[A_{(i,t),(j,s)}A_{(j,s),(k,\upsilon)}A_{(k,\upsilon),(\ell,h)}A_{(\ell,h),(i,t)}\right. \\
& \qquad\qquad\qquad \left.+A_{(i,t),(\ell,h)}A_{(\ell,h),(k,\upsilon)}A_{(k,\upsilon),(j,s)}A_{(j,s),(i,t)}\right] \\
= & \sum_{1\le(i,t)<(j,s)<(k,\upsilon)\le m_n} A^2_{(i,t),(j,s)}A^2_{(j,s),(k,\upsilon)} \\
& +\sum_{1\le(i,t)<(j,s)<(k,\upsilon)<(\ell,h)\le m_n} \left[A_{(\ell,h),(k,\upsilon)}A_{(k,\upsilon),(i,t)}A_{(i,t),(j,s)}A_{(j,s),(\ell,h)}\right. \\
& \qquad\qquad\qquad \left.+A_{(\ell,h),(j,s)}A_{(j,s),(i,t)}A_{(i,t),(k,\upsilon)}A_{(k,\upsilon),(\ell,h)}\right] \\
= & \sum_{1\le(i,t)<(j,s)<(k,\upsilon)\le m_n} A^2_{(i,t),(j,s)}A^2_{(j,s),(k,\upsilon)} \\
& +2\sum_{1\le(i,t)<(j,s)<(k,\upsilon)<(\ell,h)\le m_n} A_{(i,t),(j,s)}A_{(i,t),(k,\upsilon)}A_{(j,s),(\ell,h)}A_{(k,\upsilon),(\ell,h)}
\end{aligned}
$$

$$
\begin{aligned}
& tr\left\{LL'LL'\right\} \\
= \; & \sum_{(i,t)}\sum_{(j,s)}\sum_{(k,\upsilon)}\sum_{(\ell,h)}\left[A_{(i,t),(j,s)}\mathbb{I}\left\{(i,t)>(j,s)\right\}A_{(j,s),(k,\upsilon)}\mathbb{I}\left\{(k,\upsilon)>(j,s)\right\}\right. \\
& \qquad\qquad\qquad \left.\times A_{(k,\upsilon),(\ell,h)}\mathbb{I}\left\{(k,\upsilon)>(\ell,h)\right\}A_{(\ell,h),(i,t)}\mathbb{I}\left\{(i,t)>(\ell,h)\right\}\right] \\
= \; & \sum_{1\le(j,s)<(i,t)\le m_n} A_{(i,t),(j,s)}A_{(j,s),(i,t)}A_{(i,t),(j,s)}A_{(j,s),(i,t)} \\
& +\sum_{1\le(j,s)<(k,\upsilon)<(i,t)\le m_n} A_{(i,t),(j,s)}A_{(j,s),(k,\upsilon)}A_{(k,\upsilon),(j,s)}A_{(j,s),(i,t)} \\
& +\sum_{1\le(j,s)<(i,t)<(k,\upsilon)\le m_n} A_{(i,t),(j,s)}A_{(j,s),(k,\upsilon)}A_{(k,\upsilon),(j,s)}A_{(j,s),(i,t)} \\
& +\sum_{1\le(j,s)<(\ell,h)<(i,t)\le m_n} A_{(i,t),(j,s)}A_{(j,s),(i,t)}A_{(i,t),(\ell,h)}A_{(\ell,h),(i,t)} \\
& +\sum_{1\le(\ell,h)<(j,s)<(i,t)\le m_n} A_{(i,t),(j,s)}A_{(j,s),(i,t)}A_{(i,t),(\ell,h)}A_{(\ell,h),(i,t)} \\
& +\left(\sum_{1\le(\ell,h)<(j,s)<(k,\upsilon)<(i,t)\le m_n}+\sum_{1\le(j,s)<(\ell,h)<(k,\upsilon)<(i,t)\le m_n}\right)A_{(i,t),(j,s)}A_{(j,s),(k,\upsilon)}A_{(k,\upsilon),(\ell,h)}A_{(\ell,h),(i,t)} \\
& +\left(\sum_{1\le(\ell,h)<(j,s)<(i,t)<(k,\upsilon)\le m_n}+\sum_{1\le(j,s)<(\ell,h)<(i,t)<(k,\upsilon)\le m_n}\right)A_{(i,t),(j,s)}A_{(j,s),(k,\upsilon)}A_{(k,\upsilon),(\ell,h)}A_{(\ell,h),(i,t)} \\
= \; & \sum_{1\le(i,t)<(j,s)\le m_n} A^4_{(i,t),(j,s)}+2\sum_{1\le(i,t)<(j,s)<(k,\upsilon)\le m_n}\left(A^2_{(i,t),(j,s)}A^2_{(i,t),(k,\upsilon)}+A^2_{(i,t),(k,\upsilon)}A^2_{(j,s),(k,\upsilon)}\right) \\
& +4\sum_{1\le(i,t)<(j,s)<(k,\upsilon)<(\ell,h)\le m_n} A_{(i,t),(k,\upsilon)}A_{(j,s),(k,\upsilon)}A_{(i,t),(\ell,h)}A_{(j,s),(\ell,h)}
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& tr\left\{A^4\right\} \\
= \; & 8 \sum_{1\le(i,t)<(j,s)<(k,v)<(\ell,h)\le m_n} \left[A_{(i,t),(j,s)}A_{(j,s),(k,v)}A_{(i,t),(\ell,h)}A_{(k,v),(\ell,h)}\right. \\
& \qquad + A_{(i,t),(j,s)}A_{(i,t),(k,v)}A_{(j,s),(\ell,h)}A_{(k,v),(\ell,h)} \\
& \qquad \left. + A_{(i,t),(k,v)}A_{(j,s),(k,v)}A_{(i,t),(\ell,h)}A_{(j,s),(\ell,h)}\right] \\
& +2 \sum_{1\le(i,t)<(j,s)\le m_n} A^4_{(i,t),(j,s)} \\
& +4 \sum_{1\le(i,t)<(j,s)<(k,v)\le m_n} \left(A^2_{(i,t),(j,s)}A^2_{(i,t),(k,v)} + A^2_{(i,t),(k,v)}A^2_{(j,s),(k,v)} + A^2_{(i,t),(j,s)}A^2_{(j,s),(k,v)}\right) \\
= \; & 8S_n + 2\sum_{1\le(i,t)<(j,s)\le m_n} A^4_{(i,t),(j,s)} \\
& +4 \sum_{1\le(i,t)<(j,s)<(k,v)\le m_n} \left(A^2_{(i,t),(j,s)}A^2_{(i,t),(k,v)} + A^2_{(i,t),(k,v)}A^2_{(j,s),(k,v)} + A^2_{(i,t),(j,s)}A^2_{(j,s),(k,v)}\right)
\end{aligned}
$$

It follows from the triangle inequality, the result given in part (a) above, parts (b) and (c) of Lemma S2-1, and the symmetry of A that

$$
\begin{aligned}
|S_n| \; \le \; & \frac{1}{8}tr\left\{A^4\right\} + \frac{1}{4}\sum_{1\le(i,t)<(j,s)\le m_n} A^4_{(i,t),(j,s)} \\
& +\frac{1}{2}\sum_{1\le(i,t)<(j,s)<(k,v)\le m_n} \left(A^2_{(i,t),(j,s)}A^2_{(i,t),(k,v)} + A^2_{(i,t),(k,v)}A^2_{(j,s),(k,v)} + A^2_{(i,t),(j,s)}A^2_{(j,s),(k,v)}\right) \\
\le \; & \frac{1}{8}tr\left\{A^4\right\} + \frac{1}{4}\sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A^4_{(i,t),(j,s)} + \frac{3}{2}\sum_{(j,s)=1}^{m_n}\sum_{\substack{(i,t),(k,v)=1 \\ (i,t)\neq(j,s),(k,v)\neq(j,s)}}^{m_n} A^2_{(i,t),(j,s)}A^2_{(j,s),(k,v)} \\
= \; & O_{a.s.}\left(K_{2,n}\right) + O_{a.s.}\left(\frac{K^2_{2,n}}{n^2}\right) + O_{a.s.}\left(\frac{K^2_{2,n}}{n}\right) \\
= \; & O_{a.s.}\left(K_{2,n}\right)
\end{aligned}
$$

Finally, for part (c), we take $\left\{\eta_{(i,t)}\right\}$ to be a double-indexed sequence of $i.i.d.$ random variables with mean 0 and variance 1 and where $\eta_{(i,t)}$ and W_n are independent for all (i,t) and n. Define the random quantities

$$\begin{aligned}
\Delta_1 &= \sum_{1 \leq (i,t) < (j,s) < (k,v) \leq m_n} \Big[A_{(i,t),(j,s)} A_{(i,t),(k,v)} \eta_{(j,s)} \eta_{(k,v)} + A_{(i,t),(j,s)} A_{(j,s),(k,v)} \eta_{(i,t)} \eta_{(k,v)} \\
&\qquad + A_{(i,t),(k,v)} A_{(j,s),(k,v)} \eta_{(i,t)} \eta_{(j,s)} \Big] \\
\Delta_2 &= \sum_{1 \leq (i,t) < (j,s) < (k,v) \leq m_n} \Big[A_{(i,t),(j,s)} A_{(i,t),(k,v)} \eta_{(j,s)} \eta_{(k,v)} + A_{(i,t),(j,s)} A_{(j,s),(k,v)} \eta_{(i,t)} \eta_{(k,v)} \Big] \\
\Delta_3 &= \sum_{1 \leq (i,t) < (j,s) < (k,v) \leq m_n} A_{(i,t),(k,v)} A_{(j,s),(k,v)} \eta_{(i,t)} \eta_{(j,s)}
\end{aligned}$$

Next, note that using part (c) of Lemma S2-1 and the symmetry of A, we have

$$\begin{aligned}
E\left[\Delta_3^2 | \mathcal{F}_n^W\right] &= E\left[\left(\sum_{1 \leq (i,t) < (j,s) < (k,v) \leq m_n} A_{(i,t),(k,v)} A_{(j,s),(k,v)} \eta_{(i,t)} \eta_{(j,s)}\right)^2 | \mathcal{F}_n^W\right] \\
&= \sum_{1 \leq (i,t) < (j,s) < \{(k,v),(\ell,h)\} \leq m_n} A_{(i,t),(k,v)} A_{(j,s),(k,v)} A_{(i,t),(\ell,h)} A_{(j,s),(\ell,h)} \\
&= \sum_{1 \leq (i,t) < (j,s) < (k,v) \leq m_n} A^2_{(i,t),(k,v)} A^2_{(j,s),(k,v)} \\
&\quad +2 \sum_{1 \leq (i,t) < (j,s) < (k,v) < (\ell,h) \leq m_n} A_{(i,t),(k,v)} A_{(j,s),(k,v)} A_{(i,t),(\ell,h)} A_{(j,s),(\ell,h)} \\
&\leq \sum_{(k,v)=1}^{m_n} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t) \neq (j,s)}}^{m_n} A^2_{(i,t),(k,v)} A^2_{(k,v),(j,s)} \\
&\quad +2 \sum_{1 \leq (i,t) < (j,s) < (k,v) < (\ell,h) \leq m_n} A_{(i,t),(k,v)} A_{(j,s),(k,v)} A_{(i,t),(\ell,h)} A_{(j,s),(\ell,h)} \\
&= O_{a.s.}\left(\frac{K_{2,n}^2}{n}\right) + 2 \sum_{1 \leq (i,t) < (j,s) < (k,v) < (\ell,h) \leq m_n} A_{(i,t),(k,v)} A_{(j,s),(k,v)} A_{(i,t),(\ell,h)} A_{(j,s),(\ell,h)}
\end{aligned}$$

$$
\begin{aligned}
& E\left[\Delta_2\Delta_3|\mathcal{F}_n^W\right] \\
= \;& E\Bigg[\Bigg(\sum_{1\le(i_1,t_1)<(j_1,s_1)<(k_1,g_1)\le m_n}\Big\{A_{(i_1,t_1),(j_1,s_1)}A_{(i_1,t_1),(k_1,g_1)}\eta_{(j_1,s_1)}\eta_{(k_1,g_1)} \\
& \qquad\qquad +A_{(i_1,t_1),(j_1,s_1)}A_{(j_1,s_1),(k_1,g_1)}\eta_{(i_1,t_1)}\eta_{(k_1,g_1)}\Big\}\Bigg) \\
& \quad\times\Bigg(\sum_{1\le(i_2,t_2)<(j_2,s_2)<(k_2,g_2)\le m_n}A_{(i_2,t_2),(k_2,g_2)}A_{(j_2,s_2),(k_2,g_2)}\eta_{(i_2,t_2)}\eta_{(j_2,s_2)}\Bigg)|\mathcal{F}_n^W\Bigg] \\
= \;& \sum_{1\le(i,t)<(j,s)<(k,\upsilon)<(\ell,h)\le m_n}\Big[A_{(i,t),(j,s)}A_{(i,t),(k,\upsilon)}A_{(j,s),(\ell,h)}A_{(k,\upsilon),(\ell,h)} \\
& \qquad\qquad +A_{(i,t),(j,s)}A_{(j,s),(k,\upsilon)}A_{(i,t),(\ell,h)}A_{(k,\upsilon),(\ell,h)}\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
& E\left[\Delta_2^2|\mathcal{F}_n^W\right] \\
= \; & E\left[\left(\sum_{1\le(i,t)<(j,s)<(k,v)\le m_n}\left\{A_{(i,t),(j,s)}A_{(i,t),(k,v)}\varsigma_{(j,s)}\varsigma_{(k,v)}+A_{(i,t),(j,s)}A_{(j,s),(k,v)}\eta_{(i,t)}\eta_{(k,v)}\right\}\right)^2|\mathcal{F}_n^W\right] \\
= \; & \sum_{1\le\{(i,t),(\ell,h)\}<(j,s)<(k,v)\le m_n} A_{(i,t),(j,s)}A_{(i,t),(k,v)}A_{(\ell,h),(j,s)}A_{(\ell,h),(k,v)} \\
& + \sum_{1\le(i,t)<\{(j,s),(\ell,h)\}<(k,v)\le m_n} A_{(i,t),(j,s)}A_{(j,s),(k,v)}A_{(i,t),(\ell,h)}A_{(\ell,h),(k,v)} \\
& + \sum_{1\le(i,t)<(j,s)<(\ell,h)<(k,v)\le m_n} A_{(i,t),(j,s)}A_{(i,t),(k,v)}A_{(j,s),(\ell,h)}A_{(\ell,h),(k,v)} \\
& + \sum_{1\le(\ell,h)<(i,t)<(j,s)<(k,v)\le m_n} A_{(i,t),(j,s)}A_{(j,s),(k,v)}A_{(\ell,h),(i,t)}A_{(\ell,h),(k,v)} \\
= \; & \sum_{1\le(i,t)<(j,s)<(k,v)\le m_n} A^2_{(i,t),(j,s)}A^2_{(i,t),(k,v)} + \sum_{1\le(i,t)<(j,s)<(k,v)\le m_n} A^2_{(i,t),(j,s)}A^2_{(j,s),(k,v)} \\
& +2\sum_{1\le(i,t)<(\ell,h)<(j,s)<(k,v)\le m_n} A_{(i,t),(j,s)}A_{(i,t),(k,v)}A_{(\ell,h),(j,s)}A_{(\ell,h),(k,v)} \\
& +2\sum_{1\le(i,t)<(j,s)<(\ell,h)<(k,v)\le m_n} A_{(i,t),(j,s)}A_{(j,s),(k,v)}A_{(i,t),(\ell,h)}A_{(\ell,h),(k,v)} \\
& + \sum_{1\le(i,t)<(j,s)<(k,v)<(\ell,h)\le m_n} A_{(i,t),(j,s)}A_{(i,t),(\ell,h)}A_{(j,s),(k,v)}A_{(k,v),(\ell,h)} \\
& + \sum_{1\le(i,t)<(j,s)<(k,v)<(\ell,h)\le m_n} A_{(j,s),(k,v)}A_{(k,v),(\ell,h)}A_{(i,t),(j,s)}A_{(i,t),(\ell,h)} \\
= \; & \sum_{1\le(i,t)<(j,s)<(k,v)\le m_n} A^2_{(i,t),(j,s)}A^2_{(i,t),(k,v)} + \sum_{1\le(i,t)<(j,s)<(k,v)\le m_n} A^2_{(i,t),(j,s)}A^2_{(j,s),(k,v)} + 2S_n \\
\le \; & 2\sum_{(j,s)=1}^{m_n}\sum_{\substack{(i,t),(k,v)=1\\(i,t)\ne(j,s),(k,v)\ne(j,s)}}^{m_n} A^2_{(i,t),(j,s)}A^2_{(j,s),(k,v)} + 2S_n \\
= \; & O_{a.s.}\left(\frac{K_{2,n}^2}{n}\right) + O_{a.s.}\left(K_{2,n}\right) = O_{a.s.}\left(K_{2,n}\right)
\end{aligned}
$$

Since $\Delta_1 = \Delta_2 + \Delta_3$, it follows from part (b) of this lemma and the results given above that

$$
\begin{aligned}
E\left[\Delta_1^2|\mathcal{F}_n^W\right] &= E\left[\Delta_2^2|\mathcal{F}_n^W\right] + E\left[\Delta_3^2|\mathcal{F}_n^W\right] + 2E\left[\Delta_2\Delta_3|\mathcal{F}_n^W\right] \\
&= O_{a.s.}\left(\frac{K_{2,n}^2}{n}\right) + 4S_n \\
&= O_{a.s.}\left(\frac{K_{2,n}^2}{n}\right) + O_{a.s.}\left(K_{2,n}\right) = O_{a.s.}\left(K_{2,n}\right).
\end{aligned}
$$

Moreover, note that

$$\begin{aligned}
& \sum_{1\leq(i,t)<(j,s)<(k,v)<(\ell,h)\leq m_n} A_{(i,t),(k,v)}A_{(j,s),(k,v)}A_{(i,t),(\ell,h)}A_{(j,s),(\ell,h)} \\
= & \frac{1}{2}E\left[\Delta_3^2|\mathcal{F}_n^W\right]-\frac{1}{2}\sum_{1\leq(i,t)<(j,s)<(k,v)\leq m_n} A_{(i,t),(k,v)}^2A_{(j,s),(k,v)}^2 \\
= & \frac{1}{2}E\left[(\Delta_1-\Delta_2)^2|\mathcal{F}_n^W\right]-\frac{1}{2}\sum_{1\leq(i,t)<(j,s)<(k,v)\leq m_n} A_{(i,t),(k,v)}^2A_{(j,s),(k,v)}^2
\end{aligned}$$

so that, by making use of the triangle inequality, Loève's c_r inequality, and the symmetry of A; we obtain

$$\begin{aligned}
& \left|\sum_{1\leq(i,t)<(j,s)<(k,v)<(\ell,h)\leq m_n} A_{(i,t),(k,v)}A_{(j,s),(k,v)}A_{(i,t),(\ell,h)}A_{(j,s),(\ell,h)}\right| \\
\leq & E\left[\Delta_1^2|\mathcal{F}_n^W\right]+E\left[\Delta_2^2|\mathcal{F}_n^W\right]+\frac{1}{2}\sum_{1\leq(i,t)<(j,s)<(k,v)} A_{(i,t),(k,v)}^2A_{(j,s),(k,v)}^2 \\
\leq & E\left[\Delta_1^2|\mathcal{F}_n^W\right]+E\left[\Delta_2^2|\mathcal{F}_n^W\right]+\frac{1}{2}\sum_{(k,v)=1}^{m_n}\sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(k,v),(j,s)\neq(k,v)}}^{m_n} A_{(i,t),(k,v)}^2A_{(k,v),(j,s)}^2 \\
= & O_{a.s.}\left(K_{2,n}\right). \ \square
\end{aligned}$$

Lemma OA-12:

Let L be the lower triangular matrix such that $L_{(i,t),(j,s)}=A_{(i,t),(j,s)}\mathbb{I}\left\{(i,t)>(j,s)\right\}$. Then, under Assumptions 5-6,

$$\left\|LL'\right\|_F=O_{a.s.}\left(\sqrt{K_{2,n}}\right),$$

where $\|\cdot\|_F$ denotes the Frobenius norm, i.e., $\|A\|_F=\left[tr\left(A'A\right)\right]^{1/2}$.

Proof of Lemma OA-12:

By parts (b) and (c) of Lemma S2-1 and part (c) of Lemma OA-11, we have

$$
\begin{aligned}
& \left\|LL'\right\|_F^2 \\
= \; & tr\left\{LL'LL'\right\} \\
= \; & \sum_{1\leq(i,t)<(j,s)\leq m_n} A^4_{(i,t),(j,s)} + 2\sum_{1\leq(i,t)<(j,s)<(k,\upsilon)\leq m_n}\left[A^2_{(i,t),(j,s)}A^2_{(i,t),(k,\upsilon)} + A^2_{(i,t),(k,\upsilon)}A^2_{(j,s),(k,\upsilon)}\right] \\
& +4\sum_{1\leq(i,t)<(j,s)<(k,\upsilon)<(\ell,h)\leq m_n} A_{(i,t),(k,\upsilon)}A_{(k,\upsilon),(j,s)}A_{(j,s),(\ell,h)}A_{(\ell,h),(i,t)} \\
\leq \; & \sum_{1\leq(i,t)<(j,s)\leq m_n} A^4_{(i,t),(j,s)} + 2\sum_{1\leq(i,t)<(j,s)<(k,\upsilon)\leq m_n}\left[A^2_{(i,t),(j,s)}A^2_{(i,t),(k,\upsilon)} + A^2_{(i,t),(k,\upsilon)}A^2_{(j,s),(k,\upsilon)}\right] \\
& +4\left|\sum_{1\leq(i,t)<(j,s)<(k,\upsilon)<(\ell,h)\leq m_n} A_{(i,t),(k,\upsilon)}A_{(k,\upsilon),(j,s)}A_{(j,s),(\ell,h)}A_{(\ell,h),(i,t)}\right| \\
\leq \; & \sum_{\substack{(i,t),(j,s)=1\\(i,t)\neq(j,s)}}^{m_n} A^4_{(i,t),(j,s)} + 4\sum_{(k,\upsilon)=1}^{m_n}\sum_{\substack{(i,t),(j,s)=1\\(i,t)\neq(k,\upsilon),(j,s)\neq(k,\upsilon)}}^{m_n} A^2_{(i,t),(k,\upsilon)}A^2_{(k,\upsilon),(j,s)} \\
& +4\left|\sum_{1\leq(i,t)<(j,s)<(k,\upsilon)<(\ell,h)\leq m_n} A_{(i,t),(k,\upsilon)}A_{(k,\upsilon),(j,s)}A_{(j,s),(\ell,h)}A_{(\ell,h),(i,t)}\right| \\
= \; & O_{a.s.}\left(\frac{K^2_{2,n}}{n^2}\right) + O_{a.s.}\left(\frac{K^2_{2,n}}{n}\right) + O_{a.s.}\left(K_{2,n}\right) = O_{a.s.}\left(K_{2,n}\right),
\end{aligned}
$$

from which the required result follows. □

Lemma OA-13:

Under Assumptions 1-7, the following statements are true.

(a)
$$\frac{\Gamma' A\Gamma}{n} = \frac{\Gamma' M^{(Z_1,Q)}\Gamma}{n} + o_{a.s.}(1) = O_p(1)$$

(b)
$$\frac{D_\mu^{-1}D_\kappa F' AFD_\kappa D_\mu^{-1}}{n} = O_{a.s.}\left(\frac{(\kappa_n^{\max})^2}{(\mu_n^{\min})^2}\frac{1}{K_{1,n}^{2\varrho_f}}\right)$$

(c)
$$D_\mu^{-1}U' AUD_\mu^{-1} = o_p(1)$$

(d)
$$\frac{D_\mu^{-1}D_\kappa F' A\Gamma}{n} = O_p\left(\frac{\kappa_n^{\max}}{\mu_n^{\min}}\frac{1}{K_{1,n}^{\varrho_f}}\right), \quad \frac{\Gamma' AFD_\kappa D_\mu^{-1}}{n} = O_p\left(\frac{\kappa_n^{\max}}{\mu_n^{\min}}\frac{1}{K_{1,n}^{\varrho_f}}\right)$$

(e)

$$\frac{D_\mu^{-1} D_\kappa F' A U D_\mu^{-1}}{\sqrt{n}} = o_p(1), \frac{D_\mu^{-1} U' A F D_\kappa D_\mu^{-1}}{\sqrt{n}} = o_p(1)$$

(f)

$$\frac{\Gamma' A U D_\mu^{-1}}{\sqrt{n}} = o_p(1), \quad \frac{D_\mu^{-1} U' A \Gamma}{\sqrt{n}} = o_p(1).$$

Proof of Lemma OA-13:

In the proofs below, we shall take a, b, $c \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = \|c\|_2 = 1$. To show part (a), first write

$$\frac{\Gamma' A \Gamma}{n} = \frac{\Gamma' P^{\perp} \Gamma}{n} - \frac{\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} \Gamma}{n}$$

Now, note that by the CS inequality, part (b) of Lemma OA-1, and Assumptions 5 and 7(iii);

$$\begin{aligned}
& \left| \frac{a'\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} \Gamma b}{n} \right| \\
\leq & \sqrt{\frac{a'\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} \Gamma a}{n}} \sqrt{\frac{b'\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} \Gamma b}{n}} \\
\leq & \left(\frac{a'\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}}^2 M^{(Z,Q)} \Gamma a}{n} \right)^{1/4} \left(\frac{a'\Gamma' M^{(Z,Q)} \Gamma a}{n} \right)^{1/4} \\
& \times \left(\frac{b'\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}}^2 M^{(Z,Q)} \Gamma b}{n} \right)^{1/4} \left(\frac{b'\Gamma' M^{(Z,Q)} \Gamma b}{n} \right)^{1/4} \\
= & \sqrt{\max_{1 \leq (i,t) \leq m_n} \left| \widehat{\vartheta}_{(i,t)} \right|^2} \sqrt{\frac{a'\Gamma' M^{(Z,Q)} \Gamma a}{n}} \sqrt{\frac{b'\Gamma' M^{(Z,Q)} \Gamma b}{n}} \\
= & \sqrt{\max_{1 \leq (i,t) \leq m_n} \left| \widehat{\vartheta}_{(i,t)} \right|^2} \sqrt{\frac{a' \left(\Gamma - Z_2 \Pi_{K_{2,n}} \right)' M^{(Z,Q)} \left(\Gamma - Z_2 \Pi_{K_{2,n}} \right) a}{n}} \\
& \times \sqrt{\frac{b' \left(\Gamma - Z_2 \Pi_{K_{2,n}} \right)' M^{(Z,Q)} \left(\Gamma - Z_2 \Pi_{K_{2,n}} \right) b}{n}} \\
\leq & \sqrt{\max_{1 \leq (i,t) \leq m_n} \left| \widehat{\vartheta}_{(i,t)} \right|^2} \frac{m_n d}{n} \left\| \gamma(\cdot) - \Pi_{K_{2,n}}' Z_2(\cdot) \right\|_{\infty,d}^2 \\
= & O_{a.s.} \left(\frac{1}{K_{2,n}^{2\varrho_\gamma - 1} n} \right)
\end{aligned}$$

Since the above argument holds for all $a, b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$, we deduce that

$$\frac{\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} \Gamma}{n} = O_{a.s.}\left(\frac{1}{K_{2,n}^{2\varrho_\gamma - 1} n}\right).$$

Next, write

$$\begin{aligned} P^{\perp} &= M^{(Z_1,Q)} + P^{\perp} - M^{(Z_1,Q)} \\ &= M^{(Z_1,Q)} + P^{(Z,Q)} - P^{(Z_1,Q)} - \left(I_n - P^{(Z_1,Q)}\right) \\ &= M^{(Z_1,Q)} - M^{(Z,Q)} \end{aligned}$$

so that

$$\frac{\Gamma' P^{\perp} \Gamma}{n} = \frac{\Gamma' M^{(Z_1,Q)} \Gamma}{n} - \frac{\Gamma' M^{(Z,Q)} \Gamma}{n}$$

By a similar argument as above, we have

$$\frac{\Gamma' M^{(Z,Q)} \Gamma}{n} = O_{a.s.}\left(\frac{1}{K_{2,n}^{2\varrho_\gamma}}\right).$$

Furthermore, note that

$$\frac{\Gamma' M^{(Z_1,Q)} \Gamma}{n} \leq \frac{\Gamma' \Gamma}{n} = O_{a.s.}(1)$$

where we take $A \leq B$ for two square matrices A and B to mean that $A - B$ is negative semi-definite, or, alternatively, $B - A$ is positive semidefinite. Putting everything together and making use of Assumption 5, we have

$$\begin{aligned} \frac{\Gamma' A \Gamma}{n} &= \frac{\Gamma' M^{(Z_1,Q)} \Gamma}{n} - \frac{\Gamma' M^{(Z,Q)} \Gamma}{n} - \frac{\Gamma' M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} \Gamma}{n} \\ &= \frac{\Gamma' M^{(Z_1,Q)} \Gamma}{n} + o_{a.s.}(1) \\ &= O_p(1) \end{aligned}$$

as required.

To show part (b), note first that for any $a, b \in \mathbb{R}^d$ such that $\|a\| = \|b\| = 1$. Define $\widetilde{a} = D_\kappa D_\mu^{-1} a$ and $\widetilde{b} = D_\kappa D_\mu^{-1} b$, and note that, applying CS inequality along with Assumptions 3(ii), 4(i), 5, and

7(ii); we have

$$
\begin{aligned}
& \left|\frac{a' D_\mu^{-1} D_\kappa F' A F D_\kappa D_\mu^{-1} b}{n}\right| \\
= & \left|\frac{a' D_\mu^{-1} D_\kappa \left(F - Z_1 \Theta_{K_{1,n}}\right)' A \left(F - Z_1 \Theta_{K_{1,n}}\right) D_\kappa D_\mu^{-1} b}{n}\right| \\
\leq & \sqrt{\frac{\widetilde{a}' \left(F - Z_1 \Theta_{K_{1,n}}\right)' A A' \left(F - Z_1 \Theta_{K_{1,n}}\right) \widetilde{a}}{n}} \sqrt{\frac{\widetilde{b}' \left(F - Z_1 \Theta_{K_{1,n}}\right)' \left(F - Z_1 \Theta_{K_{1,n}}\right) \widetilde{b}}{n}} \\
= & \sqrt{\frac{\widetilde{a}' \left(F - Z_1 \Theta_{K_{1,n}}\right)' \left(P^\perp - M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\right) \left(P^\perp - M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\right) \left(F - Z_1 \Theta_{K_{1,n}}\right) \widetilde{a}}{n}} \\
& \times \sqrt{\frac{\widetilde{b}' \left(F - Z_1 \Theta_{K_{1,n}}\right)' \left(F - Z_1 \Theta_{K_{1,n}}\right) \widetilde{b}}{n}} \\
= & \sqrt{\frac{\widetilde{a}' \left(F - Z_1 \Theta_{K_{1,n}}\right)' \left(P^\perp + M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}\right) \left(F - Z_1 \Theta_{K_{1,n}}\right) \widetilde{a}}{n}} \\
& \times \sqrt{\frac{\widetilde{b}' \left(F - Z_1 \Theta_{K_{1,n}}\right)' \left(F - Z_1 \Theta_{K_{1,n}}\right) \widetilde{b}}{n}} \\
\leq & \sqrt{1 + \max_{(i,t)} \left|\widehat{\vartheta}_{(i,t)}\right|^2} \frac{\left(\kappa_n^{\max}\right)^2}{\left(\mu_n^{\min}\right)^2} \frac{d m_n}{n} \left\| f(\cdot) - \Theta_{K_{1,n}}' Z_1(\cdot) \right\|_{\infty,d}^2 \\
= & O_{a.s.}\left(\frac{\left(\kappa_n^{\max}\right)^2}{\left(\mu_n^{\min}\right)^2} \frac{1}{K_{1,n}^{2\varrho_f}}\right)
\end{aligned}
$$

Since this holds for all a, $b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$, we further deduce that

$$
\frac{D_\mu^{-1} D_\kappa F' A F D_\kappa D_\mu^{-1}}{n} = O_{a.s.}\left(\frac{\left(\kappa_n^{\max}\right)^2}{\left(\mu_n^{\min}\right)^2} \frac{1}{K_{1,n}^{2\varrho_f}}\right),
$$

as required.

To show part (c), note that, for a, $b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$, we write

$$
\begin{aligned}
a' D_\mu^{-1} U' A U D_\mu^{-1} b &= a' D_\mu^{-1} \left(\sum_{\substack{(i,t),(j,s)=1 \\ (i,t) \neq (j,s)}}^{m_n} A_{(i,t),(j,s)} U_{(i,t)} U_{(j,s)}' \right) D_\mu^{-1} b \\
&= \sum_{\substack{(i,t),(j,s)=1 \\ (i,t) \neq (j,s)}}^{m_n} A_{(i,t),(j,s)} u_{a,(i,t)} u_{b,(j,s)},
\end{aligned}
$$

where $u_{a,(i,t)} = a' D_\mu^{-1} U_{(i,t)}$ and $u_{b,(j,s)} = b' D_\mu^{-1} U_{(j,s)}$. Next, making use of the CS inequality, part

(a) of Lemma S2-1, and Assumptions 1, 2(i), 3(ii), and 5; we get

$$
\begin{aligned}
& E\left[\left(a' D_\mu^{-1} U' A U D_\mu^{-1} b\right)^2 |\mathcal{F}_n^W\right] \\
= \; & E\left[\left(\sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)} u_{a,(i,t)} u_{b,(j,s)}\right)^2 |\mathcal{F}_n^W\right] \\
= \; & \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} \sum_{\substack{(k,v),(\ell,h)=1 \\ (k,v)\neq(\ell,h)}}^{m_n} A_{(i,t),(j,s)} A_{(k,v),(\ell,h)} E\left[u_{a,(i,t)} u_{b,(j,s)} u_{a,(k,v)} u_{b,(\ell,h)} |\mathcal{F}_n^W\right] \\
= \; & \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^2 \left\{E\left[u_{a,(i,t)}^2 |\mathcal{F}_n^W\right] E\left[u_{b,(j,s)}^2 |\mathcal{F}_n^W\right] + E\left[u_{a,(i,t)} u_{b,(i,t)} |\mathcal{F}_n^W\right] E\left[u_{a,(j,s)} u_{b,(j,s)} |\mathcal{F}_n^W\right]\right\} \\
\leq \; & \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^2 \left\{E\left[u_{a,(i,t)}^2 |\mathcal{F}_n^W\right] E\left[u_{b,(j,s)}^2 |\mathcal{F}_n^W\right] + E\left[\left|u_{a,(i,t)} u_{b,(i,t)}\right| |\mathcal{F}_n^W\right] E\left[\left|u_{a,(j,s)} u_{b,(j,s)}\right| |\mathcal{F}_n^W\right]\right\} \\
\leq \; & \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^2 E\left[u_{a,(i,t)}^2 |\mathcal{F}_n^W\right] E\left[u_{b,(j,s)}^2 |\mathcal{F}_n^W\right] \\
& + \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^2 \sqrt{E\left[u_{a,(i,t)}^2 |\mathcal{F}_n^W\right]} \sqrt{E\left[u_{b,(i,t)}^2 |\mathcal{F}_n^W\right]} \sqrt{E\left[u_{a,(j,s)}^2 |\mathcal{F}_n^W\right]} \sqrt{E\left[u_{b,(j,s)}^2 |\mathcal{F}_n^W\right]} \\
\leq \; & \frac{C K_{2,n}}{(\mu_n^{\min})^4} \frac{1}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^2 \\
= \; & O_{a.s.}\left(\frac{K_{2,n}}{(\mu_n^{\min})^4}\right) \\
= \; & o_{a.s.}(1),
\end{aligned}
$$

It follows from the conditional version of the Markov's inequality that for any $\epsilon > 0$

$$\Pr\left(\left|a' D_\mu^{-1} U' A U D_\mu^{-1} b\right| \geq \epsilon |\mathcal{F}_n^W\right) \to 0 \; a.s.$$

Moreover, note that

$$\sup_n E\left[\left|\Pr\left(\left|a' D_\mu^{-1} U' A U D_\mu^{-1} b\right| \geq \epsilon |\mathcal{F}_n^W\right)\right|^2\right] < \infty$$

Hence, by a version of the dominated convergence theorem, as given by Theorem 25.12 of Billingsley

(1986), it follows that as $n \to \infty$

$$\begin{aligned} & \Pr\left(\left|a' D_\mu^{-1} U' A U D_\mu^{-1} b\right| \geq \epsilon\right) \\ = \; & E\left[\Pr\left(\left|a' D_\mu^{-1} U' A U D_\mu^{-1} b\right| \geq \epsilon | \mathcal{F}_n^W\right)\right] \to 0 \end{aligned}$$

Since the above result holds for all a, $b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$, we further deduce that

$$D_\mu^{-1} U' A U D_\mu^{-1} = o_p(1),$$

as required.

To show part (d), note that, applying the CS inequality and making argument similar to that given for part (b) above, we have

$$\begin{aligned} \left|\frac{a' D_\mu^{-1} D_\kappa F' A \Gamma b}{n}\right| & \leq \sqrt{\frac{a' D_\mu^{-1} D_\kappa F' A A' F D_\kappa D_\mu^{-1} a}{n}} \sqrt{\frac{b' \Gamma' \Gamma b}{n}} \\ & = O_p\left(\frac{\kappa_n^{\max}}{\mu_n^{\min}} \frac{1}{K_{1,n}^{\varrho_f}}\right) O_p(1) \\ & = O_p\left(\frac{\kappa_n^{\max}}{\mu_n^{\min}} \frac{1}{K_{1,n}^{\varrho_f}}\right) \end{aligned}$$

Since the above result holds for all a, $b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$, we further deduce that

$$\frac{D_\mu^{-1} D_\kappa F' A \Gamma}{n} = O_p\left(\frac{\kappa_n^{\max}}{\mu_n^{\min}} \frac{1}{K_{1,n}^{\varrho_f}}\right)$$

Moreover, it follows immediately that

$$\frac{\Gamma' A F D_\kappa D_\mu^{-1}}{n} = \left(\frac{D_\mu^{-1} D_\kappa F' A \Gamma}{n}\right)' = O_p\left(\frac{\kappa_n^{\max}}{\mu_n^{\min}} \frac{1}{K_{1,n}^{\varrho_f}}\right).$$

To show part (e), let $u_{b,(i,t)} = U_{(i,t)} D_\mu^{-1} b$ and $f_{a,(i,t)} = f'_{(i,t)} D_\kappa D_\mu^{-1} a$, and note that, by making use of the conditional serial independence assumption in Assumption 1, we have

$$
\begin{aligned}
& E\left(\left[\frac{a' D_\mu^{-1} D_\kappa F' A U D_\mu^{-1} b}{\sqrt{n}}\right]^2 |\mathcal{F}_n^W\right) \\
= & E\left(\left[\frac{1}{\sqrt{n}} \sum_{\substack{(i,t),(k,v)=1 \\ (i,t)\neq(k,v)}}^{m_n} A_{(i,t),(k,v)} f_{a,(i,t)} u_{b,(k,v)}\right]^2 |\mathcal{F}_n^W\right) \\
= & \frac{1}{n} \sum_{\substack{(i,t),(k,v)=1 \\ (i,t)\neq(k,v)}}^{m_n} \sum_{\substack{(j,s),(\ell,h)=1 \\ (j,s)\neq(\ell,h)}}^{m_n} A_{(i,t),(k,v)} A_{(j,s),(\ell,h)} f_{a,(i,t)} f_{a,(j,s)} E\left[u_{b,(k,v)} u_{b,(\ell,h)} |\mathcal{F}_n^W\right] \\
= & \frac{1}{n} \sum_{(k,v)=1}^{m_n} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(k,v),(j,s)\neq(k,v)}}^{m_n} A_{(i,t),(k,v)} A_{(j,s),(k,v)} f_{a,(i,t)} f_{a,(j,s)} b' D_\mu^{-1} E\left[U_{(k,v)} U'_{(k,v)} |\mathcal{F}_n^W\right] D_\mu^{-1} b \\
\leq & \frac{\max_{1\leq(k,v)\leq m_n} E\left[\left\|U_{(k,v)}\right\|_2^2 |\mathcal{F}_n^W\right]}{(\mu_n^{\min})^2} \frac{1}{n} \sum_{(k,v)=1}^{m_n} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(k,v),(j,s)\neq(k,v)}}^{m_n} A_{(i,t),(k,v)} A_{(j,s),(k,v)} f_{a,(i,t)} f_{a,(j,s)} \\
= & \frac{\max_{1\leq(k,v)\leq m_n} E\left[\left\|U_{(k,v)}\right\|_2^2 |\mathcal{F}_n^W\right]}{(\mu_n^{\min})^2} \frac{a' D_\mu^{-1} D_\kappa F' A A' F D_\kappa D_\mu^{-1} a}{n}
\end{aligned}
$$

Next, note that, from the proof of part (b) above, we have

$$
\frac{a D_\mu^{-1} D_\kappa F' A A' F D_\kappa D_\mu^{-1} a}{n} \leq \left(1 + \max_{(i,t)} \left|\widehat{\vartheta}_{(i,t)}\right|^2\right) \frac{(\kappa_n^{\max})^2}{(\mu_n^{\min})^2} \left\|f(\cdot) - \Theta'_{K_{1,n}} Z_1(\cdot)\right\|_{\infty,d}^2
$$

so that, applying Assumptions 2(i), 3(ii), 4(i), and 7(ii); we have

$$
\begin{aligned}
& E\left(\left[\frac{a' D_\mu^{-1} D_\kappa F' A U D_\mu^{-1} b}{\sqrt{n}}\right]^2 |\mathcal{F}_n^W\right) \\
\leq & \left(\max_{1\leq(k,v)\leq m_n} E\left[\left\|U_{(k,v)}\right\|_2^2 |\mathcal{F}_n^W\right]\right) \left(1 + \max_{(i,t)} \left|\widehat{\vartheta}_{(i,t)}\right|^2\right) \frac{(\kappa_n^{\max})^2}{(\mu_n^{\min})^4} \left\|f(\cdot) - \Theta'_{K_{1,n}} Z_1(\cdot)\right\|_{\infty,d}^2 \\
= & O_{a.s.}\left(\frac{(\kappa_n^{\max})^2}{(\mu_n^{\min})^4} \frac{1}{K_{1,n}^{2\varrho_f}}\right) = o_{a.s.}(1)
\end{aligned}
$$

It follows from the conditional version of the Markov's inequality that for any $\epsilon > 0$

$$
\Pr\left(\left|\frac{a' D_\mu^{-1} D_\kappa F' A U D_\mu^{-1} b}{\sqrt{n}}\right| \geq \epsilon |\mathcal{F}_n^W\right) \to 0 \ a.s.
$$

In addition, note that

$$\sup_n E\left[\left|\Pr\left(\left|\frac{a'D_\mu^{-1}D_\kappa F'AUD_\mu^{-1}b}{\sqrt{n}}\right|\geq\epsilon|\mathcal{F}_n^W\right)\right|^2\right]<\infty$$

Hence, by a version of the dominated convergence theorem, as given by Theorem 25.12 of Billingsley (1986), it follows that as $n\to\infty$

$$\Pr\left(\left|\frac{a'D_\mu^{-1}D_\kappa F'AUD_\mu^{-1}b}{\sqrt{n}}\right|\geq\epsilon\right)=E\left[\Pr\left(\left|\frac{a'D_\mu^{-1}D_\kappa F'AUD_\mu^{-1}b}{\sqrt{n}}\right|\geq\epsilon|\mathcal{F}_n^W\right)\right]\to 0$$

Since the above result holds for all a, $b\in\mathbb{R}^d$ such that $\|a\|_2=\|b\|_2=1$, we further deduce that

$$\frac{D_\mu^{-1}D_\kappa F'AUD_\mu^{-1}}{\sqrt{n}}=o_p\left(1\right),$$

Moreover, it follows immediately that

$$\frac{D_\mu^{-1}U'AFD_\kappa D_\mu^{-1}}{\sqrt{n}}=\left(\frac{D_\mu^{-1}D_\kappa F'AUD_\mu^{-1}}{\sqrt{n}}\right)'=o_p\left(1\right)$$

as required.

To show part (f), again, let $u_{b,(i,t)}=U_{(i,t)}D_\mu^{-1}b$ and let $\gamma_{a,(i,t)}=\gamma\left(W_{2,(i,t)}\right)'a$. Note again that, by making use of the conditional serial independence assumption in Assumption 1, we have

$$
\begin{aligned}
& E\left(\left[\frac{a'\Gamma' AUD_{\mu}^{-1}b}{\sqrt{n}}\right]^2 |\mathcal{F}_n^W\right) \\
= & E\left(\left[\frac{1}{\sqrt{n}}\sum_{\substack{(i,t),(k,\upsilon)=1\\(i,t)\neq(k,\upsilon)}}^{m_n} A_{(i,t),(k,\upsilon)}\gamma_{a,(i,t)}u_{b,(k,\upsilon)}\right]^2 |\mathcal{F}_n^W\right) \\
= & \frac{1}{n}\sum_{\substack{(i,t),(k,\upsilon)=1\\(i,t)\neq(k,\upsilon)}}^{m_n}\sum_{\substack{(j,s),(\ell,h)=1\\(j,s)\neq(\ell,h)}}^{m_n} A_{(i,t),(k,\upsilon)}A_{(j,s),(\ell,h)}\gamma_{a,(i,t)}\gamma_{a,(j,s)}E\left[u_{b,(k,\upsilon)}u_{b,(\ell,h)}|\mathcal{F}_n^W\right] \\
= & \frac{1}{n}\sum_{(k,\upsilon)=1}^{m_n}\sum_{\substack{(i,t),(j,s)=1\\(i,t)\neq(k,\upsilon),(j,s)\neq(k,\upsilon)}}^{m_n} A_{(i,t),(k,\upsilon)}A_{(j,s),(k,\upsilon)}\gamma_{a,(i,t)}\gamma_{a,(j,s)}b'D_{\mu}^{-1}E\left[U_{(k,\upsilon)}U_{(k,\upsilon)}'|\mathcal{F}_n^W\right]D_{\mu}^{-1}b \\
\leq & \frac{\max_{1\leq(k,\upsilon)\leq m_n}E\left[\left\|U_{(k,\upsilon)}\right\|_2^2|\mathcal{F}_n^W\right]}{\left(\mu_n^{\min}\right)^2}\frac{1}{n}\sum_{(k,\upsilon)=1}^{m_n}\sum_{\substack{(i,t),(j,s)=1\\(i,t)\neq(k,\upsilon),(j,s)\neq(k,\upsilon)}}^{m_n} A_{(i,t),(k,\upsilon)}A_{(j,s),(k,\upsilon)}\gamma_{a,(i,t)}\gamma_{a,(j,s)} \\
= & \frac{\max_{1\leq(k,\upsilon)\leq m_n}E\left[\left\|U_{(k,\upsilon)}\right\|_2^2|\mathcal{F}_n^W\right]}{\left(\mu_n^{\min}\right)^2}\frac{a'\Gamma' AA'\Gamma a}{n}
\end{aligned}
$$

Next, by applying part (b) of Lemma OA-1 along with Assumptions 3(iii) and 7(iii)

$$
\begin{aligned}
& \frac{a'\Gamma' AA'\Gamma a}{n} \\
= & \frac{a'\Gamma'\left(P^{\perp}-M^{(Z,Q)}D_{\widehat{\vartheta}}M^{(Z,Q)}\right)\left(P^{\perp}-M^{(Z,Q)}D_{\widehat{\vartheta}}M^{(Z,Q)}\right)\Gamma a}{n} \\
= & \frac{a'\Gamma' P^{\perp}\Gamma a}{n}+\frac{a'\left(\Gamma-Z_2\Pi_{K_{2,n}}\right)'M^{(Z,Q)}D_{\widehat{\vartheta}}M^{(Z,Q)}D_{\widehat{\vartheta}}M^{(Z,Q)}\left(\Gamma-Z_2\Pi_{K_{2,n}}\right)a}{n} \\
\leq & \frac{a'\Gamma'\Gamma a}{n}+\left(\max_{1\leq(i,t)\leq m_n}\left|\widehat{\vartheta}_{(i,t)}\right|^2\right)\left\|\gamma(\cdot)-\Pi_{K_{2,n}}'Z_2(\cdot)\right\|_{\infty,d}^2 \\
= & \frac{a'\Gamma'\Gamma a}{n}+O_{a.s.}\left(\frac{1}{K_{2,n}^{2\left(\varrho_\gamma-1\right)}n^2}\right)
\end{aligned}
$$

from which it follows by Assumption 5(ii) that

$$
\begin{aligned}
& E\left(\left[\frac{a'\Gamma' AUD_{\mu}^{-1}b}{\sqrt{n}}\right]^2 |\mathcal{F}_n^W\right) \\
\leq \;& \frac{\max_{1\leq(k,v)\leq m_n} E\left[\left\|U_{(k,v)}\right\|_2^2 |\mathcal{F}_n^W\right]}{\left(\mu_n^{\min}\right)^2}\left[\frac{a'\Gamma'\Gamma a}{n} + \left(\max_{(i,t)}\left|\widehat{\vartheta}_{(i,t)}\right|^2\right)\left\|\gamma(\cdot) - \Pi'_{K_{2,n}} Z_2(\cdot)\right\|_{\infty,d}^2\right] \\
= \;& O_{a.s.}\left(\frac{1}{\left(\mu_n^{\min}\right)^2}\right) + O_{a.s.}\left(\frac{K_{2,n}^2}{\left(\mu_n^{\min}\right)^2 K_{2,n}^{2\varrho_\gamma} n^2}\right) \\
= \;& O_{a.s.}\left(\frac{1}{\left(\mu_n^{\min}\right)^2}\right) + o_{a.s.}\left(\frac{1}{\left(\mu_n^{\min}\right)^2}\right) = o_{a.s.}(1)
\end{aligned}
$$

It further follows from the conditional version of the Markov's inequality that for any $\epsilon > 0$

$$
\Pr\left(\left|\frac{a'\Gamma' AUD_{\mu}^{-1}b}{\sqrt{n}}\right| \geq \epsilon|\mathcal{F}_n^W\right) \to 0 \; a.s.
$$

In addition, note that

$$
\sup_n E\left[\left|\Pr\left(\left|\frac{a'\Gamma' AUD_{\mu}^{-1}b}{\sqrt{n}}\right| \geq \epsilon|\mathcal{F}_n^W\right)\right|^2\right] < \infty
$$

Hence, by a version of the dominated convergence theorem, as given by Theorem 25.12 of Billingsley (1986), it follows that as $n \to \infty$

$$
\Pr\left(\left|\frac{a'\Gamma' AUD_{\mu}^{-1}b}{\sqrt{n}}\right| \geq \epsilon\right) = E\left[\Pr\left(\left|\frac{a'\Gamma' AUD_{\mu}^{-1}b}{\sqrt{n}}\right| \geq \epsilon|\mathcal{F}_n^W\right)\right] \to 0
$$

Since the above result holds for all a, $b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$, we further deduce that

$$
\frac{\Gamma' AUD_{\mu}^{-1}}{\sqrt{n}} = o_p(1),
$$

Moreover, it follows immediately that

$$
\frac{D_{\mu}^{-1}U'A\Gamma}{\sqrt{n}} = \left(\frac{\Gamma' AUD_{\mu}^{-1}}{\sqrt{n}}\right)' = o_p(1),
$$

as required. □

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