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Determining delay times for phase space reconstruction with application to the FF/DM exchange rate

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Abstract

Economists have widely applied the correlation integral of Grassberger and Procaccia (1983) to determine the dimension of a nonlinear dynamical system. A key judgmental input into this procedure is the choice of delay time for reconstructing a possible attractor. The literature, however, lacks a rigorous procedure for choosing the delay time. In this paper, I apply a simple nonparametric test for independence developed in Mizrach (1995a) to determine the appropriate lag for reconstruction and dimension estimation. In an application to the Lorenz equations, I obtain the best estimates of the correlation dimension at the lag chosen by the test. I then use the method to uncover nonlinear structure in the FF/DM exchange rate.

JEL classification: C22; F31

Keywords: Phase space reconstruction; Delay times; Nonlinear dynamics; Exchange rates

1. Introduction

Dimension is a measure of the complexity of a dynamical system. Economists are interested in estimating dimension because it can reveal whether a parsimonious representation of an economic variable is likely to be found. Following the lead of the physical sciences, nonlinear analysts have focused on what is known as the correlation

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dimension prposed by Grassberger and Procaccia (1983). GP used a construct known as the correlation integral to estimate the dimension of the attracting sets for dynamical systems. The GP methodology has proved quite reliable and has become the workhorse of the empirical literature. The GP algorithm is efficient and produces reliable estimates with relatively small sample sizes.

Nonetheless, it is possible to produce poor estimates of the correlation dimension. Brock (1989) first noted that near unit root processes may produce spuriously low correlation dimension estimates. Theiler (1986) and Ramsey et al. (1990) also point out that data limitations may lead to unreliable inference. Even when data are not limited, as in experimental situations or with a known dynamical system, a poor choice of inputs to the GP algorithm can produce seriously biased estimates of dimension.

A critical input to the GP approach is the choice of delay time. The literature, to this point, lacks a reliable method to choose the lags needed for the Takens (1980) reconstruction. This paper, using a new test for independence proposed in Mizrach (1995a), provides a simple procedure for determining the delay time.

Applications of the Grassberger–Procaccia methodology in economics and finance are numerous. Brock and Sayers (1988) have analyzed GNP data. Frank and Stengos (1989) examine precious metals prices. Scheinkman and LeBaron (1989) and Hiemstra (1992) estimate the correlation dimension of aggregate stock returns. Mayfield and Mizrach (1992) look at real time stock price data. Hsieh (1989), Guillaume (1994) investigate foreign exchange rates as I do below. These papers have produced estimates of correlation dimension ranging from 3–7 on standard data series. With the exception of the papers by Mayfield and Mizrach and Guillaume, these other estimates used filtered data rather than delays in reconstructing the dynamics.

I estimate the correlation dimension using a method proposed by Denker and Keller (1986) and Hiemstra (1992). This procedure generates points estimates of the correlation dimension and standard errors under a null of weak dependence. I calibrate the algorithm to a system of known dimension, the Lorenz system. I show that a naive choice of delay time generally biases dimension estimates downward. Using a rule of thumb commonly employed in the literature also produces biased estimates. The delay time chosen by the nonparametric test produces the most accurate estimate for the correlation dimension.

Having demonstrated the procedure on the Lorenz system, I then analyze a high frequency financial data set, the French France–German Deutschemark (FF/DM) exchange rate. I calculate the correlation dimension using the time-delayed, unfiltered data. I find nonlinear structure clearly distinguishable from white noise with dimension under 3.

The paper is organized as follows. In Section 2, I sketch Takens' approach to reconstructing a dynamical system. The importance of delay time to the reconstruction is discussed in Section 3, where I also briefly review alternative approaches to selecting the delay time. In Section 4, I develop the test for determining delay time, and I calibrate the procedure using the Lorenz system in Section 5. Section 6 contains the exchange rate analysis. I offer a summary and conclusions in Section 7.

2. Phase space reconstruction

Consider a compact n-dimensional manifold $S \in R^n$. A dynamical system is a diffeomorphism, $T : S \rightarrow S$. Time evolution follows an initial position $s_0 \in S$,

$$s_1 = T(s_0), s_2 = T(s_1) = T(T(s_0)) = T^2(s_0). \tag{1}$$

Assume that one observes a scalar x, through a measurement function, $h : S \rightarrow R$. Define the m-dimensional vector

$$x_t^m \equiv (h(s_t)), h(T(s_t)), \ldots, h(T^{m-1}(s_t))) \tag{2}$$

to illustrate the map, $\Phi(s) : S \rightarrow R^m$. Takens then demonstrated that if (T,h) are both smooth, an equivalence relation existed between the original dynamical system and the map $\Phi(s)$ so long as $m \geq 2n+1$. This can be stated formally as:

Proposition (Takens' Embedding Theorem for Diffeomorphisms). Let S be a compact manifold of dimension n, $T : S \rightarrow S$ be a C^2 diffeomorphism, and $h : S \rightarrow R$ be a smooth function. There exists an open dense set of pairs (T,h) for which the map $\Phi(s) : S \rightarrow R^m$ is an embedding for $m \geq 2n + 1$.

The embedding diffeomorphically maps the original dynamical system on S to a submanifold of $R^{m \geq 2n+1}$. The dynamics are C^2 equivalent to those on $\Phi(s)$. In particular, both dimension and entropy are preserved by the smooth, invertible change of coordinates involved in the phase space reconstruction of T.

3. The method of time delays

In their pioneering work on dynamical systems, Packard et al. (1980) realized that not all embeddings are equivalent, given a finite data sample. Consider again the m-dimensional vector (2), where we now delay by τ between each component,

$$x_t^m \equiv (x_t, x_{t+\tau}, \ldots, x_{t+(m-1)\tau}) \tag{3}$$

Packard et al. conjectured that accurate dimension estimates of strange attractors would depend upon choosing τ in such a way as to form sharp conditional probability distributions,

$$\Pr.[x_{t+(m-1)\tau} \mid x_t, x_{t+\tau}, \ldots, x_{t+(m-2)\tau}]. \tag{4}$$

Since the time series are nonlinear and often non-Gaussian, methods that look beyond linear dependence are needed to choose a delay time. I develop some of these preliminaries in Section 3.1, and present Fraser and Swinney's (1986) mutual information function in Section 3.2.

3.1. Correlation and independence

To proceed with a rigorous presentation of Fraser and Swinney's results, I must make some assumptions about the observables. Let $\{x_t\}_{t=1}^N$ be realizations of a strictly stationary and ergodic process, with joint probability density

$$f(x_t^m) = \Pr.[x_t = X_1, x_{t+\tau} = X_2, \ldots, x_{t+(m-1)\tau} = X_m]. \tag{5}$$

Consider the marginal density of the first component of the m-vector,

$$f(x_t) = \int_X \cdots \int_X f(x_t^m)\mathrm{d}x_{t+\tau} \cdots \mathrm{d}x_{t+(m-1)\tau}. \tag{6}$$

If τ chosen so as to make each element independent, the joint density would factor into the m-marginals,

$$f(x_t^m) = f(x_t)f(x_{t+\tau}) \cdots f(x_{t+(m-1)\tau}). \tag{7}$$

In Section 4, I devise a test based upon this factorization.

If the x's were Gaussian random variables, independence could be determined by the correlation between x_t and $x_{t+\tau}$,

$$\gamma(\tau) \equiv E[(x_t - \mu)(x_{t+\tau} - \mu)], \tag{8}$$

where μ is the unconditional population mean. Linear independence, however, is not sufficient to guarantee independence for non-Gaussian processes. That is, $\gamma(\tau)=0$ does not imply that I can factor the join density as in Eq. (7). Nonetheless, the zero-crossing of the autocorrelation function is a rule of thumb used for the delay time in the literature.[2]

3.2. Mutual information

The existing literature on choice of delay times is grounded in information theory and tends to focus on entropy. Define the entropy of the message x_t as

$$H(x_t) \equiv -\int_X f(x_t)\log[f(x_t)]\mathrm{d}x_t, \tag{9}$$

and the conditional entropy of $x_{t+\tau}$ given x_t,

$$H(x_{t+\tau} \mid x_t) \equiv -\int_X f(x_{t+\tau} \mid x_t)\log[f(x_{t+\tau} \mid x_t)]\mathrm{d}x_t\mathrm{d}x_{t+\tau}. \tag{10}$$

I can express the conditional entropy in a way that will be intuitively useful below,

$$\begin{aligned} H(x_{t+\tau} \mid x_t) &= -\int_X f(x_{t+\tau}, x_t)\log[f(x_{t+\tau}, x_t)]\mathrm{d}x_t\mathrm{d}x_{t+\tau} - H(x_{t+\tau}) \\ &= H(x_{t+\tau}, x_t) - H(x_{t+\tau}). \end{aligned} \tag{11}$$

This states that the conditional entropy is just the difference between the entropy of the joint density $f(x_{t+\tau}, x_t)$ and the entropy of the first marginal density. Note that the entropy tells us the average uncertainty in the message to be received, and the conditional entropy tells us the same, given a previous message. If the x_t and $x_{t+\tau}$ are independent, x_t is not at all informative about the next message. Knowing x_t resolves none of the uncertainty, and it follows that $H(x_{t+\tau} \mid x_t) = H(x_{t+\tau})$.

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[2]See e.g. Holzfuss and Mayer-Kress (1986).

Define, as in Fraser and Swinney (1986), the mutual information,

$$MI \equiv H(x_{t+\tau}) - H(x_{t+\tau} \mid x_t). \tag{12}$$

Note that if the pairs are independent, the function reaches it minimum at zero. A natural extension to this work was Fraser's (1989b) concept of marginal redundancy,

$$MR(x_{t+m\tau}) \equiv \int_X (\log[f(x_{t+mt} \mid x_t, \ldots, x_{t+(m-1)\tau}] - \log[f(x_{t+mt})])\mathrm{d}x_t^m \mathrm{d}x_{t+m\tau}. \tag{13}$$

Note again that if τ is large enough, the components of the m-vectors are independent, and the marginal redundancy will be minimized.

The algorithm proposed by Fraser and Swinney for mutual information, and generalized to higher dimensions by Fraser (1989a, 1989b) is quite cumbersone, by the authors' admission. For m>2, literally millions of points are required to construct a good estimate of the conditional densities. An alternative approach, based on work by Mizrach (1995a), is to use the distribution functions.

4. A simple nonparametric test

Consider now the distribution function of the m-vectors

$$F(x_t^m) = \Pr.[x_t < X_1, \ldots, x_{t+(m-1)\tau} < X_m]. \tag{14}$$

I will define a stochastic process to be *locally independent of order p* if the realization x_t provides no information about the process p periods ahead. Formally, this implies the equality of the conditional and unconditional distributions. Let $(p_1, \ldots, p_{m-1})$ be a set of increasing integers on $[1,L]$, $L<n-m+1$. Local independence then implies

$$\Pr.[x_{t+p_{m-1}} < \varepsilon, \ldots, x_{t+p_1} < \varepsilon, x_t < \varepsilon] = (\Pr.[x_t < \varepsilon])^m. \tag{15}$$

To estimate the joint, $F(x_t^m)$, and marginal, $F(x_t)$, distributions in Eq. (15), introduce the indicator function, $I : R \rightarrow R$,

$$I[x_t < \varepsilon] = \begin{Bmatrix} 1, \text{if} x_t < \varepsilon \\ 0, \text{otherwise} \end{Bmatrix} \equiv I(x_t, \varepsilon). \tag{16}$$

The joint unconditional probability that m leads of the x's are less than ε is given by

$$\theta(m, \varepsilon) = \int_X \prod_{i=0}^{m-1} I(x_{t+p_i}, \varepsilon)\mathrm{d}F(x_t), \tag{17}$$

where for notational convenience I set p_0=0. A consistent estimator of Eq. (17) is the statistic[3]

$$\theta(m, N, \varepsilon) = \sum_{t=1}^{N} \prod_{i=0}^{m-1} I(X_{t+p_i}, \varepsilon)/N, \tag{18}$$

where N=n−max $[p_i]$

[3]Statistics like Eq. (18) and the correlation integral of Grassberger and Procaccia (1983) are known as U-statistics. For an introduction, see Serfling (1980).

My testing procedure falls into the very general class proposed by Blum et al. (BKR, 1961) for sample distribution functions. The adjustable parameter p corresponds to the delay time, highlighting the alternative of interest.

While the distribution theory of the tests in the BKR class is often quite complicated, a simple nonparametric test for local independence (SNT) can then be constructed using consistent estimators of the first two moments of this statistic. In Mizrach (1995a), I prove the following:

Proposition. Let $\{x_t\}$ *be locally independent for any* $p_i \in [1,L]$, $i=1,\ldots,m-1$, $L<N$, *then if* $\theta(m,\varepsilon)>0$,

$$\sqrt{N}\frac{[\theta(m,N,\varepsilon)-\theta(m-1,N,\varepsilon)\theta(1,N,\varepsilon)]}{[\theta(m-1,N,\varepsilon)\theta(1,N,\varepsilon)(1-\theta(m-1,N,\varepsilon))(1-\theta(1,N,\varepsilon))]^{0.5}} \xrightarrow{d} N(0,1). \quad (19)$$

I use Eq. (19) to construct a test for local optimality of the delay time. As Packard et al. noted, a good reconstruction requires data that are approximately independent. Using the simple test, I can search for the p such that the statistic is closest to be being normally distributed. I turn to this method in the Section 5 in an application to the Lorenz equations.

5. Application to the Lorenz attractor

5.1. Phase portraits

The Lorenz equations are a three-dimensional dynamical system first conceived as a model for the climate. The system is given by

$$\begin{aligned} \mathrm{d}x/\mathrm{d}t &= 10(y-x), \\ \mathrm{d}y/\mathrm{d}t &= 28x-y-xz, \\ \mathrm{d}z/\mathrm{d}t &= xz-(8/3)z. \end{aligned} \quad (20)$$

I simulated 20,000 points for all three coordinates, discarding 10,000 transients, using a fourth order Runge–Kutta routine with a stepzise of 0.003. My objective will be reconstruct the system using a delay time representation involving only one coordinate. I will also try to estimate the dimension of the attracting set.

I first calculate the autocorrelation function for the y coordinate for lags ranging for 1 to 250. The first zero crossing of the function occurs at lag 247 at the very edge of Fig. 1. I then used the SNT of Section 4 and found a local minima at lag 154. The standardized statistics for $m=2$ with p varying from 1 to 250 are also plotted in Fig. 1.

In Fig. 2, I plot an X–Z phase plane projection of 2,500 points. I then constructed phase portraits using time delays of 1,75,155, and 250. This animated sequence appears in Figs. 3–6. One can see in the figures the gradual fleshing out of the dynamics as the delay time grows. At lag 1, the coordinates are so highly correlated that they lie nearly along a line. Structure does not really emerge until $\tau = 75$.

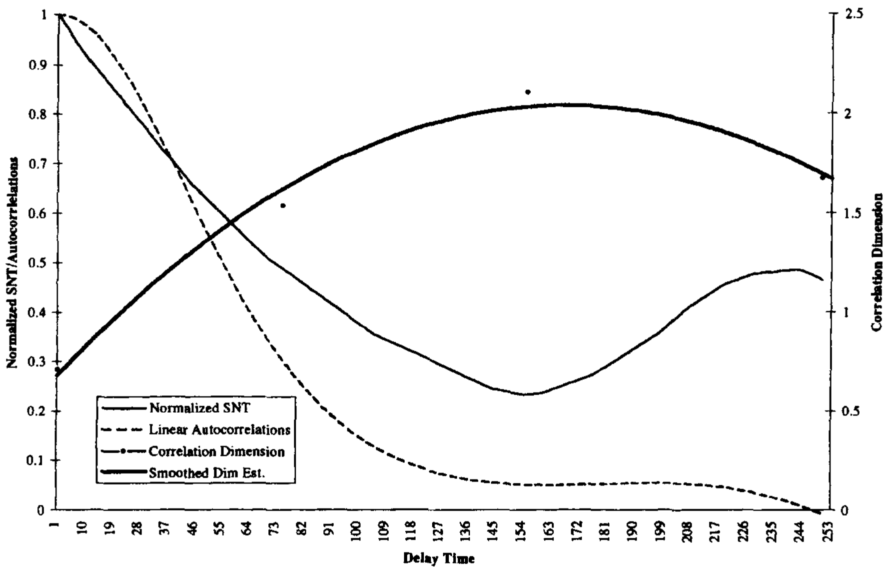

Fig. 1. Correlation dimension estimates for the Lorenz attractor.

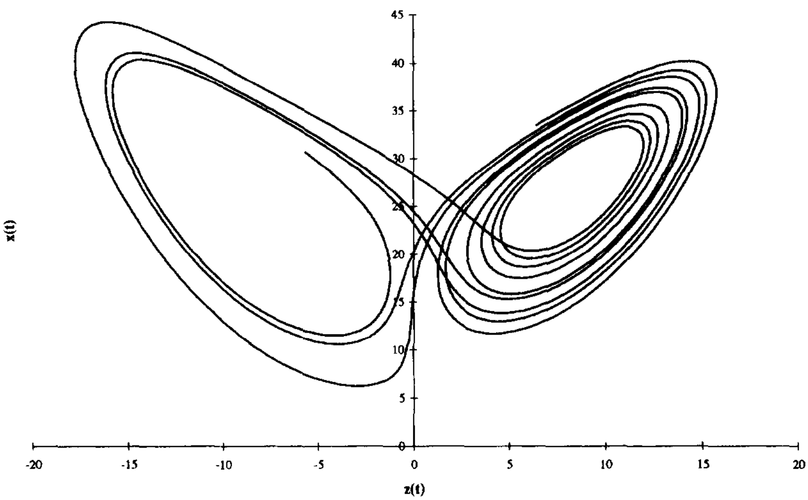

Fig. 2. X–Z phase space of Lorenz equations.

5.2. Dimension estimates

The correlation dimension is an estimate of the fractal dimension of a chaotic dynamical systems attracting set. The most commonly employed algorithm used to estimate the dimension is the correlation integral of Grassberger and Procaccia (1983).

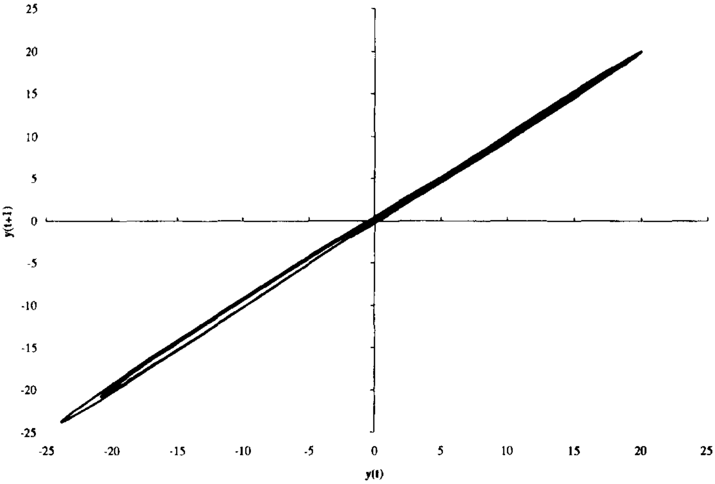

Fig. 3. Phase space reconstruction of Lorenz attractor delay time $\tau = 1$.

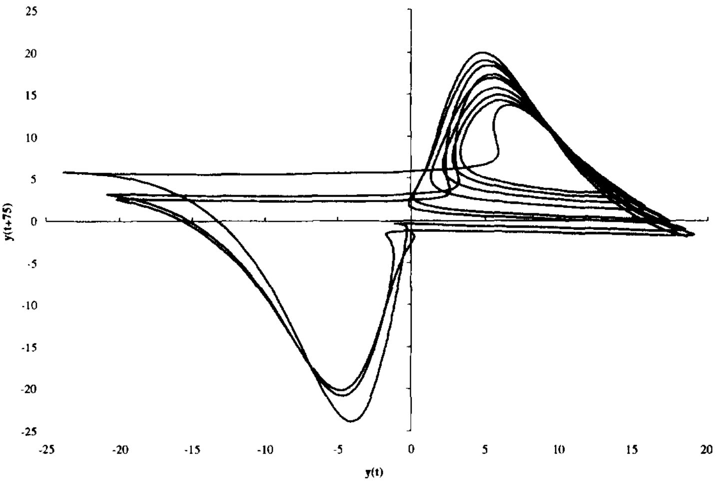

Fig. 4. Phase space reconstruction of Lorenz attractor delay time $\tau = 75$.

Define the statistic

$$C(m,N,\varepsilon) \equiv \sum_{i,j}^{N} I\left[\|x_i^m - x_j^m\| < \varepsilon\right], \tag{21}$$

where I is the indicator (or Heaviside) function of Eq. (16) and $\|.\|$ is the ℓ_∞ norm. the correlation integral is just the limit of Eq. (21),

$$C(m,\varepsilon) \equiv \lim_{N\to\infty} C(m,N,\varepsilon).$$

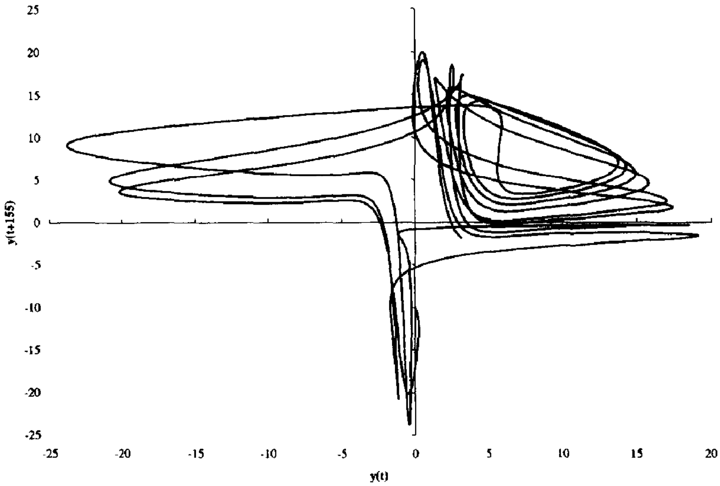

Fig. 5. Phase space reconstruction of Lorenz attractor delay time $\tau = 155$.

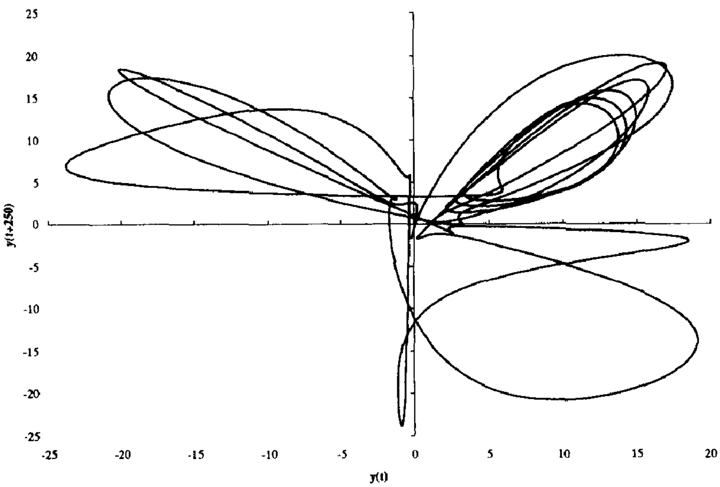

Fig. 6. Phase space reconstruction of Lorenz attractor delay time $\tau = 250$.

The integral is not of direct interest, but I make use of the scaling relation,

$$C(\varepsilon) \propto \varepsilon^{\nu},$$

where ν is the correlation dimension. GP have shown that ν is a lower bound for the fractal dimension. In practice, most analysts construct a plot of the statistics Eq. (21) for a range of ε's, and estimate ν with the logarithmic change, $\mathrm{d}\log[C(m,N,\varepsilon)]/\mathrm{d}\log[\varepsilon]$.

Table 1
GLS dimension estimates [a]

Delay	Dimension Estimate
Lorenz attractor	
1	0.711 (0.021)
75	1.538 (0.028)
155	2.108 (0.035)
250	1.675 (0.005)
FF/DM Exchange rate	
1	0.707 (0.007)
30	2.042 (0.067)
60	2.461(0.027)
90	2.189 (0.006)
120	2.330 (0.006)
150	2.278 (0.012)

[a] For the Lorenz estimates, I simulate 2,500 observations. There are 1,437 exchange rate data points. For both estimates, I use 25 autcovariances over a range of 15 ε's from 0.5 to 1.5 standard deviations. For the Lorenz data, the embedding dimension m=3, and for the exchange rate data, m=4.

Brock and Baek (1991) show that this log change is a U-statistic and compute its standard error under the assumption of i.i.d.

I follow an alternative procedure proposed by Hiemstra (1992) that accounts for weak dependence in the data. Hiemstra constructs a j-vector of the statistics Eq. (21), $y(m,N,\varepsilon)=(C(m,N,\varepsilon_1),...,C(m,N,\varepsilon_j))$, and regresses them on the log ε's. His GLS estimates of ν handle the weak dependence with Parzen weights on k lagged autocovariances. In the examples, I set j=15, ranging from 0.5 to 1.5 sample standard deviations for ε, and I set k=25.[4]

I wanted to see whether the choice of delay time would influence the dimension calculations. I report estimates in Table 1 at the four delay times in the phase portraits and plot them in Fig. 1. A smoothed curve is fit to show the dependence of the dimension estimate on time delay.

The estimates clearly show that the short delays lead to a biasing downward of the dimension estimates. The points nearly lie along a line at τ=1, and I estimate a dimension of 0.711. This is statistically different from the other estimates. This estimated correlation dimension rises smoothly until just after lag 150. At τ=155, the local minima of the SNT, I estimate ν=2.108, within two standard deviations of GP's estimate of 2.04 (GP 1983, p.347, Table 2).

The other interesting thing to note is that it is also possible to wait too long. Some structure starts to disappear in Fig. 6 as the attractor starts to fold up again for large τ. At lag 250, near the zero crossing of the autocorrelation function, the dimension estimate falls back to 1.675.

[4]Monte Carlo estimates in Hiemstra (1992) show this to be a useful range. Autocovariances beyond 25 made little or no difference to the estimates.

6. Application to the FF/DM exchange rate

A number of studies, previously cited in the introduction, have applied the method of Grassberger and Procasccia to estimate the complexity of economic and financial data. All these papers use filtered data, generally using log differences. Many also try to filter out the effects of linear dependence with ARMA models and even nonlinear dependence with GARCH models. With the exception of Mayfield and Mizrach (1992) and Guillaume (1994), none of these papers uses a delay time reconstruction.

The motivation for filtering in the economics literature is that the data are generally nonstationary. Differencing (or some other more sophisticated filter like Hodrick–Prescott) is used to render the data stationary, but as is known from a decade of unit root econometrics, many results are quite sensitive to the choice of filter. This is particularly true when it comes to dimension calculations. Chen (1993) and Ramsey et al. (1990) have arrived at very different conclusions about the correlation dimension of several monetary aggregates using different filtering techniques. It would be nice to analyze a time series of asset prices where the levels appear stationary *without* filtering.

Fortunately for our purposes, the majority of intra-European exchange rates have fluctuated in ±2.25% bands for more than 15 years in what is called the Exchange Rate Mechanism (ERM).[5] While these bands have been realigned several times, I focus on the period from January 12, 1987 to September 14, 1992 when there were no realignments in the ERM. This constitutes a sample of 1,437 daily observations in which the FF/DM exchange rate fluctuated between 3.3206 and 3.4197 FF per DM, around a central parity of 3.3539 FF/DM.

I approach these data in the same fashion as I did the Lorenz system. I analyze the autocorrelation function and graph it in Fig. 7. The zero crossing does not occur until lag 171. There is almost a year's worth of linear dependence in the levels of the data. This is a structure that I do not want to throw away.

The SNT locates a local minima at lag 60, well before the zero crossing. I then reconstruct the system using time delays at increments of 30 lags: τ=1, 30, 60, 90, 120 and 150. The dimension estimates for an embedding dimension of m=4 are reported in Table 1 and graphed in Fig. 7, along with a smoothed curve fit to those points.[6] At lag 60, I estimate a correlation dimension of ν=2.461. As with the Lorenz attractor, the dimension estimates are a concave function in the delay time with an inflection point just a little beyond the local minima of the SNT.

To substantiate a claim for nonlinear structure, I calculated correlation dimensions for a series of uniform random deviates on [0,1]. These are graphed in Fig. 7 as well. Using the standard errors in Table 1, I can easily reject that the dimension estimates for the exchange rate data equal to those for the random deviates. More important though, the estimates from the random data show no dependence on delay time.

[5] For more details on the ERM and an application of nonlinear modeling to the FF/DM exchange rate, see Mizrach (1995b).

[6] Normally, one would proceed to higher and higher embedding dimensions and see whether the correlation dimension plateaus. Data limitations prevent me from using m greater than 4.

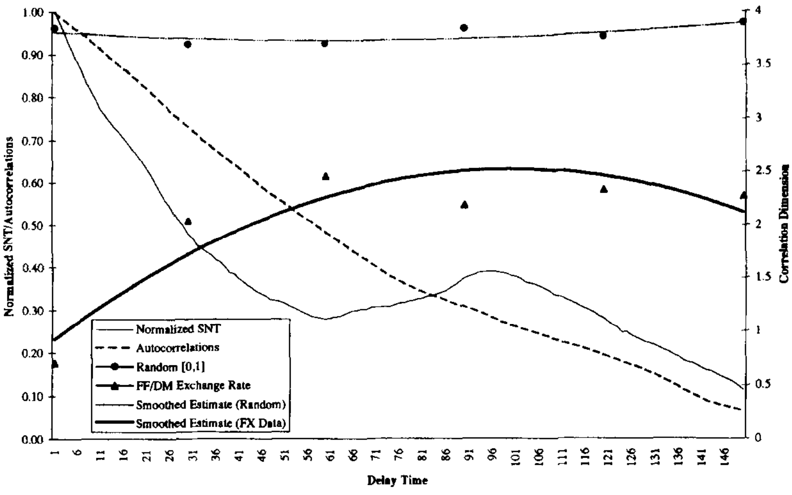

Fig. 7. Correlation dimension estimates for FF/DM exchange rate.

7. Conclusions

An unresolved question in the literature on dimension estimation has been the method of choosing a delay time for reconstruction. This paper proposed the use of a measure of nonlinear correlation which I call the simple nonparametric test (SNT). In dimension estimation of the Lorenz attractor, the SNT proved to be a more reliable indicator of delay time than a common rule of thumb, the first zero crossing of the autocorrelation function.

The utility of this procedure for economists was demonstrated in an application to the FF/DM exchange rate. I find nonlinear structure in this banded exchange rate using time-delayed data in the reconstruction.

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