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The state of economic dynamics

A review essay*

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1. Introduction

Until recently, differential equations was taught as an extension of integral calculus, with the focus on solving particular classes or systems of ordinary (scalar-valued) equations. The typical undergraduate curriculum was devoted to substitutions with the trigonometric functions, the Laplace transform, and other methods of obtaining explicit solutions. For many nonmathematicians, these recipes of solution techniques were quickly filed away and forgotten.

The modern approach, beginning with the work of the French mathematician Henri Poincaré in the late 19th century, has concentrated on qualitative properties of large classes of equations, linear and nonlinear. Poincaré initiated the topological approach to dynamic systems. His insight was that the ordinary differential equation

$$dx/dt \equiv \dot{x} = f(x,t) \tag{1}$$

could be studied from a geometric point of view in the $\dot{x}$, x phase plane with the solution, $\phi_t(x_0, t)$, being an orbit or trajectory parameterized by time, t, given an initial condition x_0.

*Book review of *Differential Equations, Stability and Chaos in Dynamic Economics* by W.A. Brock and A.G. Malliaris, ISBN 0-444-70500-7 (North-Holland, Amsterdam) $49.00.

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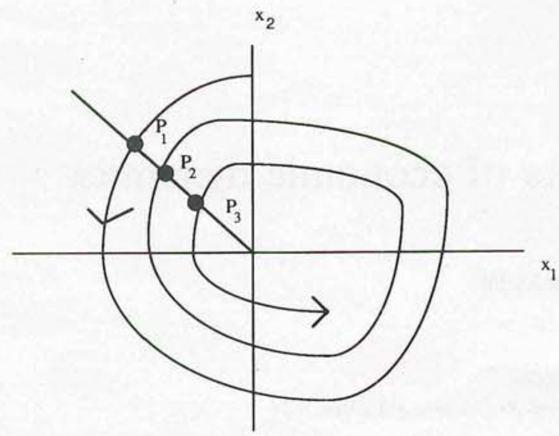

Fig. 1. Poincaré–Bendixson limit cycle.

For the two-dimensional case, Poincaré and Bendixson proved the existence of the limit cycle or closed orbit. With enough smoothness[1] to guarantee uniqueness, the Poincaré–Bendixson theorem can be seen geometrically[2] with reference to fig. 1. Consider the trajectory from P_1 to P_2 to P_3. Uniqueness ensures that the trajectory never crosses itself. Now consider a line segment of points connecting P_1, P_2, and P_3. These points form a Jordan curve which bisects the plane; the trajectory must always strike the segment from one direction. Clearly then, the orbit must continue to spiral inward. Bendixson was able to provide criteria to rule out the erratic dynamics. Peixoto (1962) completely classified the taxonomy of behavior for two-dimensional flows.

Steve Smale, a Field's medal winner now at Berkeley, conjectured that even in higher-dimensional systems there would be a finite number of closed orbits.[3] Smale was persuaded by a counterexample shown to him by Levison[4] that his conjecture was wrong. Smale then began work on his horseshoe example, culminating in his seminal 'Differentiable Dynamical Systems' [Smale (1967)]. The horseshoe was, Smale has written, 'the first structurally stable dynamical system with an infinite number of periodic solutions'. In this

[1] It is sufficient that f be Lipschitz. The proof is due to Cauchy and Peano.

[2] I draw heavily here on Coddington and Levinson (1955).

[3] The conjecture can be found in Smale (1960).

[4] Smale's own account of the conjecture can be found in the essay in Smale (1980): 'How I Got Started in Dynamical Systems'.

abstract example much of the new dynamics, unfortunately labelled as 'chaos', was born.

Guided by Smale and several mathematicians of the Soviet school,[5] the study of dynamics has adopted a global approach. The object of interest is a flow defined on some subset of Euclidean space or a differentiable manifold. Properties such as stability and uniqueness can then be deduced, even when explicit solutions are not known. The limiting behavior of flows can also be studied probabilistically. If the trajectories are defined on a metric space, ergodic theory studies the transformations that preserve measure.[6] This is a rapidly evolving field that I return to in section 4.

The book reviewed here, by William A. Brock of the University of Wisconsin at Madison and A.G. Malliaris of Loyola University of Chicago, follows a hybrid approach, incorporating some of the older methods with the new. The first part of my remarks is devoted to the classical material on solving differential equations and analysis of stability. The next section looks at the economic applications. Section 4 is devoted to chaotic dynamics. These roughly follow the book's chronology.

2. Reference material on differential equations

The first two chapters contain standard reference material on differential equations. The authors present the existence theorem of Cauchy and Peano and then establish existence and uniqueness of the method of successive approximations under weak assumptions. For linear differential equations, the focus is, not surprisingly, on explicit solutions. The general nth-order linear system is examined in great detail. The Jordan canonical form, used later in the stability analysis, is also presented.

Chapters 3 and 4 are a two-step introduction to stability analysis. The linear theory is developed in Chapter 3 and used to prove local stability for nonlinear systems, and asymptotic stability for perturbed systems. A brief digression on geometrical approaches is pursued towards the end of the chapter. Chapter 4 develops Liapunov's direct method. This chapter contains difficult material, and the authors wisely present numerous examples. I found the application of Liapunov's method to the growth method to be particularly instructive.

Brock and Malliaris develop in the last half of Chapter 4 some topological notions to study global asymptotic stability (g.a.s.). A dynamical system is said to be g.a.s. if the trajectories tend to a fixed point in the limit for all initial

[5]Don Saari has informed me that a paper by Sitnikov (1960) on the three-body problem was highly influential among Soviet mathematicians.

[6]This is the motivation of the classic work by Sinai (1969).

conditions. Fixed points are in what is called the ω-limit set[7] on a differential equation. A related concept is the nonwandering set which includes all points within a specified distance of recurrent trajectories.[8] Fixed points and limit cycles are in the nonwandering set.

Brock and Malliaris proceed with a sequence of theorems establishing g.a.s. under a variety of regularity conditions. Unfortunately, as Boldrin and Montrucchio (1986) note, stability theorems for concave programming problems depend crucially upon the rate of discounting, unless strong conditions are imposed. The theorems of Cass and Shell (1976), Rockafellar (1976), and Brock and Scheinkman (1976), developed in Chapter 4, all require the planner to be quite 'patient' for g.a.s. to hold. Absent a large discount factor, the nonwandering set can include virtually any type of dynamical behavior, including chaotic attractors.

I was somewhat disappointed that there was not even a brief development of bifurcation theory in this section. Bifurcation refers to critical parameter values at which a differential equation loses stability. Consider a family of differential equations,

$$\dot{x} = f_\mu(x), \qquad x \in R^n, \quad \mu \in R^k, \tag{2}$$

parameterized by μ. A solution to (2) is given by $f_\mu(x) = 0$. Guckenheimer and Holmes (1983) note that as μ varies the equilibria are described by smooth functions of μ away from those points of the Jacobian with zero eigenvalue. At an equilibrium (x_0, μ_0) with zero eigenvalue, branches of equilibria come together, making (x_0, μ_0) a point of bifurcation.

A prototypical example is the so-called pitchfork bifurcation in fig. 2. Consider the equation

$$\dot{x} = \mu x - x^3, \tag{3}$$

which has fixed points at $x = 0$ and $x = \pm\sqrt{\mu}$. Evaluating the Jacobian at $x = 0$, one sees that $x = 0$ is unstable for positive μ. From the bifurcation diagram, we see that stability is transferred to the points $x = \pm\sqrt{\mu}$.

To gain a more precise understanding of how dynamical systems produce seemingly random behavior, we must delve more deeply into bifurcation theory. Let us introduce the invariant set S, a subset of R^n, on which the flow

[7]The ω-limit set is the limit set of forward orbits. The backward orbits are in the α-limit set. See Brock and Malliaris, p. 114.

[8]A point p is called nonwandering for the flow $\phi_t(x)$ if for any neighborhood U of p there exists arbitrarily large t such that $\phi_t(U) \cap U \neq \varnothing$. See Guckenheimer and Holmes (1983, p. 33).

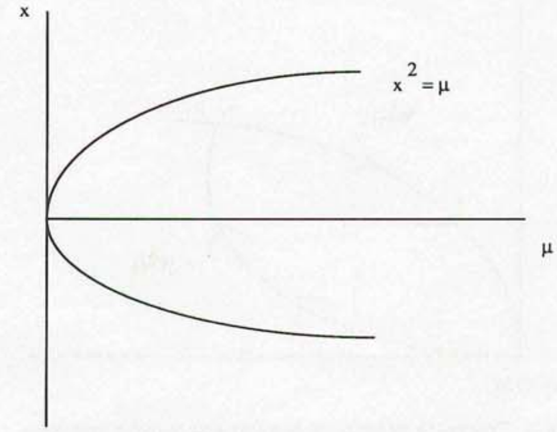

Fig. 2. Pitchfork bifurcation.

remains over time:

$$S \subset R^n \equiv \{\phi_t(x) \in S, \forall x \in S, t \in R\}. \tag{4}$$

The invariant set includes both the local stable and unstable manifolds associated with a fixed points $\bar{x}$:

$$W^{s_{loc}}(\bar{x}) \equiv \{x \in U | \phi_t(x) \to \bar{x} \text{ as } t \to \infty, \phi_t(x) \in U, \forall t \geq 0\}, \tag{5a}$$

$$W^{u_{loc}}(\bar{x}) \equiv \{x \in U | \phi_t(x) \to \bar{x} \text{ as } t \to -\infty, \phi_t(x) \in U, \forall t \leq 0\}, \tag{5b}$$

where $U \subset R^n$ is some neighborhood of $\bar{x}$.

Assume for expository purposes we are in R^2. Consider a diffeomorphism[9] f: $R^2 \to R^2$, with a compact invariant set. Let $\bar{x} = (0,0)$ be a hyperbolic fixed point. This means that the linearized system has no eigenvalues of zero real part, enabling us to determine stability directly. Suppose that the Jacobian consists of one stable and one unstable root, $0 < |\lambda_1| < 1$ and $|\lambda_2| > 1$. With reference to fig. 3, let p be the unique intersection of the stable and unstable

[9]A smooth mapping that is both one-to-one and onto and whose inverse is also smooth. This can be thought of as a Poincaré map of a higher-dimensional differential equation.

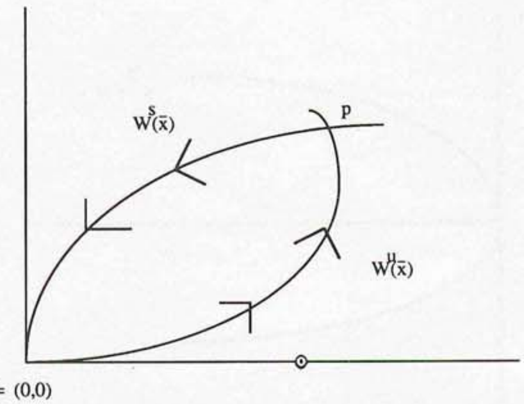

Fig. 3. Transverse intersection of stable and unstable manifolds.

manifolds. In a neighborhood of p, the dynamics of the diffeomorphism will resemble Smale's horseshoe.

Geometrically, the horseshoe involves a stretching and folding of topological space. Consider the unit square $S=[0,1]\times[0,1]$ and define a mapping $f\colon S\to R^2$, which contracts the unit square horizontally by λ_1 and expands it vertically by λ_2. You then fold the long thin strip as in fig. 4(ii). This produces two vertical bands V_1 and V_2 in the unit square. The pre-image $f^{-1}(S\cap f(S))$ consists of two horizontal strips, H_1 and H_2, as in fig. 4(iii). Repeating this transformation, $S\cap f(S)\cap f^2(S)$ produces, as in fig. 4(iv), four rectangles by removing the nonshaded segments in fig. 4(ii). We readily see that $\Lambda=\{x|f^t(x)\in S-\infty<t<\infty\}$ is a Cantor set.[10] Smale has shown that Λ is invariant under f and it is topologically equivalent to a Markov partition[11] of the neighborhood.

The horseshoe is the mathematical idealization of the erratic behavior near where stable and unstable trajectories meet.[12] Points that are initially near one another are separated by the expansion and contraction of S. While an invariant set gains stability in the horseshoe, the set is *not* an attractor. We

[10]This construction is identical to the classic Cantor set produced on $[0,1]$ by removing the middle third each time.

[11]One can divide the neighborhood of the intersection of the stable and unstable manifolds into horizontal and vertical rectangles and assign symbols to each. The trajectory will trace out a sequence of symbols as it moves from region to region. The symbol sequence describes a Markov partition.

[12]The point of intersection is called a homoclinic point. The class of Axiom A dynamical systems, which includes the horseshoe, satisfy two conditions: (i) the invariant set S is hyperbolic, and (ii) fixed points and periodic orbits are dense in S.

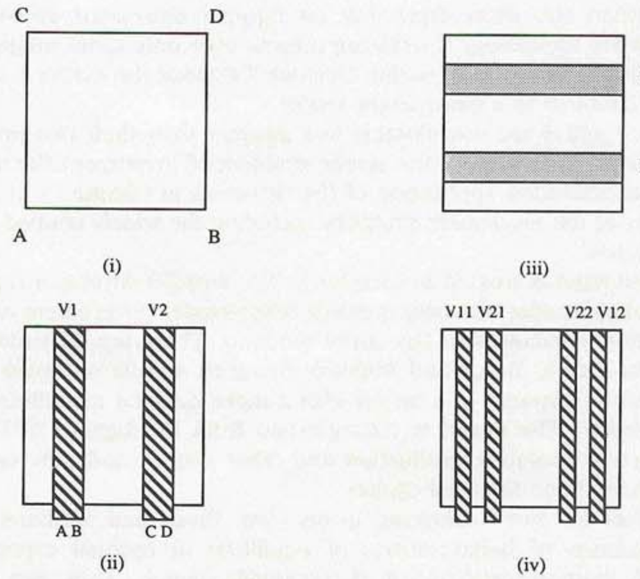

Fig. 4. The Smale horseshoe.

return to this topic in section 4. Excellent references on this material are the books by Palis and de Melo (1982), Guckenheimer and Holmes (1983), and Ruelle (1989). I have drawn freely on all of these in this section.

3. Economic applications

There are five chapters devoted directly to economic applications. Combined with the economic examples in the reference chapters, there is a wealth of material for the applied worker. These chapters should become part of the canon in economic theory.

Chapter 5 is devoted to the optimal control problem and is concerned with isolating sufficient conditions on the Hamiltonian of the control problem for global asymptotic stability. As all of the conditions require concavity of the Hamiltonian in the state and control, Brock and Malliaris take up the topic of increasing returns in Chapter 6. Since much of the revival in growth theory has focused on the importance of increasing returns,[13] this is a welcome addition to the literature. My enthusiasm is tempered by the authors' choice of studying the increasing returns in the context of the two-sector growth

[13]See, e.g., the work of Romer (1986).

model. I found the short digression on optimal one-sector growth with convex–concave technology (increasing returns over only some range of the state variable) to be the most useful. Chapter 9 extends the earlier results on the control problem to a multi-sector model.

Chapters 7 and 8 are considerably less abstract than their two predecessors. Chapter 7 is devoted to the classic problem of investment theory. It is essentially an extended application of the theorems in Chapter 5 to various formulations of the investment problem, including the widely studied adjustment cost model.

Macroeconomics is treated in Chapter 8. The authors develop a representative household model with consumption, labor supply, government expenditure, and money balances in the utility function. There are 22 variables in total, but, in return, Brock and Malliaris integrate results on optimal consumption and investment in a model with a more detailed modelling of the corporate sector. The model is distinguished from the Lucas (1978) 'tree' economy by incorporating production and labor supply, and differentiating between physical and financial capital.

The chapter has two weaknesses in my view. Brock and Malliaris fail to link the problem of indeterminacy of equilibria in rational expectations models with multiple equilibria in deterministic models. They also do not devote sufficient attention to fluctuations which, in my view, is what distinguishes macroeconomics from growth theory. The burgeoning real business cycle literature[14] is devoted to calibrating stochastic variants of Brock and Malliaris' model to realized output fluctuations. Readers interested in stochastically perturbed differential equations are referred to the earlier book of Malliaris and Brock (1982).

4. Complex dynamics

There is only one chapter on chaotic dynamics in the book, and I warn the reader that I am spending a disproportionate amount of time on this topic. The leap to ergodic theory in this chapter, I felt, was too abrupt. Pedagogical continuity would have been served by some development of abstract dynamical systems on measure spaces as in Bhatia and Szego (1970) or Mane (1983). This would also provide a link to the Markov theory, a useful step in a book devoted almost exclusively to deterministic models.

I will provide some background for the reader that begins by defining some ergodic invariants on an attractor. This motivates the statistical analysis of nonlinear dynamics that consumes the bulk of Chapter 10. The economic applications are reserved to a separate subsection.

[14] For a critical survey of the literature see McCallum (1986).

4.1. Chaos: Ideas and definitions

If a dynamical system $f\colon R^n \to R^n$ possesses an attractor, Λ, a closed indecomposable invariant set, to which nearby trajectories are mapped, we can study the global long-run dynamics of an ensemble of orbits. There exists on the attractor a unique, ergodic probability measure ρ, such that for Lebesgue almost everywhere x in the basin of attraction of Λ

$$\lim_{T\to\infty} 1/T \sum_{i=0}^{T-1} g(f^i(x)) = \int_\Lambda g(x)\,\mathrm{d}\rho, \tag{6}$$

where $g\colon R^n \to R$ is any continuous function.[15] The existence of this ergodic probability measure enables us to study a dynamical system probabilistically. The time averages of the flows will reproduce ρ, for almost any initial x.

We will call an attractor, following Ruelle and Takens (1971), strange if it possesses sensitive dependence on initial conditions. Sensitive dependence refers to the property of exponential divergence of nearby trajectories that we first observed in Smale's horseshoe. As these systems evolve through time, small discrepancies in the initial state become magnified, eventually becoming distinct trajectories. Even if the system were known with complete certainty, small measurement errors would limit your ability to predict into the future.

Since the rates of expansion and contraction will vary along the trajectory, we must analyze limiting time averages. Following Guckenheimer and Holmes (1983), define by $T_x R^n$ the set of all tangent vectors to R^n at $x \in S$, where S is an invariant set. Assume that T_x can be decomposed into subspaces: $T_x R^n = E_x^1 \oplus E_x^2 \oplus \cdots \oplus E_x^n$. For the horseshoe, with $n = 2$, E_x^1 and E_x^2 are vertical and horizontal lines formed from the iterates of f: $\bigcap_{t=0}^{\infty} f^t(S)$ and $\bigcap_{t=0}^{\infty} f^{-t}(S)$. Since x in fig. 5 is a homoclinic point, vectors in E^1 (resp. E^2) are contracted exponentially in forward (backwards) time.

Let D denote the derivative function. Consider the limit

$$\lim_{t\to\infty} 1/t \| D_x f^t u \| \equiv \lambda_j, \qquad \lambda_1 > \lambda_2 > \cdots > \lambda_n, \tag{7}$$

for every $u \in E_x^j$, $1 \le j \le n$, where $\|\cdot\|$ is the Euclidean norm. The spaces E_x^j are called the eigenspaces of f, and the λ_j are called the Liapunov exponents. Oseledec (1968) proved that this limit exists for ρ almost everywhere x. For $n = 1$, the Jacobian is just a scalar at each t, and the theorem reverts to the ordinary (nonmultiplicative) ergodic theorem. The Liapunov exponents

[15] A similar limit will pertain for flows as well: $\lim_{T\to\infty} 1/T \int_0^T g(\phi_t(x))$

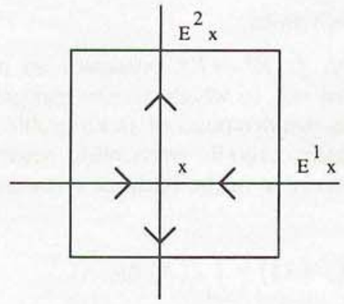

Fig. 5. Tangent vector decomposition.

tell us the average rate of expansion or contraction along the entire trajectory.

The Liapunov exponents are also directly related to entropy, a property generally associated with random, not deterministic systems. Before getting into the technicalities, let's begin an intuitive discussion of entropy. Consider a six-sided die, and define the nonnegative quantity

$$H \equiv -\sum_{i=1}^{6} p_i \log(p_i), \tag{8}$$

where p_i is the probability of the ith face appearing. assume $p_i = \frac{1}{6}$ for all i, indicating a fair die. Entropy tells us the amount of randomness in an experiment. We see from (8) that a fair die is more random than a loaded one in which some faces come up more than others. A fair twelve-sided die is more entropic than a six-sided one.

Introduce now an abstract probability space[16] $(\Omega, \mathfrak{T}, \rho)$ on which we can define a dynamical system. Ω is the outcome space. For the dice, it can be thought of as, say, all possible observed sequences in two rolls of the die: $\Omega = \{(1,1),(1,2),\ldots,(6,6)\}$. An event is some subset of points in Ω. $\mathfrak{T}$ is a σ-algebra of subsets of Ω, closed under the formation of complements and countable unions of events in $\mathfrak{T}$. ρ is a probability measure, a set function assigning probability masses to the events in $\mathfrak{T}$, $0 \le \rho(A) \le 1$, for $A \in \mathfrak{T}$, and $\rho(\Omega) = 1$. The support of ρ is the smallest $\mathfrak{T}$-set A for which $\rho(A) = 1$.

Following Eckmann and Ruelle (1985), let $A = (A_1, \ldots, A_n)$ be a finite partition of the support of ρ, a family of nonempty, disjoint sets that are ρ-measurable. Each A_i can be thought of as a single event in Ω. Now, we

[16]An excellent reference on this material is Dudley (1989).

restate (8) as the entropy of the finite field A:

$$H(A,\rho) = -\sum_{i=1}^{n} \rho(A_i)\log\rho(A_i). \tag{9}$$

Denote by $f^{-k}A_i$ the set of points mapped by f^k to A_i, and let $f^{-k}A$ be the partition $(f^{-k}A_1,\dots,f^{-k}A_n)$. Finally, define

$$A^{(t)} \equiv A \vee f^{-1}A \vee \cdots \vee f^{-t+1}A, \tag{10}$$

which is the partition generated by A in a time interval of length t. The entropy of the field A relative to f is given by

$$h(A,f,\rho) \equiv \lim_{t\to\infty} 1/tH\left(\bigvee_{k=0}^{t-1} f^{-k}A\right) = \lim_{t\to\infty} 1/tH(A^{(t)},\rho). \tag{11}$$

This measures the average uncertainty per unit of time about which element $A_i \in A$ the orbit of a dynamical system will enter under time evolution. If f were globally stable, the entropy would be zero. A final refinement shrinks the diameter of the partitions to zero,

$$h(\rho) = \lim_{d(A)\to 0} h(A,f,\rho), \tag{12}$$

yielding the measure theoretic entropy.

Pesin (1977) has shown that if f is C^2 and ρ is absolutely continuous with respect to Lebesgue measure,

$$h(\rho) = \sum \lambda_{j+}, \tag{13}$$

where λ_{j+} are the positive Liapunov exponents. For a dynamical system to have positive entropy, it must have a highly erratic trajectory. This type of nonwandering set requires the stretching and folding of topological space implied by a positive Liapunov exponent.

4.2. Economic applications

The study of complex dynamics in economics has been pursued along two lines. The first is a theoretical literature developing economic examples of chaos. The second is an empirical literature analyzing time series data for evidence of chaotic dynamics. The middle third of the chapter is devoted to chaos in macroeconomics, but it is here that I begin.

The existence of cycles in overlapping generations economies was first demonstrated by Gale (1973). Simple examples of chaotic dynamics were then presented in Stutzer (1980) and Benhabib and Day (1981). These papers rely on the theorem of Li and Yorke (1975), which is in turn a special case of the theorem of Sarkovskii (1964). A difference equation with a periodic equilibrium of order three yields, for alternative parameterizations, equilibria of any integer period. A set of aperiodic points also exists.

Grandmont (1985) provides the following illustrative example. Consider a two-period overlapping generations model. The representative agent has utility over consumption, c_t, and leisure, l_t^*,

$$U_t(c_t, l_t^*), \qquad t = 1, 2, \tag{14}$$

but he can expand labor effort, l_t, to produce consumption goods. The choice problem for the agent can be shown to depend solely on the relative money prices of goods, $\theta \equiv p_{t+1}/p_t$, which can be thought of as the real wage. A simple way to depict this graphically is with an offer curve, as in fig. 6 where the dynamics can be described as a difference equation in p_t and p_{t+1}. As θ increases, there are both substitution and income effects on labor input. To obtain periodic cycles, income effects must become dominant and make the offer curve bend backward.[17] In fig. 6, a three-cycle emerges, assuring us the existence of complex dynamics. Grandmont has offered this work as an alternative to the conventional framework that business cycle fluctuations are driven by exogenous shocks.[18]

In Arrow–Debreu economies, Sonnenschein (1972) proved that the class of excess demand correspondences includes all smooth vector-valued functions that satisfy Walras' law. Saari (1985) then demonstrated chaotic behavior in the adjustment of the equilibrium price vector. Models of the tatonnement that depend only upon the first derivative of the excess demand correspondence[19] will fail to converge on an open set of initial prices. Furthermore, the nonconvergent price sequence is uncountable, indicating highly random dynamics.

The genericity of complex dynamical phenomena is problematic for economics. It seems counterintuitive that the optimal capital shock should fluctuate randomly. While it is indeed a challenge to construct parametric

[17]Azariadis and Guesnerie (1986) have noted that the very same conditions are sufficient for the existence of sunspot equilibria.

[18]This view is developed in Lucas (1987).

[19]Newton's method is an example. This type of mechanism simply raises the price of commodities in excess demand.

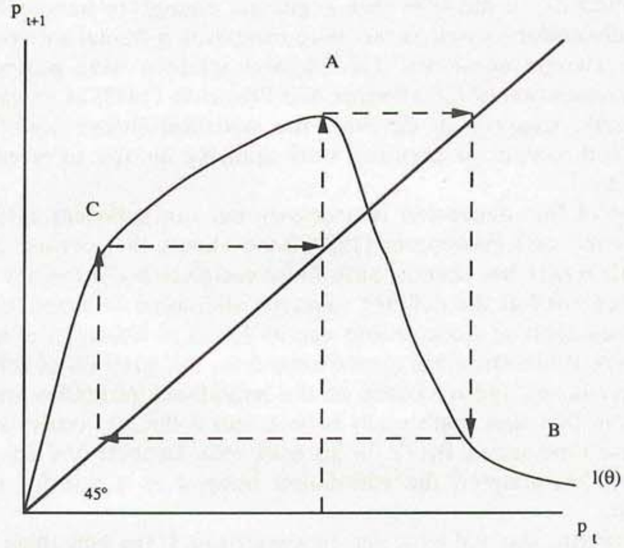

Fig. 6. A cycle of period 3.

examples apart from a small class of well-understood maps[20], the more difficult work is finding conditions to limit the asymptotic behavior to simple cases like fixed points. Some steps in this direction are in Boldrin and Montrucchio (1990).

A second line of research has been empirical. The early work of Brock (1986) sought confirmation of Grandmont's view. The question was to distinguish between random and deterministic systems. According to a theorem of Takens (1983), it is possible to construct a diffeomorphism of an attractor from a scalar time series. The embedding preserves invariants of a dynamical system, including dimension and entropy.

Dimension is a measure of complexity.[21] We want to know whether the data-generating mechanism includes a large number of state variables. Ide-

[20] The logistic equation, $x_{t+1} = \mu x_t(1 - x_t)$, a popular model in the biological sciences for modelling the ebb and flow of populations, has seen wide application. May (1976) showed that this very simple map could generate very complicated dynamics. For continuous time systems, we must look outside of the Poincaré–Bendixson plane for chaos. The climate equations of Lorenz (1963), a three-state variable model, have played the role of the logistic equation for models of flows.

[21] An excellent intuitive discussion of dimension is in Farmer (1982). Topological dimension coincides with our notion of Euclidean dimension. The capacity deals with the rate of growth of a set that 'covers' the attractor. Information dimension is a probabilistic concept that tells us the capacity of the more commonly visited regions of the attractor. The correlation dimension bounds the information dimension from below, and the capacity bounds it from above.

ally, we would like a measure that is general enough to handle Euclidean notions of dimension as well as the more complicated fractal structures often found with chaotic attractors. The physical sciences have settled on the correlation dimension of Grassberger and Procaccia (1983) as an experimental benchmark. Chapter 10 develops the statistical theory underlying the procedure and reports on empirical work applying the test to economic and financial data.[22]

A finding of low dimension is necessary but not sufficient evidence for chaos. Osborne and Provenzale (1989) have shown that certain stochastic systems with power law spectra have finite correlation dimensions. Entropy should be regarded as the defining characteristic; some evidence for positive Liapunov exponents in stock returns can be found in Eckmann et al. (1988).

The empirical literature has turned away from the question of randomness versus determinism, and refocused on the importance of nonlinearities. The authors admit that they don't really believe that a chaotic system is generating economic time series. Brock, in his work with Dechert and Scheinkman, BDS (1987), has adapted the correlation integral as a test for nonlinear dependence.

For economists, the real issue may be prediction. If the only thing we know about the data-generating mechanism is that it has a smooth explanation in terms of its own past, the data analyst is naturally led to either nonparametric formulations, orthogonal expansions, or state space modelling. Absent knowledge of the true model, knowing the dimension is only the first step in identification. The best model for prediction in a particular norm is unlikely to be of the same dimension as the attractor.[23]

5. Conclusion

Brock and Malliaris intend their book for advanced undergraduate and graduate students. The large number of worked out examples and the focus on economic applications have not subtracted from the rigor in any way. Those seeking more than an introduction to chaotic dynamics will be disappointed, but the references cited here should help the interested reader along. As a reference work, *Differential Equations, Stability and Chaos in Dynamic Economics* belongs on any shelf. The authors are to be commended for advancing the state of knowledge of economic dynamics.

[22]Several papers report low dimension estimates for asset price data. See Scheinkman and LeBaron (1989) or Mayfield and Mizrach (1990) on stock prices and Frank and Stengos (1990) for precious metals.

[23]This argument is developed rigorously in Mizrach (1990).

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