Advanced Economics Statistics

FALL 2010

Ninth-Assignment Answer Sheet

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50 full points

1. This question is based on Examples 8.3.18–8.3.20 on pp.392–393 of CB. We are going to show that for $\vartheta > \vartheta_o$ the power curve $\beta_3(\vartheta)$ never catches up with the power of curve $\beta_2(\vartheta)$.

   (a). For some sufficient statistic $T(x) = \overline{x}$ of $X_i \sim N(0, \sigma^2)$, We reject Ho: $\vartheta \leq \vartheta_o$ if $\overline{x}$ is getting far from ϑ_o in statistical terms. Then, we can reckon the power of test describing what the reject probability is when ϑ_o is true. In this case we have that $\overline{x} > \vartheta_o + \frac{\sigma Z_\alpha}{\sqrt{n}}$, thus we are looking for

$$\Pr\left(\overline{x} > \vartheta_o + \frac{\sigma Z_\alpha}{\sqrt{n}}\right)$$

   in other terms

$$\Pr\left(\frac{\overline{x} - \vartheta_o}{\frac{\sigma}{\sqrt{n}}} > Z_\alpha\right)$$

   equivalently

$$\Pr\left(\frac{\overline{x} - \vartheta}{\frac{\sigma}{\sqrt{n}}} > Z_\alpha + \frac{\vartheta_o - \vartheta}{\frac{\sigma}{\sqrt{n}}}\right) \tag{1}$$

   where ϑ is the mean of the estimator $\overline{x}$ which comes from a particular sample. We can write down the expression (1) as follows:

$$\beta_2(\vartheta) = \Pr\left(\frac{\overline{x} - \vartheta}{\frac{\sigma}{\sqrt{n}}} > Z_\alpha + \frac{\vartheta_o - \vartheta}{\frac{\sigma}{\sqrt{n}}}\right) = 1 - \Phi\left(Z_\alpha + \frac{\vartheta_o - \vartheta}{\frac{\sigma}{\sqrt{n}}}\right) \tag{2}$$

   Where Φ is the normal cumulative distribution function. Thus, we define $\beta_2(\vartheta)$ as the probability which tell us how many times out of 100 cases we can reject the Ho when that effectively is not true. This function is sketched as follows for different

1) 18
6) 10
2 10
3 10
4 10

values of ϑ.

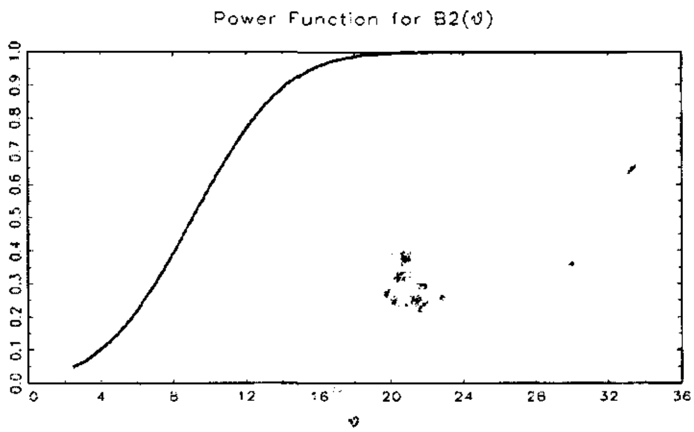

Figure 1.1

In the case of $\beta_3(\vartheta)$ we look for the probability of rejecting Ho when H_o: $\vartheta = \vartheta_o$ and H_1: $\vartheta \neq \vartheta_o$. Thus we need to look at the following cases $\overline{x} > \frac{Z_{\alpha/2}\sigma}{\sqrt{n}} + \vartheta_o$ or $\overline{x} < -\frac{Z_{\alpha/2}\sigma}{\sqrt{n}} + \vartheta_o$ .We formalize this as follows:

$$\beta_3(\vartheta) = \Pr\left(Z_{\overline{x}} > Z_{\alpha/2} + \frac{\vartheta_o - \vartheta}{\frac{\sigma}{\sqrt{n}}}\right) + \Pr\left(Z_{\overline{x}} < -Z_{\alpha/2} + \frac{\vartheta_o - \vartheta}{\frac{\sigma}{\sqrt{n}}}\right) \quad (3)$$

$$= 1 - \Phi\left(Z_{\alpha/2} + \frac{\vartheta_o - \vartheta}{\frac{\sigma}{\sqrt{n}}}\right) + \Phi\left(-Z_{\alpha/2} + \frac{\vartheta_o - \vartheta}{\frac{\sigma}{\sqrt{n}}}\right) \quad (4)$$

In the following figure we plot the function $\beta_2(\vartheta)$ and $\beta_3(\vartheta)$ for $\vartheta > \vartheta_o$, we consider the following parameters $n = 25, \sigma = 20, \alpha = .05$, and $\vartheta_o = 2.5$.

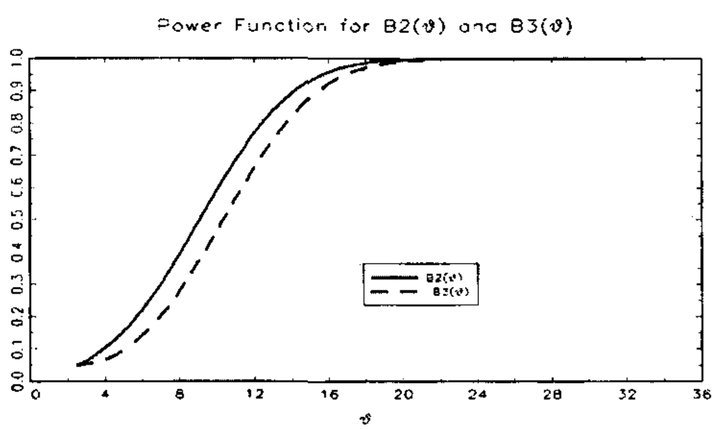

Figure 1.2

we implement al our calculations by using GAUSS programming (please see the appendix for details). In the following picture we show the difference between $\beta_2(\vartheta)$ and $\beta_3(\vartheta)$:

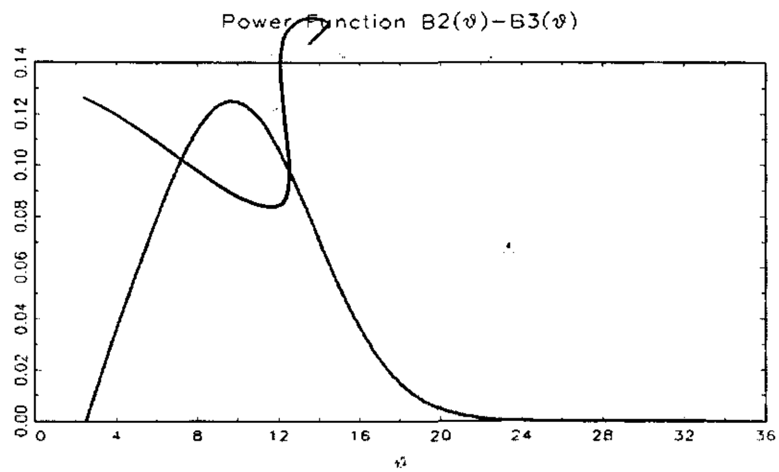

Figure 1.3

In the GAUSS program we have used the following commands for generating the power;

- `B2_thetha=1-cdfn(cdfni(1-alpha)+(thetha_0-thetha_seq)./(sigma/n^0.5));`
- `B3_thetha=1-cdfn(cdfni(1-alpha/2)+(thetha_0-thetha_seq)./(sigma/n^0.5))+ cdfn(cdfni(alpha/2)+(thetha_0-thetha_seq)./(sigma/n^0.5));`

Where `thetha_seq` is a vector wich map a subset of space of ϑ, it is $\vartheta \in (2.5, \infty)$ (See the GAUSS code in appendix).

(b). As we can see in figure 1.3 the maximum value of $f(\vartheta) = \beta_2(\vartheta)$ and $\beta_3(\vartheta)$ is between 10.5 ($= \vartheta_o + 8$) and 14.5 ($= \vartheta_o + 12$). We derive the first order condition in order to get a maximum:

$$
\begin{aligned}
f(\vartheta) &= 1 - \Phi\left(Z_\alpha + \frac{\vartheta_o - \vartheta}{\frac{\sigma}{\sqrt{n}}}\right) - \\
&\quad \left[1 - \Phi\left(Z_{\alpha/2} + \frac{\vartheta_o - \vartheta}{\frac{\sigma}{\sqrt{n}}}\right) + \Phi\left(-Z_{\alpha/2} + \frac{\vartheta_o - \vartheta}{\frac{\sigma}{\sqrt{n}}}\right)\right] \\
\frac{\partial f(\vartheta)}{\partial \vartheta} &= -\Phi'\left(Z_\alpha + \frac{\vartheta_o - \vartheta}{\frac{\sigma}{\sqrt{n}}}\right)\left(-\frac{1}{\frac{\sigma}{\sqrt{n}}}\right) + \Phi'\left(Z_{\alpha/2} + \frac{\vartheta_o - \vartheta}{\frac{\sigma}{\sqrt{n}}}\right)\left(-\frac{1}{\frac{\sigma}{\sqrt{n}}}\right) \\
&\quad - \Phi'\left(-Z_{\alpha/2} + \frac{\vartheta_o - \vartheta}{\frac{\sigma}{\sqrt{n}}}\right)\left(-\frac{1}{\frac{\sigma}{\sqrt{n}}}\right) \\
&= 0
\end{aligned}
$$

$\left.\frac{\partial f(\vartheta)}{\partial(\vartheta)}\right|_{\theta^*} = 0$ not in general

Φ denotes the cumulative distribution function. The former is equivalent to;

$$\frac{\partial f(\vartheta)}{\partial \vartheta} = -\phi\left(Z_{\alpha} + \frac{\vartheta_o - \vartheta}{\frac{\sigma}{\sqrt{n}}}\right) + \phi\left(Z_{\alpha/2} + \frac{\vartheta_o - \vartheta}{\frac{\sigma}{\sqrt{n}}}\right) - \phi\left(Z_{\alpha/2} + \frac{\vartheta_o - \vartheta}{\frac{\sigma}{\sqrt{n}}}\right) = 0 \quad (5)$$

where ϕ denotes the probability distribution function. We perform an iteration process in order to get the value of ϑ which fulfill the previous expression (5). We find the argument which solved the expression (5) by using the `solver` function in excel tools. In teh following picture we show how the first order condition look across a subspace of possible solutions:

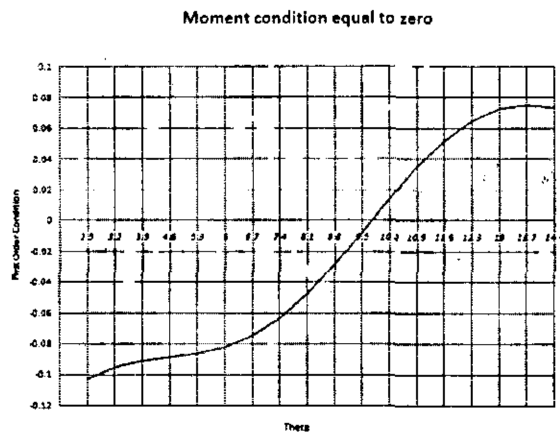

Figure 1.4

We find the value to be 9.72 by using `solver`.

2. We show the linear regression model in matricial terms;

$$Y = X\beta + \xi$$

Assuming that $X = (x_1' \;\; x_2' \;\; x_3' \;\; x_4' \;\; x_5')'$. ξ is a vector with $i.i.d$ errors. We want to test the following:

$$H_o \; : \; \begin{bmatrix} \beta_2 + \beta_3 \\ \beta_4 + \beta_5 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 0.0 \end{bmatrix} \quad (6)$$

$$H_o \; : \; \begin{bmatrix} \beta_2 + \beta_3 \\ \beta_4 + \beta_5 \end{bmatrix} \neq \begin{bmatrix} 1.0 \\ 0.0 \end{bmatrix}$$

the true parameters are $\beta = (\beta_1 \;\; \beta_2 \;\; \beta_3 \;\; \beta_4 \;\; \beta_5) = (1.0 \;\; 4.0 \;\; -1.0 \;\; 2.5 \;\; -2.5)$. Also, we know that $\sigma = 2.0$.

(i). We report the F-test statistic and the P-value for testing the proposal restriction in (6):

$$\chi_q^2(0) = \frac{(R\beta - r)'\,(R'\,(X'X)\,R')^{-1}\,(R\beta - r)}{\sigma^2} \quad (7)$$

but we need an estimate σ^2, we know that $\frac{\xi'\xi}{\sigma^2}$ is distributed as χ^2 with $T-K$ degrees of freedom. In order to test those restrictions we need to find an estimate for σ^2. Thus, we construct and F test, which is constructed as follows

$$F = \frac{\frac{\nu_1}{\text{d.f of } \nu_1}}{\frac{\nu_2}{\text{d.f of } \nu_2}}$$

where both variables ν_1 and ν_2 come from a χ^2 and d.f of ν_i denotes degrees of freedom of variable ν_i. Thus, $\nu_1 = \frac{(R\beta-r)'(R'(X'X)R')^{-1}(R\beta-r)}{\sigma^2}$ and $\nu_2 = \frac{\xi'\xi}{\sigma^2}$, the statistic is:

$$F = \frac{\frac{(R\beta-r)'(R'(X'X)R')^{-1}(R\beta-r)}{\sigma^2}}{\frac{\xi'\xi}{\sigma^2}} \cdot \frac{T-K}{q}$$

re-arranging terms

$$F = \frac{\frac{(R\beta-r)'(R'(X'X)R')^{-1}(R\beta-r)}{q}}{\frac{\xi'\xi}{T-K}}$$

The F-test value with 2 and 95 degrees of freedom is 0.909, and the p-value associated to that value is 0.407. We calculate the p-value by using the following comand (see GAUSS code in appendix):

- `cdffc(F_test,q,n-rows(betas));`

Thus, considering the value of probability (p-value) we cannot reject the null hypothesis which say the parameters can fulfill the restriction.

(ii). We compute the non-central parameter according to handout 8:

$$\delta = \frac{(R\beta_o - r)'\left(R'\left(X'X\right)R'\right)^{-1}(R\beta_o - r)}{\sigma_o^2}$$

which is evaluated at the true parameters (β_o, σ_o^2). R, r and X defined in question 1. Thus $\delta = 2.863$

(iii). The power of this test is calculated by using the following commands (See GAUSS code in appendix):

- `Power_deltha = 1-cdffnc(3.0922,q,n-5,non_central_deltha);`

by using the previous command we have 0.714 as a value related to the power. [1]

[1] All names of variables in the GAUSS lines are defined in the appendix.

3. We draw the power curve for $\delta > 0$ (please see GAUSS code in the appendix for details):

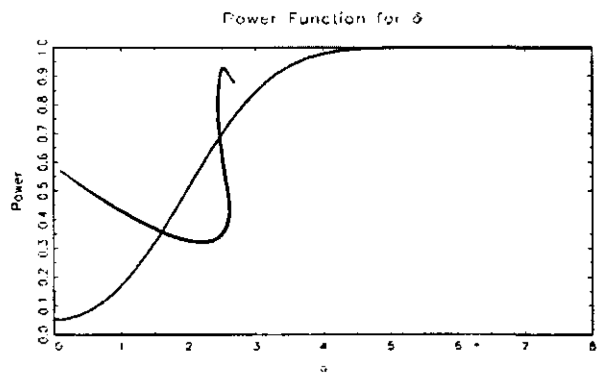

Figure 3.1

4. According to the hand out we specify the Wald Test (WT):

$$WT = g\left(\widehat{\theta}_K\right)' \left[F' I\left(\widehat{\theta}_K\right)^{-1} F\right] g\left(\theta_K\right)$$

where $\widehat{\theta}_K$ is the maximum likelihood estimate of under the alternative hypothesis; $\widehat{F}$ is F evaluated at $\widehat{\theta}_K$, and $I(\widehat{\theta}_K)$ is the sample estimate of the Fisher's information matrix evaluated at alternative hypothesis[2],.we recall from assignment 8 that the information matrix is:

$$I\left(\theta_K\right) = -E\left(\left.\frac{\partial^2 \log L}{\partial\theta\partial\theta'}\right|_{\theta_K}\right) = \begin{bmatrix} \frac{(X'X)}{\widetilde{\sigma}^2} & 0 \\ 0 & \frac{n}{2\widetilde{\sigma}^4} \end{bmatrix}$$

In other terms we have that WT is

$$WT = \frac{\left(R\widehat{\beta} - r\right)' \left(R'\left(X'X\right)R'\right)^{-1}\left(R\widehat{\beta} - r\right)}{\widehat{\sigma}^2}$$

$\widehat{\sigma}^2$ denotes the unrestricted variance which comes from the estimation, $\widehat{\beta}$ is the vector with estimated parameters. In the case of Lagrange Multiplier Test (LMT) we have

$$LMT = \frac{\left(R\widehat{\beta} - r\right)' \left(R'\left(X'X\right)R'\right)^{-1}\left(R\widehat{\beta} - r\right)}{\widetilde{\sigma}^2}$$

$\widetilde{\sigma}^2$ denotes the restricted variance which comes from imposing the restriction; in fact $\widetilde{\sigma}^2 = \widehat{\sigma}^2 + \frac{1}{n}\left(R\widehat{\beta} - r\right)' \left(R'\left(X'X\right)R'\right)^{-1}\left(R\widehat{\beta} - r\right)$, .as we see this means that LMT uses assumptions under the null hypothesis[3]; because of this $LMT \leq WT$. Thus, we can

[2] In the handout $F = [R \;\; 0]$

[3] See "*Three Classical Tests; Wald, LM(Score), and LR tests*" at `http://instruct1.cit.cornell.edu/courses/econ620/reviewm7.pdf`

contruct the Lagrange multiplier and Wald Test; the picture 4.1 gives an idea about the equivalence of those 2 tests in comparison with the Log-likelihood ratio test[4] :

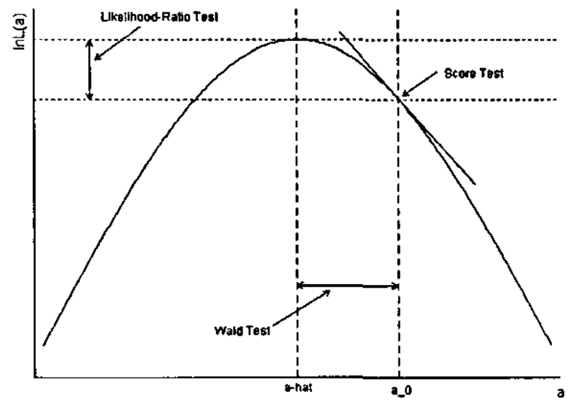

Figure 4.1

In the appendix the GAUSS code we made for performing the WT and LMT test give us tch following results:

$$\begin{aligned} LMT &= 1.785 \\ WT &= 1.817 \end{aligned}$$

See GAUSS code for details. The p-value for those statistics are 0.4096 and 0.4031 for LMT and WT respectively, those p-values were calculated by using the folowing command in GAUSS

- `cdfchic(1.785,2);`
- `cdfchic(1.817,2);`

We can say that there is no evidence which reject the null hypothesis[5].

References

[1] Mendenhall and Scheaffer. 1973. Mathematical Statistics with Applications. Duxbury Press. North Scituate, Massachusetts.

[2] Casella, G and R. Berger. 2002. Statistical Inference. Second Edition, Duxbury Advanced Studies.

[3] Mathworld website. http://mathworld.wolfram.com/

[4] GAUSS kernel density library. GAUSS.

[4] The picture is taken from http://www.ats.ucla.edu/stat/mult_pkg/faq/general/nested_tests.htm
[5] The conclusion is made by the fact that p-value is greater than 5% significance level.

[5] Mittelhamer, R., G. Judge, and D. Miller (2000). Econometric Foundations. Cambridge Unviersity Press.

[6] Chumacero, R. 2003. Maximum LIkelihood Handout. Universidad de Chile.

[7] Amemiya, T., 1985, Advanced Econometrics, Cambridge: Harvard University Press.

[8] http://www.ats.ucla.edu/stat/mult_pkg/faq/general/nested_tests.htm

[9] Three Classical Tests; Wald, LM(Score), and LR tests at http://instruct1.cit.cornell.edu/courses/econ620/reviewm7.pdf

1 Appendix

1.1 GAUSS code for Question 1

```
/*****************************************************************
   Assignment 9
   Program:
   code made by Freddy Rojas Cama
   QUESTION 1.
   *****************************************************************/
   new;
   library maxlik, pgraph;
   pqgwin many;
   cls;
   //Parameters
   n=25;
   sigma=20;
   alpha=0.05;
   thetha_0=2.5;
   step=(33-2.5)/(100-1);
   thetha_seq=seqa(2.5,step,100);
   B2_thetha=1-cdfn(cdfni(1-alpha)+(thetha_0-thetha_seq)./(sigma/n^0.5));
   B3_thetha=1-cdfn(cdfni(1-alpha/2)+(thetha_0-thetha_seq)./(sigma/n^0.5))+
   cdfn(cdfni(alpha/2)+(thetha_0-thetha_seq)./(sigma/n^0.5));
    _pcolor = { 3 1}; /* Colors for series */
    _pmcolor = { 1, 8, 2, 8, 8, 8, 8, 8, 15 };
    /*Colors for axes, title, x and y labels, date, box, and background */
    _plwidth={12}; /*Controls line thickness for main curves*/
    // _paxht=0.20; /*Controls size of axes labels*/
   _ptitlht = 0.18; /*Controls main title size */
   _pltype={6};
   //_plegctl = { 1 7 0.04 13.5};
   //_plegstr="Data Coming from Gamma distribution";
   fonts("simplex complex microb simgrma");
   //Plotting B2
   //title("\201Power Function for B2(\204\110\201)");
   title("\201Power Function for B2(\204\113\201)");
   //xlabel("\204p");
   //xlabel("\204\110");
   xlabel("\204\113");
   xy(thetha_seq,B2_thetha);
   //Plotting B3
   title("\201Power Function for B3(\204\113\201)");
   xlabel("\204\113");
   xy(thetha_seq,B3_thetha);
   //plotting the difference
```

```
title("\201Power Function B2(\204\113\201)-B3(\204\113\201)");
xlabel("\204\113");
xy(thetha_seq,B2_thetha-B3_thetha);
//Plotting B2 and B3
xlabel("\204\113");
title("\201Power Function for B2(\204\113\201) and B3(\204\113\201)");
_pltype={6 1};
_plegctl={2 4 4.5 2};
_plegstr="B2(\204\113\201) \000 B3(\204\113\201)";
xy(thetha_seq,B2_thetha~B3_thetha);
```

1.2 GAUSS code for Questions 2, 3 and 4

```
/*******************************************************************"
"Assignment 9";
"Program:";
"code made by Freddy Rojas Cama QUESTION 2,3 and 4";
"*******************************************************************/
new;
library maxlik, pgraph;
pqgwin many;
cls;
chDir C:\Users\Freddy\freddy_Backup\Freddy\Persona\Rutgers 2010\Semester I\adv_Statistics\Ni
load data[100,6]=assign9.txt;
y=data[.,1];
x=data[.,2:6];
n=rows(y);
sigma=2;
betas=y/x;
e=y-x*betas;
sga0=e'e/(n-rows(betas));
r={1,0};
RR={0 1 1 0 0, 0 0 0 1 1};
q=rows(RR);
F_test=(RR*betas-r)'*inv(RR*inv(x'x)*RR')*(RR*betas-r)/((sga0)*q);
"Assignment 9";
"GAUSS Program:";
"code made by Freddy Rojas Cama";
" ";
" ";
"***Question 2.  Part i***";
"The F test is";;
F_test;
"the P-value is";;
cdffc(F_test,q,n-rows(betas));
//part (ii)
```

```
" ";
" ";
"***Question 2.  Part ii***";
betas0={1, 4, -1, 2.5, -2.5};
non_central_deltha=(RR*betas0-r)'*inv(RR*inv(x'x)*RR')*(RR*betas0-r)/(sigma^2);
"non central F-test";;
non_central_deltha;
//part (iii)
" ";
" ";
"***Question 2.  Part iii***";
power_deltha = 1-cdffnc(3.0922,q,n-5,non_central_deltha);
"The power of the test";;
power_deltha;
/*******************************************************************************
//question 3
*******************************************************************************/
" ";
" ";
"***Question 3***";
p=cdffc(3.0922,q,n-5);
"verifying P-value at 5%";;
p;
step=(8-0)/(100-1);
dlta=seqa(0,step,100);
p_dltha=zeros(rows(dlta),1);
j=1;
do while j lt rows(dlta)+1;
p_dltha[j,1] = 1-cdffnc(3.9,1,n-5,dlta[j,1]);
j=j+1;
endo;
graphset;
 _pcolor = { 9 }; /* Colors for series */
 _pmcolor = { 1, 8, 2, 8, 8, 8, 8, 8, 15 };
 /*Colors for axes, title, x and y labels, date, box, and background */
 _plwidth={12}; /*Controls line thickness for main curves*/
 // _paxht=0.20; /*Controls size of axes labels*/
 _ptitlht = 0.18; /*Controls main title size */
_pltype={6};
_plegctl = { 1 5 0.04 13};
fonts("simplex complex microb simgrma");
title("\201Power Function for \204\100");
ylabel("\201Power");
xlabel("\204\100");
xy(dlta,p_dltha);
```

```
/*******************************************************************************
//question 4
*******************************************************************************/
" ";
" ";
"***Question 4***";
n=rows(y);
sigma=2;
betask=y/x;
e=y-x*betask;
sgak=e'e/(n-rows(betas));
r={1,0};
RR={0 1 1 0 0, 0 0 0 1 1};
q=rows(RR);
e=y-x*betas;
sga0=e'e/(n-rows(betas));
//betasH=;
//WALD TEST chi-square with q degrees of freedom
WT=(RR*betas-r)'*inv(RR*inv(x'x)*RR')*(RR*betas-r)/((sgak));
//Lagrange Multiplier test
sgaH=(RR*betas-r)'*inv(RR*inv(x'x)*RR')*(RR*betas-r)/n+sgak;
LMT=(RR*betas-r)'*inv(RR*inv(x'x)*RR')*(RR*betas-r)/(sgaH);
"Wald Test";; WT;
"LM Test";; LMT;
"pvalue WT";; cdfchic(1.817,2);
"pvalue LMT";; cdfchic(1.785,2);
```