Advanced Economics Statistics

FALL 2010

Sixth-Assignment Answer Sheet

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1. Let X and Y be random variables. Let us consider the ratio of random variables:

$$\frac{X}{Y}, \qquad \text{where } \Pr(Y=0)=0$$

In general the expected value of a ratio of random variables is not the ratio of the expected values:

$$E\left(\frac{X}{Y}\right) \neq \frac{E(X)}{E(Y)}$$

but there are cases where

$$E\left(\frac{X}{Y}\right) = \frac{E(X)}{E(Y)} \tag{1}$$

holds. We found the most general condition where equation (1) holds. In general, we have for whatever variable:

$$\begin{aligned} cov(Z,Y) &= E(Z-\mu_z)(Y-\mu_y) \\ &= E(ZY - Z\mu_y - Y\mu_z + \mu_y\mu_z) \\ &= E(ZY) - E(Z)\mu_y - E(Y)\mu_z + \mu_y\mu_z \\ &= E(ZY) - \mu_z\mu_y \end{aligned}$$

If correlation between Z and Y is zero i.e. $\texttt{corr}(Z,Y)=0$ then, $E(ZY)=\mu_z\mu_y = E(Z)E(Y)$ so independence. If $Z=\frac{X}{Y}$, so

$$E\left(\frac{X}{Y}Y\right) = E(X) = E\left(\frac{X}{Y}\right)E(Y)$$

The conclusion therefore is given by:

$$E\left(\frac{X}{Y}\right) = \frac{E(X)}{E(Y)}$$

such that: $E\left(\frac{X}{Y}Y\right) = E(X) = \frac{E(X)}{E(Y)}E(Y)$.

2. This exercise asks for showing that `t-student pdf` converges to the normal pdf as the degree of freedom $p\to\infty$ (see problem 5.18 (c) on p.258 of CB). as the problem specify we fill in the gaps given by the answer in the proposed answer sheet (see ch5sol.pdf). Thus, we consider the following expression:

$$\frac{\left(\frac{p-1}{2}\right)^{\frac{p-1}{2}+\frac{1}{2}}\exp\left(-\frac{p-1}{2}\right)}{\left(\frac{p-1}{2}\right)^{\frac{p-1}{2}+\frac{1}{2}}\exp\left(-\frac{p-1}{2}\right)\sqrt{p\pi}} \longrightarrow \frac{\exp\left(-\frac{1}{2}\right)\exp\left(\frac{1}{2}\right)}{\sqrt{\pi}\sqrt{2}} = \frac{1}{\sqrt{2\pi}}$$

First, we consider the entire expression for the `t-student pdf`:

$$f_x(x) = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\sqrt{p\pi}} \frac{1}{\left(1+\frac{X^2}{p}\right)^{\frac{p+1}{2}}} \tag{2}$$

we can re-arrange the above expression without the term $\frac{1}{\left(1+\frac{X^2}{p}\right)^{\frac{p+1}{2}}}$. So, we can express terms in numerator and denominator of expression (2) as follows:

$$\begin{aligned}\Gamma\left(\frac{p+1}{2}\right) &= \left(\frac{p+1}{2}-1\right)! \\ &= \sqrt{2\pi\left(\frac{p+1}{2}-1\right)}\left(\frac{\frac{p+1}{2}-1}{e^1}\right)^{\frac{p+1}{2}-1}\end{aligned}$$

and

$$\begin{aligned}\Gamma\left(\frac{p}{2}\right) &= \left(\frac{p}{2}-1\right)! \\ &= \sqrt{2\pi\left(\frac{p}{2}-1\right)}\left(\frac{\frac{p}{2}-1}{e^1}\right)^{\frac{p}{2}-1}\end{aligned}$$

Basically, in above steps we have used the Stirling's formula[1]. By replacing those expressions into the next equation we have:

$$f_x(x) = \frac{\sqrt{2\pi\left(\frac{p+1}{2}-1\right)}\left(\frac{\frac{p+1}{2}-1}{e^1}\right)^{\frac{p+1}{2}-1}}{\sqrt{2\pi\left(\frac{p}{2}-1\right)}\left(\frac{\frac{p}{2}-1}{e^1}\right)^{\frac{p}{2}-1}\sqrt{p\pi}} \frac{1}{\left(1+\frac{X^2}{p}\right)^{\frac{p+1}{2}}}$$

re-arranging terms:

$$\begin{aligned}&= \frac{\left(\frac{p+1}{2}-1\right)^{\frac{1}{2}}\left(\frac{p+1}{2}-1\right)^{\frac{p+1}{2}-1}e^{-\frac{p+1}{2}+1}}{\left(\frac{p}{2}-1\right)^{\frac{1}{2}}\left(\frac{p}{2}-1\right)^{\frac{p}{2}-1}e^{-\frac{p}{2}+1}\sqrt{p\pi}} \frac{1}{\left(1+\frac{X^2}{p}\right)^{\frac{p+1}{2}}} \\ &= \frac{\left(\frac{p+1}{2}-1\right)^{\frac{p+1}{2}-\frac{1}{2}}e^{-\frac{1}{2}}}{\left(\frac{p}{2}-1\right)^{\frac{p}{2}-\frac{1}{2}}\sqrt{p\pi}} \frac{1}{\left(1+\frac{X^2}{p}\right)^{\frac{p+1}{2}}}\end{aligned}$$

then we reduce terms in parentheses in above expression by just performing mathematical operations. Thus, $f_x(x)$ is expressed as follows:

$$= \frac{\left(\frac{p-1}{2}\right)^{\frac{p-1}{2}+\frac{1}{2}}e^{-\frac{1}{2}}}{\left(\frac{p-2}{2}\right)^{\frac{p-2}{2}+\frac{1}{2}}\sqrt{p\pi}} \frac{1}{\left(1+\frac{X^2}{p}\right)^{\frac{p+1}{2}}} \tag{3}$$

We know that expression (3) must converge to some expression different from infinity[2]

[1] Stirling's formula for approximate the factorials is expressed as: $n! \approx \sqrt{2\pi n}\left(\frac{n}{\exp(1)}\right)^n$. See `mathworld`.

[2] In other words $\lim_{p\longrightarrow\infty} f_x(x) = M < \infty$.

when p goes to infinity. In that sense, we realize that term $\frac{p-1}{2}$ is $O(p)$.[3] Thus, in order to stabilize the numerator and denominator when $p \longrightarrow \infty$ we must divide both terms by some expression $O(p)$. We propose $\left(\frac{p-2}{2}\right)^{\frac{p-2}{2}+\frac{1}{2}}$, thus expression (3) is equivalent to the following:

$$= \frac{\frac{1}{\sqrt{p\pi}} \frac{\left(\frac{p-1}{2}\right)^{\frac{p-1}{2}+\frac{1}{2}}}{\left(\frac{p-2}{2}\right)^{\frac{p-2}{2}+\frac{1}{2}}} e^{-\frac{1}{2}}}{\frac{\left(\frac{p-2}{2}\right)^{\frac{p-2}{2}+\frac{1}{2}}}{\left(\frac{p-2}{2}\right)^{\frac{p-2}{2}+\frac{1}{2}}}} \frac{1}{\left(1+\frac{X^2}{p}\right)^{\frac{p+1}{2}}}$$

Term $\left(\frac{p-1}{2}\right)^{\frac{p-1}{2}+\frac{1}{2}}$ can be expressed as $\left(\frac{p-1}{2}\right)^{\frac{p-2}{2}+\frac{1}{2}}\left(\frac{p-1}{2}\right)^{\frac{1}{2}}$. Thus, re-arranging terms;

$$= \frac{\frac{1}{\sqrt{p\pi}} \left(\frac{\frac{p-1}{2}}{\frac{p-2}{2}}\right)^{\frac{p-2}{2}+\frac{1}{2}} \left(\frac{p-1}{2}\right)^{\frac{1}{2}} e^{-\frac{1}{2}}}{\frac{\left(\frac{p-2}{2}\right)^{\frac{p-2}{2}+\frac{1}{2}}}{\left(\frac{p-2}{2}\right)^{\frac{p-2}{2}+\frac{1}{2}}}} \frac{1}{\left(1+\frac{X^2}{p}\right)^{\frac{p+1}{2}}}$$

reducing expressions in both numerator and denominator:

$$= \frac{1}{\sqrt{p\pi}} \left(\frac{p-1}{p-2}\right)^{\frac{p-2}{2}+\frac{1}{2}} \left(\frac{p-1}{2}\right)^{\frac{1}{2}} e^{-\frac{1}{2}} \frac{1}{\left(1+\frac{X^2}{p}\right)^{\frac{p+1}{2}}}$$

re-arranging the $\sqrt{p}$ term (which is outside of parentheses) into $\left(\frac{p-1}{2}\right)^{\frac{1}{2}}$

$$= \frac{1}{\sqrt{\pi}} \left(\frac{p-1}{p-2}\right)^{\frac{p-2}{2}+\frac{1}{2}} \left(\frac{p-1}{2p}\right)^{\frac{1}{2}} e^{-\frac{1}{2}} \frac{1}{\left(1+\frac{X^2}{p}\right)^{\frac{p+1}{2}}}$$

$$= \frac{1}{\sqrt{\pi}} \left(\frac{p-1}{p-2}\right)^{\frac{p-2}{2}+\frac{1}{2}} \left(\frac{1}{2}-\frac{1}{2p}\right)^{\frac{1}{2}} e^{-\frac{1}{2}} \frac{1}{\left(1+\frac{X^2}{p}\right)^{\frac{p+1}{2}}}$$

by using the trick "*add and substract 1*" and then re-arranging terms:

$$= \frac{1}{\sqrt{\pi}} \left(1-1+\frac{p-1}{p-2}\right)^{\frac{p-2}{2}+\frac{1}{2}} \left(\frac{1}{2}-\frac{1}{2p}\right)^{\frac{1}{2}} e^{-\frac{1}{2}} \frac{1}{\left(1+\frac{X^2}{p}\right)^{\frac{p+1}{2}}}$$

$$= \frac{1}{\sqrt{\pi}} \left(1+\frac{-(p-2)+(p-1)}{p-2}\right)^{\frac{p-2}{2}+\frac{1}{2}} \left(\frac{1}{2}-\frac{1}{2p}\right)^{\frac{1}{2}} e^{-\frac{1}{2}} \frac{1}{\left(1+\frac{X^2}{p}\right)^{\frac{p+1}{2}}}$$

$$= \frac{1}{\sqrt{\pi}} \left(1+\frac{1}{p-2}\right)^{\frac{p-2}{2}+\frac{1}{2}} \left(\frac{1}{2}-\frac{1}{2p}\right)^{\frac{1}{2}} e^{-\frac{1}{2}} \frac{1}{\left(1+\frac{X^2}{p}\right)^{\frac{p+1}{2}}}$$

[3] $O(n)$ refers to Landau notation.

Equivalently, we have the following (useful) expression which are equivalent to above expression[4];

$$= \frac{1}{\sqrt{\pi}} \left(1 + \frac{1}{p-2}\right)^{\frac{p-2}{2}} \left(1 + \frac{1}{p-2}\right)^{\frac{1}{2}} \left(\frac{1}{2} - \frac{1}{2p}\right)^{\frac{1}{2}} e^{-\frac{1}{2}} \frac{1}{\left(1 + \frac{X^2}{p}\right)^{\frac{p+1}{2}}}$$

one more arrangement

$$= \frac{1}{\sqrt{\pi}} \left(1 + \frac{\frac{1}{2}}{\frac{p-2}{2}}\right)^{\frac{p-2}{2}} \left(1 + \frac{1}{p-2}\right)^{\frac{1}{2}} \left(\frac{1}{2} - \frac{1}{2p}\right)^{\frac{1}{2}} e^{-\frac{1}{2}} \frac{1}{\left(1 + \frac{\frac{X^2}{2}}{\frac{p}{2}}\right)^{\frac{p}{2}} \left(1 + \frac{X^2}{p}\right)^{\frac{1}{2}}}$$

we know that $\frac{1}{p-2}$, $\frac{1}{2p}$ and $\frac{X^2}{p}$ are $o(p)$. Then, we have the following expression:

$$= \frac{1}{\sqrt{\pi}} \left(1 + \frac{\frac{1}{2}}{\frac{p-2}{2}}\right)^{\frac{p-2}{2}} (1 + o(p))^{\frac{1}{2}} \left(\frac{1}{2} - o(p)\right)^{\frac{1}{2}} \frac{e^{-\frac{1}{2}}}{\left(1 + \frac{\frac{X^2}{2}}{\frac{p}{2}}\right)^{\frac{p}{2}} (1 + o(p))^{\frac{1}{2}}} \tag{4}$$

By using the lemma , let $a_1, a_2, a_3, ...a_n$ be a sequence of numbers converging to a, that is

$$\lim_{n \to \infty} a_n = a$$

then

$$\lim_{n \to \infty} \left(1 + \frac{a_n}{n}\right)^n = \exp^a$$

By using above lemma and $p \to \infty$:

$$\lim_{p \longrightarrow \infty} \left(1 + \frac{\frac{1}{2}}{\frac{p-2}{2}}\right)^{\frac{p-2}{2}} = \exp\left(\frac{1}{2}\right)$$

Also,

$$\lim_{p \longrightarrow \infty} \left(1 + \frac{\frac{X^2}{2}}{\frac{p}{2}}\right)^{\frac{p}{2}} = \exp\left(\frac{X^2}{2}\right)$$

So, when $p \to \infty$, expression (4) is

$$\underset{p \longrightarrow \infty}{fx}(x) = \frac{1}{\sqrt{\pi}} e^{\frac{1}{2}} \left(\frac{1}{2}\right)^{\frac{1}{2}} \frac{e^{-\frac{1}{2}}}{\exp\left(\frac{X^2}{2}\right)}$$

re-arranging terms

$$\underset{p \longrightarrow \infty}{fx}(x) = \frac{1}{\sqrt{\pi}} \left(\frac{1}{2}\right)^{\frac{1}{2}} \frac{1}{\exp\left(\frac{X^2}{2}\right)}$$

[4] In terms of our purpose, we will see that this arrangement is helpful.

thus;

$$\underset{p\longrightarrow\infty}{fx(x)} = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$$

Since we prove that the following expression with $p \rightarrow \infty$ is given by:

$$\frac{1}{\left(1+\frac{x^2}{p}\right)^{\frac{p+1}{2}}} = \frac{1}{\left(1+\frac{\frac{X^2}{2}}{\frac{p}{2}}\right)^{\frac{p}{2}}\left(1+\frac{x^2}{p}\right)^{\frac{1}{2}}} = \exp\left(-\frac{x^2}{2}\right)$$

Therefore:

$$\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\sqrt{p\pi}} = \frac{1}{\sqrt{2\pi}}$$

3. Considering the example 6.2.17 on pp.282-283 of Casella and Berger (CB).

   (i). By using (5.4.7) on p.230 of CB, we obtain:

$$g\left(x_{(1)}, x_{(n)}|\theta\right) = \begin{cases} = n(n-1)\left(x_{(n)} - x_{(1)}\right)^{n-2} & \theta < x_{(1)} < x_{(n)} < \theta + 1 \\ 0 & \text{otherwise} \end{cases}$$

   By using theorem 5.4.6 in CB:

$$fx_{(1)}, x_{(n)}(u,\nu) = \frac{n!}{(i-1)!(j-1-i)!((n-j)!)} f_X(u) f_X(v) \left[F_X(u)\right]^{i-1} \left[F_X(v) - F_X(u)\right]^{j-i-1}\left[1 - F_X(v)\right]^{n-j}$$

   For $-\infty < u < v < \infty$.So if $i = 1$ and $j = n$, then the expression reduces to:

$$fx_{(1)}, x_{(n)}(u,\nu) = \frac{n!}{(n-2)!} f_X(u) f_X(v)\left[F_X(v) - F_X(u)\right]^{n-2}$$

   And being $f_X(u) = \frac{1}{\theta+1-\theta} = 1$, $f_X(v) = \frac{1}{\theta+1-\theta} = 1$, and $\left[F_X(v) - F_X(u)\right] = x_{(n)} - x_{(1)}$, therefore we can re write the above expression as follows:

$$\begin{aligned} fx_{(1)}, x_{(n)}(u,\nu) &= \frac{n!}{(n-2)!}\left[x_{(n)} - x_{(1)}\right]^{n-2} \\ &= n(n-1)\left[x_{(n)} - x_{(1)}\right]^{n-2} \end{aligned}$$

   (ii). Obtaining the conditional pdf of $x_{(n)}$ given $x_{(1)}$:

$$g\left(x_{(n)}, x_{(1)}|\theta\right) = g\left(x_{(n)}|x_{(1)},\theta\right) h\left(x_{(1)}\right)$$

$$g\left(x_{(n)}|x_{(1)},\theta\right) = \frac{n(n-1)\left(x_{(n)} - x_{(1)}\right)^{n-2}}{h\left(x_{(1)}\right)}$$

   So,

$$\begin{aligned} h\left(x_{(1)}\right) &= fx_{(1)}(x) = \int_{x_{(1)}}^{\theta+1} n(n-1)\left(x_{(n)} - x_{(1)}\right)^{n-2} dx_{(n)} \\ &= n\left[\theta + 1 - x_{(1)}\right]^{n-1} \end{aligned}$$

$$g\left(x_{(n)}|x_{(1)},\theta\right) = \frac{n\,(n-1)\left(x_{(n)}-x_{(1)}\right)^{n-2}}{n\left[\theta+1-x_{(1)}\right]^{n-1}} = \frac{(n-1)\left(x_{(n)}-x_{(1)}\right)^{n-2}}{\left[\theta+1-x_{(1)}\right]^{n-1}}$$

We checked above answer with results in exercises 5.27 and 5.28 in CB.

4. We get 2000 draws from $N(0,1)$ and we get the distribution of an order statistic:

   i. We show in figure 4.1 the exact pdf of ordered statistic $x_{(95\%)}$ and $x_{(50\%)}$.

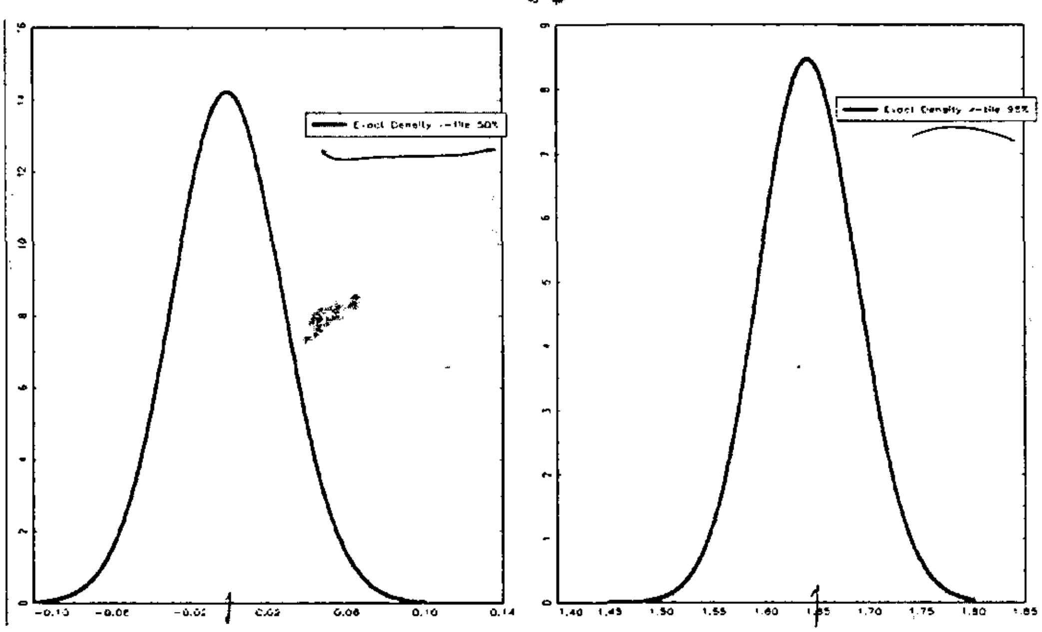

Figure 4.1 Exact pdf for $x_{95\%}$ and $x_{50\%}$

   ii. We show in table 4.1 the mean, standard deviation, skewness and kurtosis obtained by Simpson's rule for each ordered statistic (see Gauss code);

Table 4.1: Statistics for percentile 95 and 50%

	$x_{95\%}$	$x_{50\%}$
Mean	1.64120	-0.00062
s.d	0.04699	0.02795
skewness	0.02897	0.00107
Kurtosis	2.94603	2.95190

   iii. In this section we verify that a kernel density is close to the exact density. We drew

1000 replications of $x_{95\%}$ *and* $x_{50\%}$

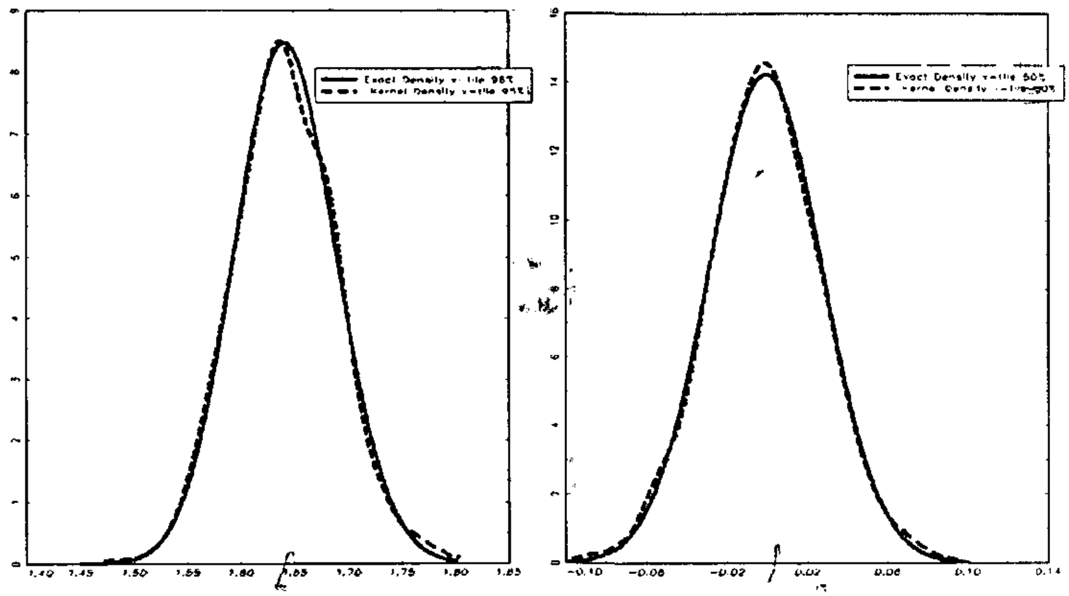

Figure 4.2: Approaching the exact density by kernels

iv. We obtain the mean, standard deviation, skewness, and kurtosis using 1,000 replications (see Gauss code)

Table 4.1: Statistics for quartile 95 and 50%

	$x_{95\%}$	$x_{50\%}$
Mean	1.64138	-0.00106
s.d	0.04713	0.02791
skewness	0.11123	0.01737
Kurtosis	3.25477	3.27173

Sample Stats or Exact?

v. We generate x_t that is an autoregressive process of order 1, then we report the $x_{(95\%)}$ and $x_{(50\%)}$ under different values of ϕ. Discussion of the results are presented in item *v.iii*

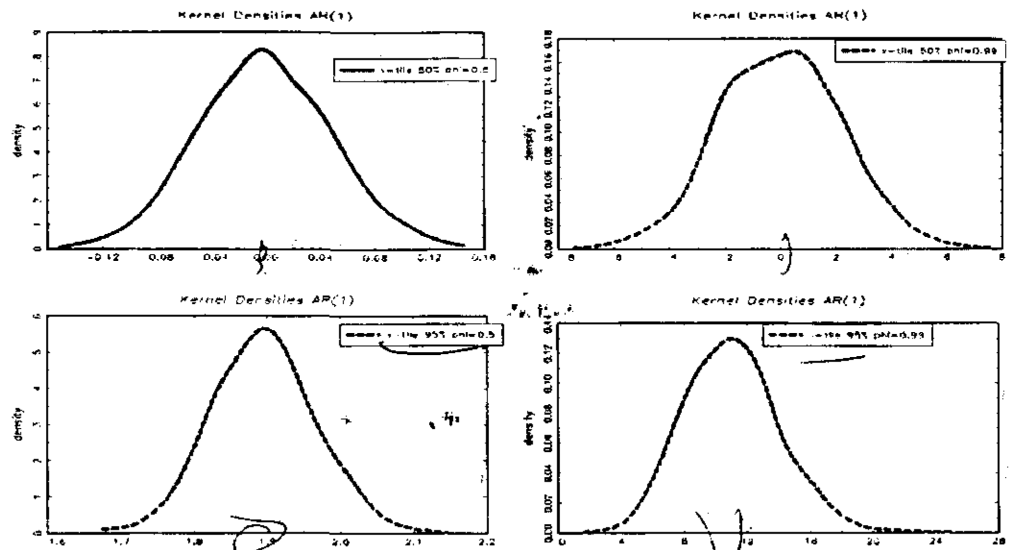

Figure 4.3: $x_{95\%}$ and $x_{50\%}$ under different AR(1) DGP

vi. We obtain on one graph three pdf's: The exact pdf of $x_{(r)}$, r = 95%-tile and $\phi = 0$; The kernel density estimate of $x_{(r)}$, r = 95%-tile and $\phi = 0$; The kernel density estimate of $x_{(r)}$, r =95%-tile and $\phi = .99$.

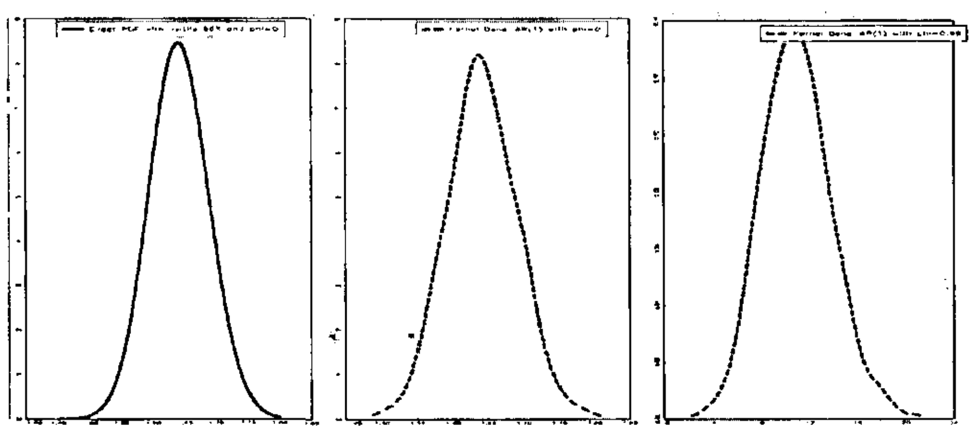

Figure 4.4: Exact PDF and Kernel Density under different values of ϕ

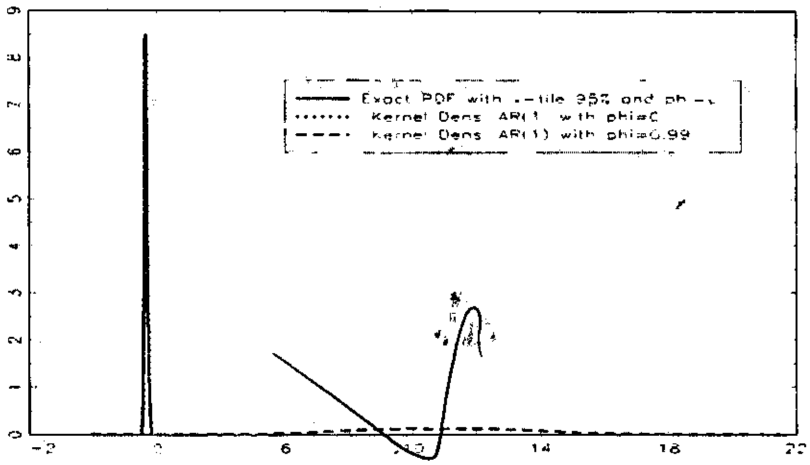

Figure 4.5: Exact PDF and Kernel Density under different values of ϕ (alternative graph)

We can observe, that because we are dealing with a parameter ϕ close to one, the mean of the order statistics at the 95% x-tile is increasing because the variance of that "quasi" random walk increases with respect to a process AR(1) with $\phi = 0.5$ (which mean and variance match the exact density at the 95% x-tile, this because ϕ with a value less than 1 in absolute value means a stationary process.

vii. We fill in the following table according previous results;

Table 4.3: Summary statistics of Order statistics $x_i \sim N(0,1)$						
	Exact		Autocorrelated case			
	r = 50%-tile	r = 95%-tile	r = 50%-tile		r = 95%-tile	
	$x_{(r)} = 0$	$x_{(r)} = 1.645$	$\phi = 0.5$	$\phi = 0.99$	$\phi = 0.5$	$\phi = 0.99$
mean	-0.00062	1.64200	-0.00082	-0.04584	1.89655	10.86409
std	0.02795	0.04699	0.04961	2.20565	0.06937	3.01318
skewness	0.00106	0.02897	-0.14334	-0.07420	0.10838	0.18650
kurtosis	2.95190	2.94603	2.92907	3.22650	3.10907	3.34283

viii. In this section we discuss the results we have got. At upper and left side in figure 3.1 we show the distribution for the percentile 50% with a parameter $\phi = 0.5$, as we see the mean of the distribution is zero. At upper and right side, we have the distribution for the percentile 50% with a parameter $\phi = 0.99$, as we see the mean of the distribution is also zero; there is no change in the mean with respect to the white noise case, this happens because the unconditional mean is $E(x) = E\left(\frac{\varepsilon}{1-\phi}\right) = E(\varepsilon) = 0$ However, we have differences respect to the white noise case when we analize the 95-th percentile; At the bottom and left side in figure 4.3, we have the distribution for the percentile 95% with a parameter $\phi = 0.5$, as we see the mean of the distribution is not around 1.644[5], this because the data generation

[5] Cumulative-inverse calculation under a normal with mean zero and variance equal to 1.

proces (DGP) comes from an AR(1) with an autoregresive parameter different from zero, thus the variance of this process is $\frac{\sigma^2}{1-\phi^2}$ which is greater than the variance (σ^2) of a white noise, for this reason the statistic at 95-th percentile is located at the right of the 95-th percentile in a white noise realization. The graph at the bottom and right side confirm the former for the distribution at the percentile 50% with a parameter $\phi = 0.99$, in this case the mean has suddenly increased as a result of increasing ϕ up to 0.99. we must recall that as long as we approach to a random-walk DGP the variance will increase without any boundness. All these characteristics of graphs presented in figure 4.3 match the statistics in table 4.3. We also realize that a kernel density from N(0,1) draws is a very good approximation for the exact density calculated from the Simpson's Rule at 95% percentile of order statistics (see figure 4.2).

References

[1] Casella, G and R. Berger. 2002. Statistical Inference. Second Edition, Duxbury Advanced Studies.

[2] Mendenhall and Scheaffer. 1973. Mathematical Statistics with Applications. Duxbury Press. North Scituate, Massachusetts.

[3] `Mathworld website. http://mathworld.wolfram.com/`

[4] `GAUSS kernel density library. GAUSS.`

[5] M.P Wand & M.C Jones. 1995. Kernel Smoothing. Monographs on Statistics and Applied Probability. Chapman & Hall, 1995.

1 Appendix

1.1 Question 4: GAUSS code

```
new;
   cls;
   /*=============================================================================================
   /* Code by Hiraki Tsurumi (Assign #6-10:  asgn06-10 hard copy) but modified by
Freddy Rojas Cama */
   // Last update October 30th 2010
   // Rutgers University - Phd program in Economics
   /*=============================================================================================
   library pgraph kernel;
   pqgwin many;
   n=2000;
   nrept=1001;
   print "Parameter Setting";
   print "";
   print "n=" n;
   /*******************************************************
   @Question i :  The exact pdf of x(r) by Simpson's rule.@
   *******************************************************/
   {pea_95,mar_95,mu_95,mode_95,med_95,sd_95,skew_95,kurto_95}=exactd(0.95);
   {pea_50,mar_50,mu_50,mode_50,med_50,sd_50,skew_50,kurto_50}=exactd(0.5);
   graphset;
   begwind;
   window(1,2,0);
    _pcolor = { 9 }; /* Colors for series */
    _pmcolor = { 1, 8, 2, 8, 8, 8, 8, 8, 15 };
    /*Colors for axes, title, x and y labels, date, box, and background */
    _plwidth={12}; /*Controls line thickness for main curves*/
    // _paxht=0.20; /*Controls size of axes labels*/
    _ptitlht = 0.18; /*Controls main title size */
   _pltype={6};
   _plegctl = { 1 5 0.04 13};
   _plegstr="Exact Density x-tile 50%";
   xlabel("x 50%");
   xy(pea_50,mar_50);
   nextwind;
    _pcolor = { 9 }; /* Colors for series */
    _pmcolor = { 1, 8, 2, 8, 8, 8, 8, 8, 15 };
    /*Colors for axes, title, x and y labels, date, box, and background */
    _plwidth={12}; /*Controls line thickness for main curves*/
    // _paxht=0.20; /*Controls size of axes labels*/
    _ptitlht = 0.18; /*Controls main title size */
   _pltype={6};
```

```
_plegctl = { 1 5 1.67 7.5};
_plegstr="Exact Density x-tile 95%";
xlabel("x 95%");
xy(pea_95,mar_95);
endwind;
pause(5);
/*****************************************************************************************
*Question ii:  Mean, standard deviation, skewness, and kurtosis obtaned by Simpson's
*rule.
*****************************************************************************************/
{pea_95,mar_95,mu_95,mode_95,med_95,sd_95,skew_95,kurto_95}=exactd(0.95);
{pea_50,mar_50,mu_50,mode_50,med_50,sd_50,skew_50,kurto_50}=exactd(0.5);
print "Question ii / Exact Density";
print "With x=tile 95%";
print "mean, mode, median,std,skewness,kurtosis";
print mu_95 mode_95 med_95 sd_95 skew_95 kurto_95;
print "With x=tile 50%";
print "mean, mode, median,std,skewness,kurtosis";
print mu_50 mode_50 med_50 sd_50 skew_50 kurto_50;
/*****************************************************************************************
 *@Question iii:  Obtain 1,000 replications of x(r), & obtain the kernel density.
Verify@
 *@that the kernel density is close to the exact density.@
*****************************************************************************************/
{x1_95,den1_95}=draws_kden(0.95,nrept);
{x1_50,den1_50}=draws_kden(0.50,nrept);
graphset;
begwind;
window(1,2,0);
 _pcolor = { 9 5}; /* Colors for series */
 _pmcolor = { 1, 8, 2, 8, 8, 8, 8, 8, 15 };
 /*Colors for axes, title, x and y labels, date, box, and background */
 _plwidth={12}; /*Controls line thickness for main curves*/
 // _paxht=0.20; /*Controls size of axes labels*/
 _ptitlht = 0.18; /*Controls main title size */
_pltype={6 3};
_plegctl = { 1 7 1.67 7.5};
_plegstr="Exact Density x-tile 95% \000 Kernel Density x-tile 95%";
xlabel("x 95%");
xy(pea_95~x1_95,mar_95~den1_95);
nextwind;
 _pcolor = { 9 5}; /* Colors for series */
 _pmcolor = { 1, 8, 2, 8, 8, 8, 8, 8, 15 };
 /*Colors for axes, title, x and y labels, date, box, and background */
 _plwidth={12}; /*Controls line thickness for main curves*/
```

```
 // _paxht=0.20; /*Controls size of axes labels*/
 _ptitlht = 0.18; /*Controls main title size */
_pltype={6 3};
_plegctl = { 1 7 0.04 13.5};
_plegstr="Exact Density x-tile 50% \000 Kernel Density x-tile 50%";
xlabel("x 50%");
xy(pea_50~x1_50,mar_50~den1_50);
endwind;
/*****************************************************************************
*Question iv: Obtain the mean, standard deviation, skewness, and kurtosis using
*1,000 replications.
*****************************************************************************/
{x1_95,den1_95}=draws_kden(0.95,nrept);
{x1_50,den1_50}=draws_kden(0.50,nrept);
{mean1_95,std1_95,skew1_95,kurtos1_95}=stats(x1_95);
{mean2_50,std2_50,skew2_50,kurtos2_50}=stats(x1_50);
print "";
print "Question iv / Density Density";
print "With x=tile 95%";
print "mean,std,skewness,kurtosis";
print mean1_95 std1_95 skew1_95 kurtos1_95;
print "With x=tile 50%";
print "mean,std,skewness,kurtosis";
print mean2_50 std2_50 skew2_50 kurtos2_50;
/****************************************************************************
@Question v: Obtain the mean, standard deviation, skewness, & kurtosis using
1,000 replications.@
****************************************************************************/
/*Percentile 50% phi=0.5 & phi=0.99*/
/*Arguments for procedure called "graph_ar" are phi,repetitions and percentile*/
{x2_5_5,den2_5_5}=graph_ar(0.5,n,0.5);
{x2_99_5,den2_99_5}=graph_ar(0.99,n,0.5);
/*Percentile 95% phi=0.5 & phi=0.99*/
{x2_5_95,den2_5_95}=graph_ar(0.5,n,0.95);
{x2_99_95,den2_99_95}=graph_ar(0.99,n,0.95);
graphset;
begwind;
title("Kernel Densities AR(1)");
ylabel("density");
xlabel("x");
window(2,2,0);
 _pcolor = { 9 }; /* Colors for series */
 _pmcolor = { 1, 8, 2, 8, 8, 8, 8, 8, 15 };
 /*Colors for axes, title, x and y labels, date, box, and background */
 _plwidth={12}; /*Controls line thickness for main curves*/
```

```
 // _paxht=0.20; /*Controls size of axes labels*/
 _ptitlht = 0.18; /*Controls main title size */
_pltype=6;
_plegctl = { 1 6 0.05 7 };
_plegstr="x-tile 50% phi=0.5";
xy(x2_5_5,den2_5_5);
nextwind;
 _pcolor = { 9 }; /* Colors for series */
 _pmcolor = { 1, 8, 2, 8, 8, 8, 8, 8, 15 };
 /*Colors for axes, title, x and y labels, date, box, and background */
 _plwidth={12}; /*Controls line thickness for main curves*/
 // _paxht=0.20; /*Controls size of axes labels*/
 _ptitlht = 0.18; /*Controls main title size */
 _plegctl = { 1 6 2 0.16 };
_pltype=3;
_plegstr="x-tile 50% phi=0.99";
xy(x2_99_5,den2_99_5);
nextwind;
 _pcolor = { 9 }; /* Colors for series */
 _pmcolor = { 1, 8, 2, 8, 8, 8, 8, 8, 15 };
 /*Colors for axes, title, x and y labels, date, box, and background */
 _plwidth={12}; /*Controls line thickness for main curves*/
 // _paxht=0.20; /*Controls size of axes labels*/
 _ptitlht = 0.18; /*Controls main title size */
 _plegctl = { 1 6 1.8 5.2 };
_pltype=3;
_plegstr="x-tile 95% phi=0.5";
xy(x2_5_95,den2_5_95);
nextwind;
 _pcolor = { 9 }; /* Colors for series */
 _pmcolor = { 1, 8, 2, 8, 8, 8, 8, 8, 15 };
 /*Colors for axes, title, x and y labels, date, box, and background */
 _plwidth={12}; /*Controls line thickness for main curves*/
 // _paxht=0.20; /*Controls size of axes labels*/
 _ptitlht = 0.18; /*Controls main title size */
 _plegctl = { 1 6 13 0.125 };
_pltype=3;
_plegstr="x-tile 95% phi=0.99";
xy(x2_99_95,den2_99_95);
endwind;
/***********************************************************************
*Question vi Obtain on one graph three pdf's:
*The exact pdf of x(r), r = 95%-tile and phi = 0;
*(ii) The kernel density estimate of x(r), r = 95%-tile and phi = 0;
*(iii) The kernel density estimate of x(r), r = 95%-tile and phi = .99
```

```
*************************************************************************/
{x2_0,den2_0}=graph_ar(0,n,0.95);
{x2_99,den2_99}=graph_ar(0.99,n,0.95);
{pea,mar,mu,mode,med,sd,skew,kurto}=exactd(0.95);
graphset;
begwind;
window(1,3,0);
 _pcolor = { 9 }; /* Colors for series */
 _pmcolor = { 1, 8, 2, 8, 8, 8, 8, 8, 15 };
 /*Colors for axes, title, x and y labels, date, box, and background */
 _plwidth={12}; /*Controls line thickness for main curves*/
 _paxht=0.60; /*Controls size of axes labels*/
 _ptitlht = 0.18; /*Controls main title size */
_pltype=6;
_plegctl = { 1 7 1.45 8.6 };
_plegstr="Exact PDF with x-tile 95% and phi=0";
xy(pea,mar);
nextwind;
 _pcolor = { 9 }; /* Colors for series */
 _pmcolor = { 1, 8, 2, 8, 8, 8, 8, 8, 15 };
 /*Colors for axes, title, x and y labels, date, box, and background */
 _plwidth={12}; /*Controls line thickness for main curves*/
 // _paxht=0.20; /*Controls size of axes labels*/
 _ptitlht = 0.18; /*Controls main title size */
 _plegctl = { 1 7 1.55 8.60};
_pltype=3;
_plegstr="Kernel Dens:  AR(1) with phi=0";
xy(x2_0,den2_0);
nextwind;
 _pcolor = { 9 }; /* Colors for series */
 _pmcolor = { 1, 8, 2, 8, 8, 8, 8, 8, 15 };
 /*Colors for axes, title, x and y labels, date, box, and background */
 _plwidth={12}; /*Controls line thickness for main curves*/
 // _paxht=0.20; /*Controls size of axes labels*/
 _ptitlht = 0.18; /*Controls main title size */
 _plegctl = { 1 7 8.0 0.133 };
_pltype=3;
_plegstr="Kernel Dens:  AR(1) with phi=0.99";
xy(x2_99,den2_99);
endwind;
/*********************************************************************
*Second version:  Question vi Obtain on one graph three pdf's:
*The exact pdf of x(r), r = 95%-tile and phi = 0;
*(ii) The kernel density estimate of x(r), r = 95%-tile and phi = 0;
*(iii) The kernel density estimate of x(r), r = 95%-tile and phi = .99
```

```
    ***************************************************************************/
    {x2_0,den2_0}=graph_ar(0,n,0.95);
    {x2_99,den2_99}=graph_ar(0.99,n,0.95);
    {pea,mar,mu,mode,med,sd,skew,kurto}=exactd(0.95);
    graphset;
    begwind;
     _pcolor = { 9 5 6 }; /* Colors for series */
     _pmcolor = { 1, 8, 2, 8, 8, 8, 8, 8, 15 };
     /*Colors for axes, title, x and y labels, date, box, and background */
     _plwidth={12}; /*Controls line thickness for main curves*/
     _paxht=0.60; /*Controls size of axes labels*/
     _ptitlht = 0.18; /*Controls main title size */
    _pltype={6 2 3};
    _plegctl = { 1 5 6 6 };
    _plegstr="Exact PDF with x-tile 95% and phi=0 \000 Kernel Dens:  AR(1) with phi=0
\000 Kernel Dens:  AR(1) with phi=0.99";
    xy(pea~x2_0~x2_99,mar~den2_0~den2_99);
    endwind;
    /************************************************
    *Question vii
    ************************************************/
    /****Autocorrelated order statistics****/
    /*Percentile 50% phi=0.5 & phi=0.99*/
    {x2_5_50,den2_5_50}=graph_ar(0.5,n,0.5);
    {x2_99_50,den2_99_50}=graph_ar(0.99,n,0.5);
    {mean_5_50,std_5_50,skew_5_50,kurtos_5_50}=stats(x2_5_50);
    {mean_99_50,std_99_50,skew_99_50,kurtos_99_50}=stats(x2_99_50);
    /*Percentile 95% phi=0.5 & phi=0.99*/
    {x2_5_95,den2_5_95}=graph_ar(0.5,n,0.95);
    {x2_99_95,den2_99_95}=graph_ar(0.99,n,0.95);
    {mean_5_95,std_5_95,skew_5_95,kurtos_5_95}=stats(x2_5_95);
    {mean_99_95,std_99_95,skew_99_95,kurtos_99_95}=stats(x2_99_95);
    print "";
    print "Question vii / Autocorrelation Section";
    print "With x=tile 50 and phi=0.5%";
    print "mean,std,skewness,kurtosis";
    print mean_5_50 std_5_50 skew_5_50 kurtos_5_50;
    print "With x=tile 50 and phi=0.99%";
    print "mean,std,skewness,kurtosis";
    print mean_99_50 std_99_50 skew_99_50 kurtos_99_50;
    print "With x=tile 95 and phi=0.5%";
    print "mean,std,skewness,kurtosis";
    print mean_5_95 std_5_95 skew_5_95 kurtos_5_95;
    print "With x=tile 95 and phi=0.99%";
    print "mean,std,skewness,kurtosis";
```

```
print mean_99_95 std_99_95 skew_99_95 kurtos_99_95;
print "";
print "Question vii / Exact Density";
print "With x=tile 50";
print "mean, median,std,skewness,kurtosis";
print mu_50 med_50 sd_50 skew_50 kurto_50;
print "";
print "With x=tile 95";
print "mean, median,std,skewness,kurtosis";
print mu_95 med_95 sd_95 skew_95 kurto_95;
/***Kernel Estimation***/
proc(2)=kden(v);
local g,h,j,nn,res;
nn=rows(v);
h=1.06*stdc(v)/nn^.2;
g=0;j=1;
do while j<=nn;
g=g|meanc(pdfn((v-v[j])/h))/h;
j=j+1;
endo;
res=sortc(v~g[2:nn+1],1);
retp(res[.,1],res[.,2]);
endp;
proc(2)=graph_ar(phi1,n,percent);
local i,xar,xarod,x0,t,xxar,xt,nmed,x2,den2;
nmed=percent*n;
i=1;
xar={};
do while i<=nrept;
xarod={};
x0=0;
t=1;
do while t<=n;
xt=phi1*x0+rndn(1,1);
x0=xt;
xarod=xarod|xt;
t=t+1;
endo;
xxar=sortc(xarod,1);
xar=xar|xxar[nmed];
i=i+1;
endo;
{x2,den2}=kden(xar);
retp(x2,den2);
endp;
```

```
proc(8)=exactd(percent);
local nmed,i,x,upa, lowa, nna,nna1, ha, pea, wp1, wp2, wp3, wp4, nn2, wp, sur,
sur1, vol, mar, mu, mode, med, var, sd, skew, kurto,seed;
seed=123456789;
nmed=percent*n;
nna=1000; nna1=nna+1;
if percent==0.95;
upa=1.8; lowa=1.45;
elseif percent==0.5;
upa=.1; lowa=-0.1;
endif;
ha=(upa-lowa)/(nna);
pea=seqa(lowa,ha,nna1);
wp1={1 4};
wp2={2 4};
wp4=1;
nn2=nna/2-1;
wp3=ones(1,nn2).*.wp2;
wp=wp1~wp3~wp4;
wp=wp*(ha/3);
sur=zeros(nna1,1);
sur1=sur;
i=1;
do while i<=nna1; x=pea[i];
sur[i]=ln(pdfn(x))+(nmed-1)*ln(cdfn(x))+(n-nmed)*ln(1-cdfn(x));
i=i+1;
endo;
/*Marginal Posteior pdf, mean and sd equation (2)*/
sur=sur-maxc(sur);
sur=exp(sur);
vol=wp*sur;
@print "volume" vol;@
mar=sur/vol;
mu=wp*(pea.*mar);
var=wp*(((pea-mu)^2).*mar);
sd=sqrt(var);
skew=wp*(((pea-mu)^3).*mar);
skew=skew/sd^3;
kurto=wp*(((pea-mu)^4).*mar);
kurto=kurto/sd^4;
mode=pea[maxindc(mar)];
med=pea[nn2+1];
graphset;
@print "cdfni(percent)" cdfni(percent);@
@print "mean, mode, median,sd,skew,kurtosis";@
```

```
@print mu~mode~med~sd~skew~kurto;@
retp(pea,mar,mu,mode,med,sd,skew,kurto);
endp;
proc(2)=draws_kden(percent,nrept);
local seed, xorder,x, i, xod, x1, den1, nmed;
seed=123456789; nmed=percent*n;
xorder={};
 i=1;
 do while i<=nrept;
 x=rndns(n,1,seed);
 xod=sortc(x,1);
 xorder=xorder|xod[nmed];
 i=i+1;
 endo;
{x1,den1}=kden(xorder);
retp(x1,den1);
endp;
proc(4)=stats(x1);
local mean,std,skew,kurtos;
mean=meanc(x1);
std=stdc(x1);
skew=meanc((x1-mean)^3/std^3);
kurtos=meanc((x1-mean)^4/std^4);
retp(mean,std,skew,kurtos);
endp;
```