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Advanced Economics Statistics

FALL 2010

Fifth-Assignment Answer Sheet
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1. This question calls for drawing and plotting a function of random variables whose are based on a chi squared distribution.

   (a) This question calls for drawing 8 *i.i.d* normal random variables $\{z\}_{i=1}^{8}$, then we create a new variable based on those z draws; we take into account the following formula $s = \sum z_i^2$. Thus, s_i is distributed as a central chi-squared variable with 8 degrees of freedom. We replicate the drawing generation process 4000 times. We plot the kernel density and the exact distribution of those draws in the figure 1. (see appendix for Gauss code)[1].

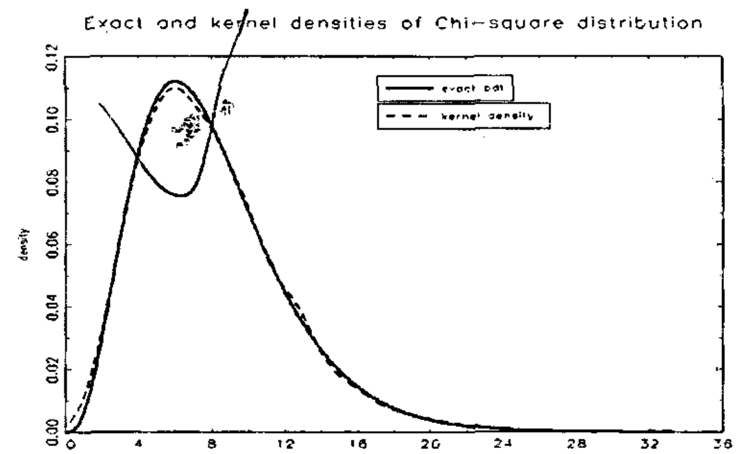

Figure 1: Exact pdf and kernel

   As we see the exact pdf of a central chi-squared variable with 8 degrees of freedom is closed to the kernel density of these 4000 draws.

   (b) The kernel density and the pdf are virtually identical When we want to do the approximation using a kernel function $g_h(x) = \frac{1}{nh}\sum_{\ell=1}^{n} K\left(\frac{x-x_\ell}{h}\right)$ we notice that we need a multivariate K, this because we have 8 draws which come from *i.i.d* normal random variables. In this case, the multivariate function K is approached by using a multivariate normal distribution function f:

$$f_{z_1^2, z_2^2, \ldots, z_8^2} = f_{z_1^2} \cdot f_{z_2^2} \ldots f_{z_8^2} \cdot |J|$$

[1] We took Tsurumi's code which he attached on sakai and we modified it in our convinience.

Because of independent property of those draws we define the product of each f_i times the jacobian as the multivariate function f. In this case the jacobian is:

$$|J| = \begin{vmatrix} \frac{\partial z_1^2}{\partial v_1} & \frac{\partial z_1^2}{\partial v_2} & \cdots & \frac{\partial z_1^2}{\partial v_8} \\ \frac{\partial z_2^2}{\partial v_1} & \frac{\partial z_2^2}{\partial v_2} & \cdots & \frac{\partial z_2^2}{\partial v_8} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial z_8^2}{\partial v_1} & \frac{\partial z_8^2}{\partial v_2} & \cdots & \frac{\partial z_8^2}{\partial v_8} \end{vmatrix}$$

where $z_i^2 = v_i$, in this case the partial derivatives $\frac{\partial z_i^2}{\partial v_j} = 1$ when $i = j$ and 0 otherwise, this because of the linear property of the expression of $s = \sum z_i^2$, we have that $|J| = 1$:

$$|J| = \begin{vmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{vmatrix} = 1$$

Thus, we perform the kernel density approaching K using f. In that case, we have that $K(\cdot) = \prod_{i=1}^{8} K_{z_i^2} = \prod_{i=1}^{8} f_{z_i^2}$. The kernel function is described as follows:

$$g_h(s) = \frac{1}{nh} \sum_{i=1}^{n} \prod_{j=1}^{8} f_{z_j^2} = \frac{1}{nh} \sum_{\ell=1}^{n} \prod_{i=1}^{8} f\left(\frac{z_i^2 - z_{i\ell}^2}{h}\right)$$

as f is a normal distribution we have the following

$$\begin{aligned} g_h(s) &= \frac{1}{nh} \sum_{\ell=1}^{n} \prod_{i=1}^{8} f\left(\frac{z_i^2 - z_{i\ell}^2}{h}\right) \\ &= \frac{1}{nh} \sum_{\ell=1}^{n} f\left(\frac{\sum_{i=1}^{8} z_i^2 - \sum_{i=1}^{8} z_{i\ell}^2}{h}\right) \end{aligned}$$

So, $s = \sum z_i^2$ and $s_\ell = \sum z_{i\ell}^2$.

$$g_h(s) = \frac{1}{nh} \sum_{\ell=1}^{n} f\left(\frac{s - s_\ell}{h}\right)$$

Thus, because of the linearity of the (*final*) variable (s) we do not need to do the steps in order to get into the *jacobian-of-transformation* process Thus, we directly use the kernel on the $s = \sum z_i^2$.

2. We use Simpson's rule in oder to obtain the mean, mode, variance, skewness and kurtosis (`stats`) of the chi-squared variable with 8 degrees of freedom and we compare them with the exact estimates of `stats`. We report results based on different grids and *upper-limits*

choices. We fix the lower limit to zero, because $s_i > 0$.

Table 2.1: Exact and Simpson's rule for grid size =10

	Exact	upper limits		
		30	40	50
Mean	8.000	7.940	7.960	8.211
Mode	6.000	6.000	8.000	5.000
Variance	16.000	16.932	17.934	17.159
Skewness	1.000	0.815	0.7988	0.882
Kurtosis	4.500	3.905	3.555	3.492

Considering a grid size of 100

Table 2.2: Exact and Simpson's rule for grid size =100

	Exact	upper limits		
		30	40	50
Mean	8.000	7.993	8.000	8.000
Mode	6.000	6.000	6.000	6.000
Variance	16.000	15.872	15.996	15.999
Skewness	1.000	0.967	0.999	1.000
Kurtosis	4.500	4.261	4.485	4.500

considering a grid size of 300

Table 2.1: Exact and Simpson's rule for grid size =300

	Exact	upper limits		
		30	40	50
Mean	8.000	7.993	8.000	8.000
Mode	6.000	6.000	6.000	6.000
Variance	16.000	15.873	15.996	16.000
Skewness	1.000	0.9670	0.998	1.000
Kurtosis	4.500	4.261	4.484	4.500

From above result we can say that the most important is to cover the support of the variable when we are using Simpson's rule; we have even got identical results with a grid of 100 points and an upper limit between 40 and 50. We fill out the final table in terms of the range

Table 2.4: Final results

Number of grids and limits of Integration...	
Grids	100-300
Lower Limit	0
Upper limit	40-50

If we want to be strict with the resolution of this question (considering no ranges), we can say that the grids and the upper limit of integration whose reproduce the exact moments are 100 and 50 respectively.

3. Let $X \sim \chi_1^2(0)$ and $Y \sim \chi_1^2(0)$ where $\chi_s^2(0)$ denotes the central chi-squared variable with a degrees of freedom. Derive the pdf of:

$$z = \frac{X}{X+Y} \tag{1}$$

and clearly identify the distribution of z . We denote $q = Y$ so,

$$\begin{aligned} f_{z,q} &= f_{x,y}\,|J| \\ &= \frac{1}{\Gamma\left(\frac{p}{2}\right)2^{p/2}}X^{p/2-1}\exp\left(-\frac{x}{2}\right)\cdot\frac{1}{\Gamma\left(\frac{p}{2}\right)2^{p/2}}Y^{p/2-1}\exp\left(-\frac{y}{2}\right)|J| \end{aligned}$$

Where $q = Y$ and the equation (1) is given by:

$$\begin{aligned} z &= \frac{X}{X+q} \\ zX + zq &= X \\ X &= \frac{zq}{1-Z} \end{aligned}$$

Here $X \sim \chi_p^2$ and $Y \sim \chi_p^2$ with p degrees of freedom Then, the jacobian $|J|$ is:

$$\begin{aligned} \begin{vmatrix} \frac{\partial X}{\partial Z} & \frac{\partial X}{\partial q} \\ \frac{\partial Y}{\partial Z} & \frac{\partial Y}{\partial q} \end{vmatrix} &= \begin{vmatrix} \frac{q}{1-z} + \frac{zq}{(1-Z)^2} & \frac{z}{1-z} \\ 0 & 1 \end{vmatrix} = \left(\frac{q}{1-z} + \frac{zq}{(1-z)^2}\right) \\ &= \frac{(1-z)q + zq}{(1-z)^2} = \frac{q}{(1-z)^2} > 0 \end{aligned}$$

By replacing terms into (1), we have:

$$\begin{aligned} f_{z,q} &= \frac{1}{\Gamma\left(\frac{p}{2}\right)2^{p/2}}X^{p/2-1}\exp\left(-\frac{x}{2}\right)\cdot\frac{1}{\Gamma\left(\frac{p}{2}\right)2^{p/2}}q^{p/2-1}\exp\left(-\frac{q}{2}\right)\frac{q}{(1-z)^2} \\ &= \left(\frac{1}{\Gamma\left(\frac{p}{2}\right)2^{p/2}}\right)^2\left(\frac{zq}{1-z}\right)^{p/2-1}\exp\left(-\frac{1}{2}\frac{zq}{1-z}\right)q^{p/2-1}\exp\left(-\frac{q}{2}\right)\frac{q}{(1-z)^2} \\ &= \left(\frac{1}{\Gamma\left(\frac{p}{2}\right)2^{p/2}}\right)^2\left(\frac{z}{1-z}\right)^{p/2-1}\exp\left(-\frac{1}{2}\frac{zq}{1-z} - \frac{q}{2}\right)\frac{q^{p-1}}{(1-z)^2} \\ &= \left(\frac{1}{\Gamma\left(\frac{p}{2}\right)2^{p/2}}\right)^2\left(\frac{z}{1-z}\right)^{p/2-1}\exp\left(-\frac{1}{2}\frac{zq}{1-z} - \frac{q}{2}\right)\frac{q^{p-1}}{(1-z)^2} \\ &= \left(\frac{1}{\Gamma\left(\frac{p}{2}\right)2^{p/2}}\right)^2\left(\frac{z}{1-z}\right)^{p/2-1}\exp\left(\frac{-zq-(1-z)q}{2(1-z)}\right)\frac{q^{p-1}}{(1-z)^2} \\ &= \left(\frac{1}{\Gamma\left(\frac{p}{2}\right)2^{p/2}}\right)^2\left(\frac{z}{1-z}\right)^{p/2-1}\exp\left(-\frac{q}{2(1-z)}\right)\frac{q^{p-1}}{(1-z)^2} \end{aligned}$$

Where $\Gamma(\cdot)$ is the gamma function, and $\Gamma(a) = (a-1)!$, and $|J|$ is the determinant of the jacobian matrix. Besides, we know that $p = 1$, so we may re write the last equation as follows:

$$f_{z,q} = \left(\frac{1}{\Gamma\left(\frac{1}{2}\right)2^{1/2}}\right)^2 \left(\frac{z}{1-z}\right)^{-1/2} \exp\left(-\frac{q}{2(1-z)}\right)\frac{1}{(1-z)^2}$$

Now we got the marginal distribution $f_z = \int f_{z,q}\, dq$ as follows:

$$f_z = \left(\frac{1}{\Gamma\left(\frac{1}{2}\right)2^{1/2}}\right)^2 \left(\frac{z}{1-z}\right)^{-1/2} \frac{1}{(1-z)^2}\int_q \exp\left(-\frac{q}{2(1-z)}\right)dq$$

By using the fact that, the integral of this *exponential* distribution $\int_q \exp\left(-\frac{q}{2(1-Z)}\right)dq$ is equal to $2(1-Z)$.[2]

$$\begin{aligned} f_z &= \left(\frac{1}{\Gamma\left(\frac{1}{2}\right)}\right)^2 \left(\frac{z}{1-z}\right)^{-1/2}\frac{1}{(1-z)} \qquad (2)\\ &= \left(\frac{1}{\Gamma\left(\frac{1}{2}\right)}\right)^2 z^{-1/2}(1-z)^{-1/2} \\ &= \frac{\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} z^{-1/2}(1-z)^{-1/2} \end{aligned}$$

By replacing the equation (1) into the last expression, we can re write the equation (2) as follows:

$$f_z = \frac{\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\left(\frac{X}{X+Y}\right)^{-1/2}\left(1-\frac{X}{X+Y}\right)^{-1/2}$$

What do you want from me??

By denoting $\alpha = \frac{1}{2}, \beta = \frac{1}{2}$ then we may have that f_z can be written as follows:

$$f_z = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\left(\frac{X}{X+Y}\right)^{\alpha-1}\left(1-\frac{X}{X+Y}\right)^{\beta-1}$$

Therefore f_z is a Beta distribution with parameter $\alpha = \frac{1}{2}, \beta = \frac{1}{2}$. We plot the distribution (histogram) which come from $z = \frac{X}{X+Y}$ and ~~the exact pdf~~ of a Beta distribution which

[2] By using the fact $\int \frac{1}{2(1-z)}\exp\left(-\frac{x}{2(1-z)}\right)dx = 1$ and $\Gamma(1) = 1$.

$\alpha = \frac{1}{2}, \beta = \frac{1}{2}$ and we confirm that above procedure is ok (see figure 2).

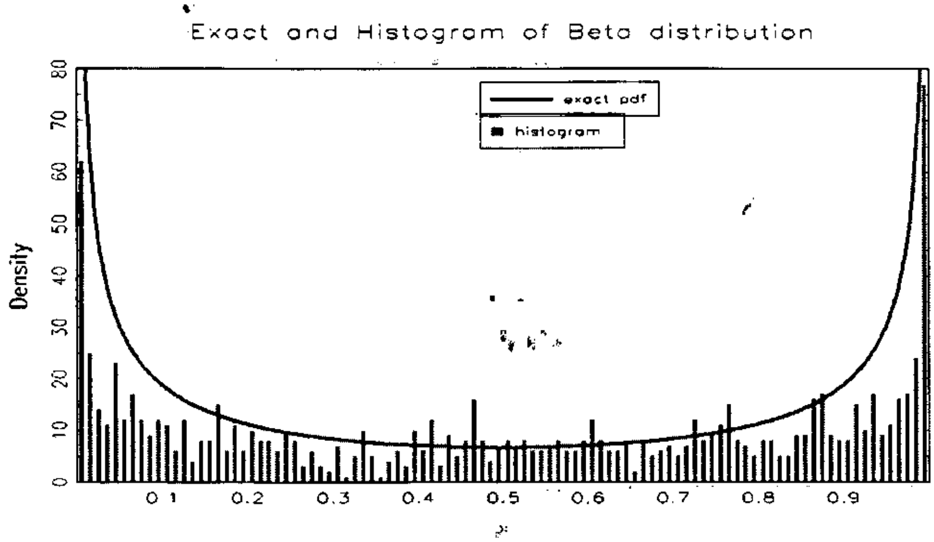

Figure 2: Exact pdf and histogram of Beta(0.5,0.5)

4. This question asks for the derivation of the pdf of the non central χ^2 random variable.

It's only asking about the pdf of the Non central χ^2 with df = 1

$$f(x_1, x_2, ...x_k) = \prod_{i=1}^{k} f(x_i)$$
$$= (2\pi)^{-k/2} \exp\left(-\frac{1}{2}\sum_{i=1}^{k}(x_i - u)^2\right)$$

The spherically simmetry given by: $x = x_1^2 + x_2^2 + ...x_k^2$ depends on the mean only through the square length.

$$\lambda = \mu_1^2 + \mu_2^2 + ...\mu_k^2$$
$$\mu_1 = \text{sqrt}(\lambda) \quad \text{and } \mu_j = 0 \text{ when } j \neq 1$$

According to the definition 1.5.1 in Muirhead (1982) "A $mx1$ random vector X is aid to have a spherical distribution if X and HX have some distribution for all mxm orthogonal matrices H. Muirhead gives the multivariate normal distribution $\mathcal{N}(0, \sigma^2 I_m)$ as an example of a spherical distribution.We set $\sigma_1 = \sigma_2 = ...\sigma_k = 1$ we are dealing with a spherically, symmetric distribution, up to a location shift". Since we know that $x = x_1^2$, $\text{sqrt}(x) = x_1, \mu = \text{sqrt}(\lambda)$, we have:

$Y = X^2,$

$X \sim N(\mu, 1)$

a single RV,

$$f(x_1, x_2, ...x_k) = \prod_{i=1}^{k} f(x_i)$$
$$= (2\pi)^{-k/2} \exp\left(-\frac{1}{2}\sum_{i=1}^{k}(x_i^2 - 2ux_i + u^2)\right)$$

so df=1

for k=1

The cdf of X is determined by:

$$\begin{aligned} F(X<\widetilde{x}) &= F\left(x_{i=1}^2<\widetilde{x}\right) \\ &= F\left(-\sqrt{\widetilde{x}}<x_{i=1}<\sqrt{\widetilde{x}}\right) \\ &= F\left(x_{i=1}\leq\sqrt{\widetilde{x}}\right)-F\left(x_{i=1}\leq-\sqrt{\widetilde{x}}\right) \\ &= F\left(\sqrt{\widetilde{x}}\right)-F\left(-\sqrt{\widetilde{x}}\right) \end{aligned}$$

where $\widetilde{x}$ is the threshold in the distribution, So

$$\begin{aligned} f(x) &= \frac{\partial F(\widetilde{x})}{\partial x} \\ &= \phi\left(\sqrt{\widetilde{x}}\right)\frac{\partial\sqrt{\widetilde{x}}}{\partial x}-\phi\left(-\sqrt{\widetilde{x}}\right)\frac{-\partial\sqrt{\widetilde{x}}}{\partial x} \\ &= f\left(\sqrt{\widetilde{x}}\right)\frac{1}{2}x^{-1/2}+\phi\left(-\sqrt{\widetilde{x}}\right)\frac{1}{2}x^{-1/2} \\ &= \frac{1}{2\sqrt{x}}\left(\phi\left(\sqrt{\widetilde{x}}\right)+\phi\left(-\sqrt{\widetilde{x}}\right)\right) \end{aligned}$$

ok

We have used the (*partial*) derivate (at the end of this solution we show that $\sqrt{\widetilde{x}}=x_{i=1}=\sqrt{x}+\sqrt{h}$)

$$\frac{\partial\sqrt{\widetilde{x}}}{\partial x}=\frac{\partial\left(\sqrt{x}-\sqrt{h}\right)}{\partial x}=\frac{1}{2}x^{-1/2}$$

So, ϕ is the normal pdf distribution;

$$=\frac{1}{2\sqrt{x}}\left(\phi\left(\sqrt{\widetilde{x}}\right)+\phi\left(-\sqrt{\widetilde{x}}\right)\right) \tag{3}$$

Given $x=(x_{i=1}-u)^2$,we have that $\sqrt{x}+u=x_{i=1}$,and we may re-define $u=\sqrt{h}$.Therefore $\sqrt{x}+\sqrt{h}=x_{i=1}$ (In this case we consider $x_{i=1}\geq 0$, thus, we have: $\sqrt{\widetilde{x}}=x_{i=1}=\sqrt{x}+\sqrt{h}$). And if $x_{i=1}<0$,we have $x=(-x_{i=1}-u)^2=(x_{i=1}+u)^2$ and finally we have that $\sqrt{x}-\sqrt{h}=x_{i=1}$ or written the other way: $-\sqrt{\widetilde{x}}=x_{i=1}=\sqrt{x}-\sqrt{h}$. We replace this into expression (3):

$$=\frac{1}{2\sqrt{x}}\left(\phi\left(\sqrt{x}+\sqrt{h}\right)+\phi\left(\sqrt{x}-\sqrt{h}\right)\right) \tag{4}$$

5. We derive the equation (4) on p.16 of the lecture notes on chapter 4.

$$
\begin{aligned}
f_X(x,1,\lambda) &= \frac{1}{2\sqrt{x}}\left[\phi\left(\sqrt{x}-\sqrt{\lambda}\right)+\phi\left(\sqrt{x}+\sqrt{\lambda}\right)\right] \\
&= \frac{1}{2\sqrt{x}}\frac{1}{\sqrt{2\pi\sigma}}\begin{bmatrix} \exp\left(-\frac{\left(\sqrt{x}-\sqrt{\lambda}\right)^2}{2}\right) \\ +\exp\left(-\frac{\left(\sqrt{x}+\sqrt{\lambda}\right)^2}{2}\right)\end{bmatrix} \\
&= \frac{1}{2\sqrt{x}}\frac{1}{\sqrt{2\pi\sigma}}\begin{bmatrix} \exp\left(-\frac{\left(x-2\sqrt{x}\sqrt{\lambda}+\lambda\right)}{2}\right) \\ +\exp\left(-\frac{\left(x+2\sqrt{x}\sqrt{\lambda}+\lambda\right)}{2}\right)\end{bmatrix} \\
&= \frac{1}{2\sqrt{x}}\frac{1}{\sqrt{2\pi\sigma}}\exp\left(-\frac{(x+\lambda)}{2}\right)\begin{bmatrix} \exp\left(\sqrt{x}\sqrt{\lambda}\right) \\ +\exp\left(-\sqrt{x}\sqrt{\lambda}\right)\end{bmatrix}
\end{aligned}
$$

We use the hyperbolic function properties, *Wikipedia*[3] says: $\exp(x) = \cosh(x) + \sinh(x)$, $\exp(-x) = \cosh(x) - \sinh(x)$. This is based on `Euler's Formula` $\cosh(\alpha) + x\sinh(\alpha) = \exp(i\alpha)$.[4]
Based on the former expressions we finally have:

$$
\begin{aligned}
X(x,1,\lambda) &= \frac{1}{2\sqrt{x}}\frac{1}{\sqrt{2\pi\sigma}}\exp\left(-\frac{(x+\lambda)}{2}\right) \\
&\quad\cdot\begin{bmatrix} \cosh\left(\sqrt{x}\sqrt{\lambda}\right)+\sinh\left(\sqrt{x}\sqrt{\lambda}\right) \\ +\cosh\left(\sqrt{x}\sqrt{\lambda}\right)-\sinh\left(\sqrt{x}\sqrt{\lambda}\right)\end{bmatrix} \\
&= \frac{1}{\sqrt{x}}\frac{1}{\sqrt{2\pi\sigma}}\exp\left(-\frac{(x+\lambda)}{2}\right)\cosh\left(\sqrt{x}\sqrt{\lambda}\right)
\end{aligned}
$$

6. In the answer to excersise 4.47 CB states "by similar argument for $z > 0$ we get $\mathcal{P}(Z > z) = \mathcal{P}(X > z)$. Fill in the intermediate step to get: $\mathcal{P}(Z > z) = \mathcal{P}(X > z)$.The question is related to: a) Show that Z has a normal distribution, b) Show that the joint distribution of Z and Y is not bivariate normal. Hint: Z and Y has the same sign. In the case for $\mathcal{P}(Z \leq z)$, where z is a threshold and negative and the fact that: x if $xy > 0$ and $-x$ if $xy < 0$. X and Y are $\mathcal{N}(0,1)$ for $z < 0$.

$$
\mathcal{P}(Z \leq z) = \mathcal{P}(X \leq z \wedge XY > 0) + \mathcal{P}(-X \leq z \wedge XY > 0)
$$

Here is $z < 0$, $Y < 0$

Need to Explain why this is -2

$$
\mathcal{P}(Z \leq z) = \mathcal{P}(X \leq z \wedge Y < 0) + \mathcal{P}(X \geq -z \wedge Y < 0)
$$

Because X and Y are independent, we have that:

$$
\begin{aligned}
\mathcal{P}(Z \leq z) &= \mathcal{P}(X \leq z)\ \mathcal{P}(Y < 0) + \mathcal{P}(X \geq -z)\mathcal{P}(Y < 0) \\
&= \frac{\mathcal{P}(X \leq z)}{2} + \frac{\mathcal{P}(X \geq -z)}{2}
\end{aligned}
$$

[3] Search for: `hyperbolic function properties`.
[4] See `mathworld`.

I need to explain why?? it's obvious

By symmetry we have:

$$\mathcal{P}(Z \leq z) = \mathcal{P}(X \leq z)$$

In the case for z positive and $\mathcal{P}(Z > z) = \mathcal{P}(X > z)$:

$$\begin{aligned}
\mathcal{P}(Z > z) &= \mathcal{P}(X > z \wedge XY > 0) + \mathcal{P}(-X > z \wedge XY > 0) \\
&= \mathcal{P}(X > z \wedge Y > 0) + \mathcal{P}(-X > z \wedge Y > 0) \\
&= \mathcal{P}(X > z)\ \mathcal{P}(Y > 0) + \mathcal{P}(X < -z)\,\mathcal{P}(Y > 0) \\
&= \frac{\mathcal{P}(X > z)}{2} + \frac{\mathcal{P}(X < -z)}{2}
\end{aligned}$$

As before we have the following:

$$\mathcal{P}(Z > z) = \mathcal{P}(X > z)$$

7. Computing the variance of X by using the conditional variance identity of the equation 4.4.4 on page 167 of C.B.

$$V(x) = E(V(x|y)) + V(E(x|y)) \tag{5}$$

Where $V(x)$ and $V(x|y)$ are the unconditional and conditional variances respectively. So, $X \sim \chi^2$ con $p + 2k$ degrees of freedom and $X|k \sim \chi^2$ con $p + 2k$ degrees of freedom.

$$\begin{aligned}
E(x|k) &= p + 2k \\
V(E(x|k)) &= V(p) + 4V(k)
\end{aligned}$$

Where $V(p) = 0$ because p is a constant. Furthermore, k is distributed as $Poisson(\lambda)$ is equal to $V(k) = \lambda$ and $V(E(x|k)) = 4\lambda$. Now $V(x|y) = E(x^2|y) - (E(x|y))^2 = 2(p + 2\lambda)$ and $E(V(x|y)) = E(2p) + 4E(\lambda) = 2p + 4\lambda$. By adding the last expressions, we have the following:

$$V(x) = E(V(x|y)) + V(E(x|y)) = 2p + 4\lambda + 4\lambda = 2p + 8\lambda$$

8. We compute $E(x^2)$ directly

$$\begin{aligned}
E(x^2) &= \int_0^\infty x^2 \sum_{k=0}^\infty \frac{X^{p/2+k-1}}{\Gamma\left(\frac{p}{2}+k\right) 2^{p/2+k}} \exp\left(-\frac{x}{2}\right) \frac{\lambda^k \exp(-\lambda)}{k!} dx \qquad (6) \\
&= \sum_{k=0}^\infty \int_0^\infty \left(x^2 \frac{X^{p/2+k-1}}{\Gamma\left(\frac{p}{2}+k\right) 2^{p/2+k}} \exp\left(-\frac{x}{2}\right) dx \right) \frac{\lambda^k \exp(-\lambda)}{k!}
\end{aligned}$$

Because $\frac{X^{p/2+k-1}}{\Gamma\left(\frac{p}{2}+k\right)2^{p/2+k}} \exp\left(-\frac{x}{2}\right)$ is the pdf of [redacted] central chi squared distribution with $p + 2k$ degrees of freedom, we can get the moment[5] $E(x^2)$ as follows:

$$E(x^2_{chi}) = \int_0^\infty x^2 \frac{X^{p/2+k-1}}{\Gamma\left(\frac{p}{2}+k\right) 2^{p/2+k}} \exp\left(-\frac{x}{2}\right) dx = 2(p + 2k) + (p + 2k)^2 \tag{7}$$

[5] See Mathworld. Because $E(x) = p + 2k$ and given that $V(x) = E(x^2) - (E(x))^2$. Also, $E(x^2) = V(x) + (E(x))^2 = 2(p + 2k) + (p + 2k)^2$.

By using the eq. (7), we can express the eq. (6) as follows:

$$\begin{aligned}
E\left(x^2\right) &= \sum_{k=0}^{\infty}\left(2\left(p+2k\right)+\left(p+2k\right)^2\right)\frac{\lambda^k \exp\left(-\lambda\right)}{k!} \\
&= 2p\sum_{k=0}^{\infty}\frac{\lambda^k \exp\left(-\lambda\right)}{k!}+4\sum_{k=0}^{\infty}k\frac{\lambda^k \exp\left(-\lambda\right)}{k!}+p^2\sum_{k=0}^{\infty}\frac{\lambda^k \exp\left(-\lambda\right)}{k!}+ \\
&\quad 4p\sum_{k=0}^{\infty}k\frac{\lambda^k \exp\left(-\lambda\right)}{k!}+4\sum_{k=0}^{\infty}k^2\frac{\lambda^k \exp\left(-\lambda\right)}{k!} \\
E\left(x^2\right) &= 2p+4\lambda+p^2+4p\lambda+4\left(\lambda+\lambda^2\right)
\end{aligned}$$

Where $\sum_{k=0}^{\infty}\frac{\lambda^k \exp(-\lambda)}{k!}=1$ and $E\left(k\right)=\lambda$ and $V\left(k\right)=\lambda$ which corresponds to the Poisson Distribution. Since $V\left(x\right)=E\left(x^2\right)-\left(E\left(x\right)\right)^2$, we have that $E\left(x^2\right)=V\left(x\right)+\left(E\left(x\right)\right)^2=\lambda+\lambda^2$. Therefore, we have that:

$$V\left(x\right)=E\left(x^2\right)-\left(E\left(x\right)\right)^2=2p+4\lambda+p^2+4p\lambda+4\left(\lambda+\lambda^2\right)-\left(p+2\lambda\right)^2$$

Because we know that:

$$\begin{aligned}
E\left(x\right) &= \sum_{k=0}^{\infty}E\left(x_{chi}\right)\frac{\lambda^k \exp\left(-\lambda\right)}{k!}=\sum_{k=0}^{\infty}\left(p+2k\right)\frac{\lambda^k \exp\left(-\lambda\right)}{k!} \\
&= p\sum_{k=0}^{\infty}\frac{\lambda^k \exp\left(-\lambda\right)}{k!}+2\sum_{k=0}^{\infty}k\frac{\lambda^k \exp\left(-\lambda\right)}{k!} \\
&= \left(p+2\lambda\right)
\end{aligned}$$

Given the last equation, we finally have that:

$$V\left(x\right)=E\left(x^2\right)-\left(E\left(x\right)\right)^2=2p+8\lambda$$

References

[1] Casella, G and R. Berger. 2002. Statistical Inference. Second Edition, Duxbury Advanced Studies.

[2] Mendenhall and Scheaffer. 1973. Mathematical Statistics with Applications. Duxbury Press. North Scituate, Massachusetts.

[3] Mathworld website. http://mathworld.wolfram.com/

[4] GAUSS kernel density library. GAUSS.

[5] M.P Wand & M.C Jones. 1995. Kernel Smoothing. Monographs on Statistics and Applied Probability. Chapman & Hall, 1995.

1 Appendix

1.1 Question 1.(a)

```
/*==================================================================================================
   /* Code by Hiraki Tsurumi (Assign #4-10:  asgn04-10.pro) but modified by Freddy
Rojas Cama */
   // Last update October 10th 2010
   // Rutgers University - Phd program
   /*===============================================================================================
   /********************************************
   * Question N 1 and N 2
   ********************************************/
   new;
   library pgraph;
   pqgwin auto;
   graphset;
   n_r=4000;
   n_c=8;
   z_ij=rndn(n_r,n_c).^2;
   z_i=sumc(z_ij');
   format /m1 /rd 6,8;
   {x1,den1}=kden(z_i);
   nn=1000;
   up=maxc(z_i);
   low=0;
   h=(up-low)/(nn-1);
   xx=seqa(low,h,nn);
   k=n_c;
   fx=(2^(k/2)*gamma(k/2))^-1*(xx.^(k/2-1)).*exp(-xx./2);
   /*===drawing on one graph two functions with different vector dimensions====*/
    graphset;
    begwind;
    _protate=0;
    _pcolor = { 9 5 }; /* Colors for series */
    _pmcolor = { 1, 8, 2, 8, 8, 8, 8, 8, 15 };
    /*Colors for axes, title, x and y labels, date, box, and background */
    _plwidth={12 12 }; /*Controls line thickness for main curves*/
    _paxht=0.10; /*Controls size of axes labels*/
    _ptitlht = 0.18; /*Controls main title size */
    _plegctl = { 2 7 2 4.0 };
    title("Exact and kernel densities of Chi-square distribution");
    ylabel("density");
    xlabel("x");
    makewind(7.8, 7.8, 0, 0, 0);
    _pltype=6;
```

```
    _plegctl={2 4 4.5 5.4};
    _plegstr="exact pdf";
    xy(xx,fx);
    _pcolor = { 5 }; /* Colors for series */
    _pmcolor = { 1, 8, 2, 8, 8, 8, 8, 8, 15 };
    _plegctl={2 4 4.5 5.0};
    _plwidth={8}; /*Controls line thickness for main curves*/
    _plegstr="kernel density ";
    _pltype=3;
    xy(x1,den1);
    endwind;
   /*====computation of mean, variance, skewness, and kurtosis see mathworld ====*/
   //exact
   mean=k;
   var=2*k;
   skew=(8/k)^0.5;
   kurtosis=12/k+3;
   mode=maxc((k-2)|0);
   cls;
   print;
   " ";
   " ---------------------------------------------------------------------------";
   " ";
   " ";; " Statistics from draws whose come from Chi-square formula ";
   print "exact mean, mode, variance, skewness, and kurtosis ";
   mean|mode|var|skew|kurtosis;
   " ";
   /*===computation of mean, mode, variance, skewness, and kurtosis by Simpson's
rule =====*/
    upa=30; lowa=0.00;
    nn=100; nn1=nn+1;
    ha=(upa-lowa)/nn;
    pea=seqa(lowa,ha,nn1);
    wp1={1 4}; wp2={2 4}; wp4=1;
    nn2=nn/2-1;
    wp3=ones(1,nn2).*.wp2;
    wp=wp1~wp3~wp4;
    wp=wp*(ha/3);
    sur=(2^(k/2)*gamma(k/2))^-1*(pea.^(k/2-1)).*exp(-pea./2);
    /*--marginal posterior pdf, mean and sd----*/
    vol=wp*sur;
    //print "volume" vol;
    mar=sur;
    mu=wp*(pea.*mar);
    var=wp*(((pea-mu)^2).*mar);
```

```
 sd=sqrt(var);
 skew=wp*(((pea-mu)^3).*mar);
 skew=skew/sd^3;
 kurto=wp*(((pea-mu)^4).*mar);
 kurto=kurto/sd^4;
 mode=pea[maxindc(mar)];
 par=mu|mode|var|skew|kurto;
" ";; " Statistics from draws whose come from Simpson's rule Formula ";
 print " mean mode var skew and kurtosis " par;
/*====sample mean, variance, skewness, and kurtosis ====*/
 x1=z_i;
 smean=meanc(x1);
 sd=stdc(x1);
 svar=sd^2;
 skw=meanc((x1-smean)^3)/sd^3;
 skurt=meanc((x1-smean)^4)/sd^4;
 sx1=sortc(x1,1);
 smode=sx1[maxindc(den1)];
print;
" ";
" ";; " Statistics from draws whose come from moment estimates ";
 print "sample mean, mode, variance, skewness, and kurtosis ";
 print smean|smode|svar|skw|skurt;
end;
/* */
/* kernel density estimation */
/* */
 proc(2)=kden(v);
 local g,h,j,nn,res;
 nn=rows(v);
 h=1.06*stdc(v)/nn^.2;
 print "h " h;
 g=0;
 j=1;
 do while j <= nn;
 g=g|meanc(pdfn((v-v[j])/h))/h;
 j=j+1;
 endo;
 res=sortc(v~g[2:nn+1],1);
 retp(res[.,1],res[.,2]);
 endp;
```