Advanced Economics Statistics

FALL 2010

Fourth-Assignment Answer Sheet
by Freddy Rojas Cama

1. This question calls for plotting a kernel density estimate and the exact pdf on the same graph.

   (i) We drew $n = 6000$ beta random variables by setting $\alpha = 148$ and $\beta = 4$ (see appendix for details).

   (ii) We evaluate the exact pdf at 1000 grid points (see appendix for details).

   (iii) The graph on which the kernel density and the exact pdf are drawn:

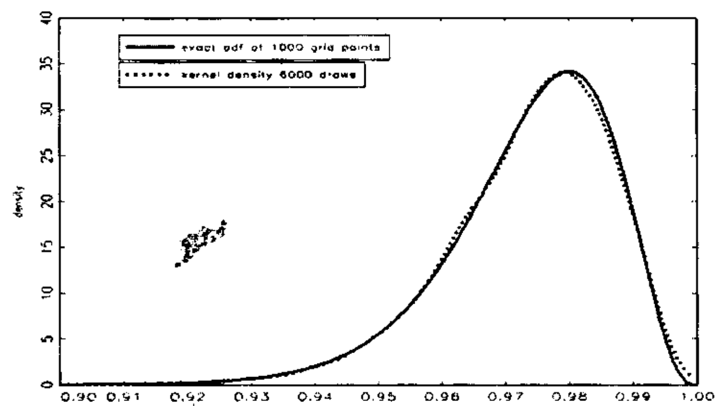

Figure 1. Exact and Kernel Density estimates for Beta(148, 4)

1 10
2 10
3 7
4 10
5 10
6 9

(56)

2. This question calls for computing the mean, variance, skewness and kurtosis (the stats) by using analitycal, numerical and sampling-based estimates.

   (i) The exact mean, mode, variance, skewness and kurtosis of a beta distribution(148,4) are[1]:

$$E(x) = \gamma_1 = \frac{\alpha}{\alpha+\beta}$$

the variance

$$E\left(x^2\right) - [E(x)]^2 = \gamma_2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

skewness:

$$\frac{E\left((x) - E(x)\right)^3}{\left(E\left(x^2\right) - [E(x)]^2\right)^{\frac{3}{2}}} = \gamma_3$$

$$\gamma_3 = \frac{2(\beta-\alpha)\sqrt{1+\alpha+\beta}}{\sqrt{\alpha\beta}\,(2+\alpha+\beta)}$$

[1] See http://mathworld.wolfram.com/BetaDistribution.html

Kurtosis

$$\frac{E\left((x)-E(x)\right)^4}{\left(E\left(x^2\right)-\left[E(x)\right]^2\right)^{\frac{4}{2}}}=\gamma_4$$

$$\gamma_4=\frac{6\left[\alpha^3+\alpha^2(1-2\beta)+\beta^2(1+\beta)-2\alpha\beta(2+\beta)\right]}{\alpha\beta(2+\alpha+\beta)(3+\alpha+\beta)}+3$$

and the mode

$$\text{mode}=\frac{\alpha-1}{\alpha+\beta-2}$$

(ii) We compute the stats by using the Simpsons's rule (see the appendix).

(iii) We compute the sample estimates of stats using the 6000 draws of beta random variables (see the appendix). Also, we compute expressions given in item (i) in this section for calculating the exact moments of the distribution. Thus we have the following results

Table 1: Report

	Exact	Simpson's rule	Sample Estimates
Mean	0.97368	0.97368	0.97404
Mode	0.98000	0.98000	0.98660
Variance	0.00017	0.00017	0.00016
Skewness	-0.95073	-0.95073	-0.88053
Kurtosis	4.30837	4.30837	3.92832

As we see the estimates by using the exact, numerical and sampling-based methods are closed by each other. In above table, it seems that we have a sort of unaccuracy in the sampling-based estimates of "stats" but the consistency of the estimators is proved as the number of draws goes to infinity.[2]

3. On the lecture #3 (page 12) a proof of Stein's lemma is shown. Alternatively, Goldstein shows a standard way to prove this Lemma. The Stein's Lemma states that the following expression must hold:

$$E\left(Z\,f(Z)\right)=E\left(f'(Z)\right)$$

if and only if $Z\sim\mathcal{N}(0,1)$. In order to prove this condition, Goldstein breaks $E\left(f'(Z)\right)$ in two parts, they are:.$\int_0^\infty f'(z)\,\phi(z)\,dz$ and $\int_{-\infty}^0 f'(z)\,\phi(z)\,dz$.Where $\phi(z)$ is the standard normal function. On page 16 of Goldstein's lecture, the *key* step for reaching the proof is:

$$\int_0^\infty f'(z)\,\phi(z)\,dz=\int_0^\infty f'(z)\int_z^\infty y\phi(y)\,dydz \tag{1}$$

We prove the expression (1) as follows: By inspection, we know that $y\phi(y)=-\phi'(y)$.

[2]We redo the experiment by using 20000 draws and the sampling-based estimates approach to the exact estimates.

by whose inspection? Show me... —3

So, applying integral to this expression, we have:

$$\begin{aligned}\int_z^\infty y\phi(y)\,dy &= \int_z^\infty -\phi'(y)\,dy \\ &= -\int_z^\infty d[\phi(y)]\,dy \\ &= -\phi(y)|_z^\infty \\ &= \phi(z)\end{aligned}$$

Thus, the condition (1) is hold. ■

4. This question calls to get the coefficient of kurtosis for a normal variable by using characteristic function and Stein's Lemma.

   (i) by using the characteristic function, we need to perform the following expression:

$$\frac{\partial^n \varphi(t)}{\partial t}|_{t=0} = i^n E(x^n) \tag{2}$$

In this case, for the calculation of kurtosis we set up $n = 4$. We use the characteristic function setting $\mu = 0$ to obtain the coefficient de kurtosis. In this case, we have that $\varphi(t) = e^{-\frac{1}{2}\sigma^2 t^2}$. So, in order to get the coefficient of kurtosis, we make the following steps:

$$\frac{\partial \varphi(t)}{\partial t} = -e^{-\frac{1}{2}\sigma^2 t^2}(\sigma^2 t)$$

The second (partial) derivative is:

$$\frac{\partial^2 \varphi(t)}{\partial t \partial t} = e^{-\frac{1}{2}\sigma^2 t^2}(\sigma^4 t^2) - e^{-\frac{1}{2}\sigma^2 t^2}(\sigma^2)$$

The third (partial) derivative is:

$$\frac{\partial^3 \varphi(t)}{\partial t \partial t \partial t} = -e^{-\frac{1}{2}\sigma^2 t^2}(\sigma^6 t^3) + 2e^{-\frac{1}{2}\sigma^2 t^2}(\sigma^4 t) + e^{-\frac{1}{2}\sigma^2 t^2}(\sigma^4 t)$$

The fourth (partial) derivative is:

$$\begin{aligned}\frac{\partial^4 \varphi(t)}{\partial t \partial t \partial t \partial t} &= e^{-\frac{1}{2}\sigma^2 t^2}(\sigma^8 t^4) - 3e^{-\frac{1}{2}\sigma^2 t^2}(\sigma^6 t^2) \\ &\quad -2e^{-\frac{1}{2}\sigma^2 t^2}(\sigma^6 t^2) + 2e^{-\frac{1}{2}\sigma^2 t^2}(\sigma^4) \\ &\quad -e^{-\frac{1}{2}\sigma^2 t^2}(\sigma^6 t^2) + e^{-\frac{1}{2}\sigma^2 t^2}(\sigma^4)\end{aligned}$$

So, we evaluate the fourth derivative at $t = 0$.

$$\frac{\partial^4 \varphi(t)}{\partial t \partial t \partial t \partial t}|_{t=0} = 3\sigma^4 \tag{3}$$

So, considering expression (2) and (3) we have that $E(x^4) = 3\sigma^4$. The kurtosis ($k[x]$) is calculated as $k[x] = \frac{\sum(x-\widehat{\mu})^4}{\widehat{\sigma}^4}$, in terms of expectations $k[x] = \frac{E(x-\mu)^4}{\sigma^4}$. So, considering that $\mu = 0$, we have

$$k[x] = 3$$

In this case, the excess of kurtosis is defined as $k[x] = \frac{E(x)^4}{\sigma^4} - 3 = 0$

(ii) By using the Stein's Lemma, we have that $E(zf(z)) = E(f'(z))$ if and only if $Z \sim \mathcal{N}(0,1)$, being in our case that $z = \frac{x}{\sigma}$.[3] We are looking for $E(x^4)$ therefore $f(z) = z^3 = \frac{x^3}{\sigma^3}$, in this case we have that;

$$\begin{aligned} E(zf(z)) &= E(f'(z)) \\ E(z^4) &= 3E(z^2) \\ E(z^4) &= 3 \end{aligned}$$

Because $E(z^2) = 1$. Then, we have that $E(z^4) = \frac{E(x^4)}{\sigma^4}$, Then, we have

$$E(x^4) = 3\sigma^4$$

The kurtosis ($k[x]$) is calculated as $k[x] = \frac{\sum(x-\widehat{\mu})^4}{\widehat{\sigma}^4}$, in terms of expectations $k[x] = \frac{E(x-\mu)^4}{\sigma^4}$. So, considering that $\mu = 0$, we have

$$k[x] = 3$$

In this case the excess of kurtosis as defined as $k[x] = \frac{E(x)^4}{\sigma^4} - 3 = 0$

5. The question calls for completing the steps of the solution in exercise 3.33 of Casella and Berger. We verify that gamma$(\alpha, \frac{1}{\alpha})$ is an exponential family. The probability distribution function is:

$$\text{gamma}(\alpha, \frac{1}{\alpha}) = \frac{x^{\alpha-1}\exp(-\alpha x)}{\Gamma(\alpha)\left(\frac{1}{\alpha}\right)^{\alpha}} \tag{4}$$

so we must to arrange the above terms in order to get the following structure:

$$f(x|\theta) = h(x)\,c(\theta)\exp\left(\sum_{i=1}^{k} w_i(\theta)\,t_i(x)\right) \tag{5}$$

thus, re-arranging terms to expression (4):

$$\begin{aligned} \text{gamma}(\alpha, \frac{1}{\alpha}) &= \frac{x^{\alpha-1}\exp(-\alpha x)}{\Gamma(\alpha)\left(\frac{1}{\alpha}\right)^{\alpha}} \\ &= \frac{x^{\alpha}\exp(-\alpha x)}{x\Gamma(\alpha)\left(\frac{1}{\alpha}\right)^{\alpha}} \\ &= \frac{\exp(\alpha\log x - \alpha x)}{x\Gamma(\alpha)\left(\frac{1}{\alpha}\right)^{\alpha}} \end{aligned}$$

where we use the fact that $\exp(\alpha\log x) = x^{\alpha}$,

$$\begin{aligned} \text{gamma}(\alpha, \frac{1}{\alpha}) &= \frac{\exp(\alpha\log x - \alpha x)}{x\Gamma(\alpha)\left(\frac{1}{\alpha}\right)^{\alpha}} \\ &= \frac{1}{x}\frac{\alpha^{\alpha}}{\Gamma(\alpha)}\exp(\alpha\log x - \alpha x) \end{aligned}$$

[3] Recall $\mu = 0$ only for kurtosis-calculation purposes.

so, in this case we have reproduced the structure of expression (5), thus we have the following (considering the parameters of the functin as arguments in that expression): $h(x) = \frac{1}{x}, c(\alpha) = \frac{\alpha^{\alpha}}{\Gamma(\alpha)}, w_1(\alpha) = \alpha, t_1(\alpha) = \log x, w_2(\alpha) = \alpha$ and $t_1(\alpha) = -x$.

6. This question calls for filling in intermediate steps in order to get a complete derivation of theorem 3.6.7 (Casella and Berger). Likewise, this question asks for obtaining the variance of χ^2_p.

   (i) We must show that

$$\Gamma\left(\frac{p}{2}\right) 2^{\frac{p}{2}} = \frac{\Gamma\left(\frac{(p+2)}{2}\right) 2^{\frac{p+2}{2}}}{p}$$

   By using the fact that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ we work on the right side of above expression[4]:

$$\begin{aligned} \Gamma\left(\frac{p}{2}\right) 2^{\frac{p}{2}} &= \frac{\Gamma\left(\frac{p}{2}+1\right) 2^{\frac{p+2}{2}}}{p} \\ &= \frac{\frac{p}{2}\Gamma\left(\frac{p}{2}\right) 2^{\frac{p+2}{2}}}{p} \\ &= \Gamma\left(\frac{p}{2}\right) 2^{\frac{p+2}{2}-1} \\ &= \Gamma\left(\frac{p}{2}\right) 2^{\frac{p}{2}} \end{aligned}$$

   (ii) We obtain the variance of χ^2_p by using the theorem 3.6.7. In this case $E\left[h\left(\chi^2_p\right)\right] = E\left[\left(\chi^2_p\right)^2\right]$

$$\begin{aligned} E\left[\left(\chi^2_p\right)^2\right] &= pE\left[\frac{\left(\chi^2_{p+2}\right)^2}{\chi^2_{p+2}}\right] \\ &= pE\left[\chi^2_{p+2}\right] \end{aligned}$$

   by using the characteristic function to get the $E\left[\chi^2_p\right]$;

$$E\left[\left(\chi^2_p\right)^2\right] = p(p+2)$$

   in any case we know that the expectation of a chi-square variable is the number of degrees of freedom[5]. We compute the variance

$$\begin{aligned} E\left[\left(\chi^2_p\right)^2\right] - \left[E\left[\left(\chi^2_p\right)\right]\right]^2 &= p(p+2) - p^2 \\ &= 2p \end{aligned}$$

[4] $\Gamma(\alpha+1) = \alpha!$, so, $\alpha\Gamma(\alpha) = \alpha \cdot (\alpha-1)! = \alpha!$

[5] see mathworld.com

References

[1] Casella, G and R. Berger. 2002. Statistical Inference. Second Edition, Duxbury Advanced Studies.

[2] Mendenhall and Scheaffer. 1973. Mathematical Statistics with Applications. Duxbury Press. North Scituate, Massachusetts.

[3] Mathworld website. http://mathworld.wolfram.com/

[4] Beta Distribution in Mathworld website. http://mathworld.wolfram.com/BetaDistribution.html

[5] GAUSS kernel density library. GAUSS.

1 Appendix

1.1 Gauss Program, Question 1.(i),(ii) and (iii)

```
/*==================================================================================
   Asignment 03 (code sent by Hiraki Tsurumi)
   Modified by Freddy Rojas Cama
   Last update October 1st, 2010
   ================================================================================*/
   /****************************************/
   // Question N 1 (i),(ii) and (iii)
   /****************************************/
   //Preliminaries and housecleaning
   new;
   cls;
   library pgraph;
   //#include density.src;
   pqgwin auto;
   graphset;
   alpha=148;
   beta=4;
   //Grid Points for exact pdf
   nn=1000;
   up=1.0;
   low=0.5;
   h=(up-low)/(nn-1);
   xx=seqa(low,h,nn);
   lnconst=lnfact(alpha+beta-1)-lnfact(alpha-1)-lnfact(beta-1);
   const=exp(lnconst);
   fx=const*xx^(alpha-1).*(1-xx)^(beta-1);
   //Draws from Beta Function
   n=6000;
   x=rndbeta(n,1,alpha,beta);
   h0=1.06*stdc(x)/(n^.2);
   h2=0.01;
   h3=0.15;
   {x0,den0}=kden(x,h0);
   {x2,den2}=kden(x,h2);
   {x3,den3}=kden(x,h3);
   {x4,den4}=kernele(x,h0);
   // setting up graphs
   graphset;
   begwind;
   _protate=0;
    _pcolor = { 9 5 }; /* Colors for series */
    _pmcolor = { 1, 8, 2, 8, 8, 8, 8, 8, 15 };
```

```
 /*Colors for axes, title, x and y labels, date, box, and background */
 _plwidth={12 12 }; /*Controls line thickness for main curves*/
 _paxht=0.10; /*Controls size of axes labels*/
 _ptitlht = 0.22; /*Controls main title size */
 _plegctl = { 2 7 2 4.0 };
title("Exact and kernel densities");
ylabel("density");
xlabel("x");
xtics(.9,1.0,.01,0);
ytics(0,40,5,0);
makewind(7.8, 7.8, 0, 0, 0);
_pltype=6;
_plegctl={2 4 1.5 5.4};
_plegstr="exact pdf at 1000 grid points";
xy(xx,fx);
_plegctl={2 4 1.5 5.0};
_plegstr="kernel density 6000 draws";
_pltype=2;
xy(x0,den0);
endwind;
/* ==================================================*/
/* kernel density estimation; Tsurumi's original code*/
/* but modified by Freddy Rojas */
/* ==================================================*/
proc(2)=kden(v,h);
local g,j,nn,res;
nn=rows(v);
" ";
" ";
@print "h ";@
@h;@
g=0;
j=1;
do while j <= nn;
g=g|meanc(pdfn((v[j]-v)./h))./h;
j=j+1;
endo;
res=sortc(v~g[2:nn+1],1);
retp(res[.,1],res[.,2]);
endp;
/* ==============================================*/
/* Epachenikov kernel density estimation; */
/* ==============================================*/

proc(2)= kernele(z,h);
```

```
local a,res,t,g,z_v,zv,zv_,j;
j=1;
g=0;
do while j <= rows(z);
 zv=(z[j]-z)./h;

 t=(abs(zv).<sqrt(5));
 a=code(t,sqrt(5)|1);
 zv_=t.*((3/4)*(1-(1/5).*(zv.^2))./a);
 g=g|meanc(zv_)./h;
j=j+1;
endo;
 res=sortc(z~g[2:rows(z)+1],1);
 retp(res[.,1],res[.,2]);
endp;
```

1.2 Gauss Program, Question 2.(i), (ii) and (iii)

```
/*=================================================================================
   Asignment 03 (code sent by Hiraki Tsurumi)
   Modified by Freddy Rojas Cama
   Last update October 1st, 2010
   ==================================================================================*/
   /****************************************/
   // Question N 2 (i),(ii) and (iii)
   /****************************************/
   //Preliminaries and housecleaning
   new;
   cls;
   library pgraph;
   //#include density.src;
   pqgwin auto;
   graphset;
   alpha=148;
   beta=4;
   /*===computation of mean, mode, variance, skewness, and kurtosis
   by Simpson's rule =====*/
   upa=1; lowa=.5;
   nn=6000; nn1=nn+1;
   ha=(upa-lowa)/nn;
   pea=seqa(lowa,ha,nn1);
   wp1={1 4}; wp2={2 4}; wp4=1;
   nn2=nn/2-1;
   wp3=ones(1,nn2).*.wp2;
   wp=wp1~wp3~wp4;
```

```
wp=wp*(ha/3);
sur=pea^(alpha-1).*(1-pea)^(beta-1);
/*--marginal posterior pdf, mean and sd----*/
vol=wp*sur;
//print "volume" vol;
mar=sur/vol;
mu=wp*(pea.*mar);
var=wp*(((pea-mu)^2).*mar);
sd=sqrt(var);
skew=wp*(((pea-mu)^3).*mar);
skew=skew/sd^3;
kurto=wp*(((pea-mu)^4).*mar);
kurto=kurto/sd^4;
mode=pea[maxindc(mar)];
par=mu~mode~var~skew~kurto;
" ";
" ";; " Statistics from draws whose come from Simpon's rule calculations ";
print " mean mode var skew and kurtosis ";
par;
//Draws from Beta Function
n=6000;
x1=rndbeta(n,1,alpha,beta);
h0=1.06*stdc(x1)/(n^.2);
{x_,den1}=kden(x1,h0);
/*====sample mean, variance, skewness, and kurtosis ====*/
smean=meanc(x1);
sd=stdc(x1);
svar=sd^2;
skw=meanc((x1-smean)^3)/sd^3;
skurt=meanc((x1-smean)^4)/sd^4;
smode=x1[maxindc(den1)];
" ";
" ";
print;
" ";; " Statistics from draws whose come from Beta Formula ";
print "sample mean, mode, variance, skewness, and kurtosis ";
print smean~smode~svar~skw~skurt;
/* ==================================================*/
/* kernel density estimation; Tsurumi's original code*/
/* but modified by Freddy Rojas */
/* ==================================================*/
proc(2)=kden(v,h);
local g,j,nn,res;
nn=rows(v);
" ";
```

```
" ";
@print "h ";@
@h;@
g=0;
j=1;
do while j <= nn;
g=g|meanc(pdfn((v[j]-v)./h))./h;
j=j+1;
endo;
res=sortc(v~g[2:nn+1],1);
retp(res[.,1],res[.,2]);
endp;
```