Advanced Economics Statistics

FALL 2010

Third-Assignment Answer Sheet
by Freddy Rojas Cama

1. This question calls for replicating the exercise in the kernel-density estimation handout

   (i). In the following graph we show the approximation of the `beta` probability function made by a kernel density[1]. We reproduce Figure 2[2] for $\alpha = 4$, $\beta = 1$. We set $n = 3,000$ rather than $n = 1,000$. The exact distribution `beta` is gotten from: $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha+1}(1-x)^{\beta-1}$ where Γ denotes the `gamma` distribution function. Then, we do the approximation using a kernel function $f_h(x) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x-x_i}{h}\right)$.

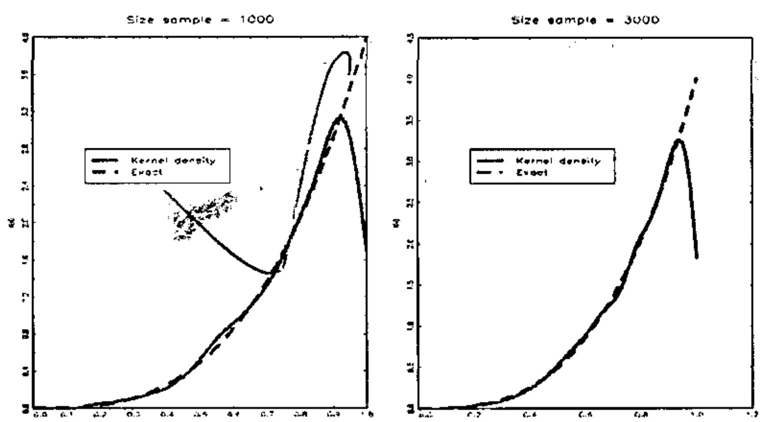

Figure 1: Kernel density estimates of `Beta(4,1)`

   In any case the kernel density (based on Gaussian probability function) is not accurate in the neighborhood close to 1. In general, kernel based on a normal probability function gives small weights to extreme values, because of this, the proposed kernel *is a bad approximation for the* `beta` *distribution. Particularly, at the left-side of* the distribution the kernel does the approximation very well because the weights are small in that extreme value.

   (ii). For $\alpha = \beta = 4$ we change the bandwith to h= 0.01 and h= 0.15. For h= 0.01 the difference between the results based on these 2 bandwiths exhibits the trade-off between accuracy and variance (see below figure 2). For example, the kernel density with h= 0.01 shows that this approximation can replicate very well the variance of the exact distribution but it is poorly doing the approximation between the points across the support. In contrast, a kernel with h= 0.15 shows a smoothing function with a poor accuracy in variance. Figure 2 also shows kernel densities with `h=1.06*stdc(x)/(n^.2)`

[1] The bandwith was fixed to `h=1.06*stdc(x)/(n^.2)`
[2] See *kernel-density estimation handout*

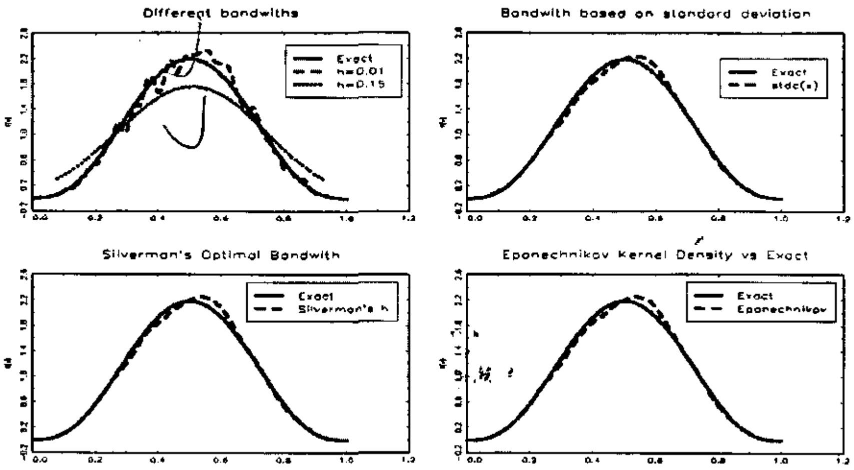

Figure 2: Kernel Density for Beta(4,4)

2. This question calls to get the distribution of mean sample of different draws -whose come from different distributions- by using the characteristic function

   (i). Item i calls to get the distribution of $\overline{X} = \frac{1}{n}\sum_i^n X_i$ using the characteristic function. X_i is NID$(\mu, 2)$, and 'NID' denotes "*normally and independently distributed*" and $\overline{X}$ is the sample mean. First at all, we need to get the characteristic function by using the probability distribution function, each step is shown in the appendix. In the following lines we proceed in order to get the inverse result.

$$P(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(-itx)\,\varphi_x(t)\,dt \tag{1}$$

There is an easier way than this

$$\begin{aligned} P(x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(-itx)\exp\left(-\frac{\sigma^2 t^2}{2} + uit\right) dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left(-\frac{\sigma^2 t^2}{2} + uit - itx\right) dt \end{aligned}$$

thus, we have that,

$$P(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left(-\frac{\sigma^2 t^2}{2} + it(u - x)\right) dt \tag{2}$$

By adding and substracting $\left(\frac{x-u}{\text{sqrt}(2)\sigma}\right)^2$

$$\begin{aligned}
P(x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\left(-\frac{\sigma^2t^2}{2}-it\left(x-u\right)+\frac{\left(x-u\right)^2}{2\sigma^2}-\frac{\left(x-u\right)^2}{2\sigma^2}\right)dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\left(-\left(\frac{\sigma t}{\text{sqrt}\left(2\right)}+\frac{i\left(x-u\right)}{\text{sqrt}\left(2\right)\sigma}\right)^2-\frac{\left(x-u\right)^2}{2\sigma^2}\right)dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\left(-\left(\frac{\sigma^2 t}{\text{sqrt}\left(2\right)\sigma}+\frac{i\left(x-u\right)}{\text{sqrt}\left(2\right)\sigma}\right)^2-\frac{\left(x-u\right)^2}{2\sigma^2}\right)dt \\
&= \frac{1}{\text{sqrt}\left(2\pi\right)}\exp\left(-\frac{\left(x-u\right)^2}{2\sigma^2}\right)\int_{-\infty}^{+\infty}\frac{1}{\text{sqrt}\left(2\pi\right)}\exp\left(-\left(\frac{\sigma^2 t}{\text{sqrt}\left(2\right)\sigma}-\frac{i\left(u-x\right)}{\text{sqrt}\left(2\right)\sigma}\right)^2\right)dt \\
&= \frac{1}{\text{sqrt}\left(2\pi\right)}\exp\left(-\frac{\left(x-u\right)^2}{2\sigma^2}\right)\int_{-\infty}^{+\infty}\frac{1}{\text{sqrt}\left(2\pi\right)}\exp\left(-\frac{\sigma^4\left(t-\sigma^{-2}i\left(u-x\right)\right)^2}{2\sigma^2}\right)dt \\
&= \frac{1}{\text{sqrt}\left(2\pi\right)\sigma}\exp\left(-\frac{\left(x-u\right)^2}{2\sigma^2}\right)\int_{-\infty}^{+\infty}\frac{1}{\text{sqrt}\left(2\pi\right)}\sigma\exp\left(-\frac{\left(t-\sigma^{-2}i\left(u-x\right)\right)^2}{\frac{2}{\sigma^2}}\right)dt
\end{aligned}$$

thus, we have

$$P(x)=\frac{1}{\text{sqrt}\left(2\pi\right)\sigma}\exp\left(-\frac{\left(x-u\right)^2}{2\sigma^2}\right) \tag{3}$$

We used the fact that $\int_{-\infty}^{+\infty}\frac{1}{\text{sqrt}(2\pi)}\sigma\exp\left(-\frac{\left(t-\sigma^{-2}i(u-x)\right)^2}{\frac{2}{\sigma^2}}\right)dt=1$. Where $t\sim N$ with mean $\sigma^{-2}i\left(u-x\right)$ and variance σ^{-2}. Now, we should derive the distribution for the mean sample by using the above steps. We have that the characteristic function[3] for the mean of draws *-from any distribution-* can be expressed as:

$$\varphi_{\bar{x}}\left(t\right)=\left(\varphi_x\left(\frac{t}{n}\right)\right)^n \tag{4}$$

thus, in order to get the distribution of the mean sample we consider to replace

[3] see `mathworld`.

$\tilde{t} = \frac{t}{n}$ instead of t in (1)

$$\begin{aligned} P(\bar{x}) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(-i\bar{x}t)\, \varphi_{\bar{x}}(\tilde{t})\, dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(-i\bar{x}t) \exp\left[n \left(-\frac{\left(\frac{\sigma}{n}\right)^2 t^2}{2} + ui\frac{t}{n} \right) \right] dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(-i\bar{x}t) \exp\left(-\frac{\left(\frac{\sigma^2}{n}\right) t^2}{2} + uit \right) dt \end{aligned}$$

Now, if we see the similarities between (2) and above expression and considering result (3). we have that only the variance has changed, this is $\sigma^2_{\bar{X}} = \frac{\sigma^2}{n}$

$$P(x) = \frac{1}{\operatorname{sqrt}(2\pi)} \frac{\sqrt{n}}{\sigma} \exp\left(-\frac{n(x-u)^2}{\sigma^2} \right)$$

Here the distribution converges more rapidly than before because its variance now is defined by $\frac{\sigma^2}{n}$. In the case when $n \to \infty$ the probability function has a variance (of the mean sample) with limit zero $\left(\frac{\sigma^2}{n} \to 0\right)$.

(ii). Using the characteristic function of a Cauchy(0,1) random variable we obtain the distribution of the mean sample. First we get the probability function from the characteritic function by using a Cauchy draw

$$\varphi_x(t) = \exp(x_0 it - \gamma |t|)$$

And

$$\begin{aligned} P(x) &= \frac{1}{\gamma\pi} \left(1 + \left(\frac{x - x_0}{\gamma} \right)^2 \right) \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(-ixt)\, Q_x(t)\, dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(-ixt) \exp(x_0 it - \gamma |t|)\, dt \end{aligned}$$

here giving some shape in order to get the exponential function;

$$\begin{aligned} &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(x_0 it - \gamma |t| - ixt)\, dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(x_0 it - \gamma |t| - ixt)\, dt \end{aligned}$$

Cauchy(0,1) means $x_0 = 0$ $\gamma = 1$. This would have made things much easier!

We are going to resolve the integral in two parts, $A = \int\limits_{0}^{+\infty} \exp\left(x_0 it - \gamma\left|t\right| - ixt\right)dt$ and $B = \int\limits_{-\infty}^{0} \exp\left(x_0 it - \gamma\left|t\right| - ixt\right)dt$ as follows:

$$\begin{aligned} A &= \frac{1}{2\pi}\int\limits_{0}^{+\infty} \exp\left(x_0 it - \gamma\left|t\right| - ixt\right)dt \\ &= \frac{1}{2\pi}\frac{i\left(x-x_0\right)+\gamma}{i\left(x-x_0\right)+\gamma}\int\limits_{0}^{+\infty} \exp\left(-t\right)\left(-x_0 i + \gamma + ix\right)dt \\ &= \frac{1}{2\pi}\frac{1}{i\left(x-x_0\right)+\gamma} \end{aligned}$$

Since that the exponential function is given by $\beta\int\limits_{0}^{+\infty} \exp\left(-\left(t\beta\right)dt\right) = 1$.

$$\begin{aligned} B &= \frac{1}{2\pi}\int\limits_{-\infty}^{0} \exp\left(x_0 it - \gamma\left|t\right| - ixt\right)dt \\ &= \frac{1}{2\pi}\int\limits_{-\infty}^{0} \exp\left(t\right)\left(x_0 i + \gamma - ix\right)dt \\ &= \frac{1}{2\pi}\frac{1}{i\left(x_0-x\right)+\gamma} = -\frac{1}{2\pi}\frac{1}{i\left(x-x_0\right)-\gamma} \end{aligned}$$

Now we take $A+B$ into one simple expression:

$$\begin{aligned} A+B &= \frac{1}{2\pi}\frac{1}{i\left(x-x_0\right)+\gamma} - \frac{1}{2\pi}\frac{1}{i\left(x-x_0\right)-\gamma} \\ &= \frac{1}{2\pi}\left(\frac{1}{i\left(x-x_0\right)+\gamma} - \frac{1}{i\left(x-x_0\right)-\gamma}\right) \\ &= \frac{1}{2\pi}\left(\frac{i\left(x-x_0\right)-\gamma-\left(i\left(x-x_0\right)+\gamma\right)}{-\left(x-x_0\right)^2-\gamma^2}\right) \\ &= \frac{1}{\pi}\left(\frac{-\gamma}{-\left(x-x_0\right)^2-\gamma^2}\right) \end{aligned}$$

So, re-arranging terms:

$$P(x) = \frac{1}{\pi\gamma}\left(\frac{1}{1+\left(\frac{x-x_0}{\gamma}\right)^2}\right)$$

Now, we should derive the distribution of the mean sample. In the case of Cauchy Distribution and considering expression (4):

$$\begin{aligned} P(\bar{x}) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(-i\bar{x}t)\,\varphi_{\bar{x}}(t)\,dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(-i\bar{x}t)\exp\left[n\left(x_0 i\frac{t}{n} - \gamma\left|\frac{t}{n}\right|\right)\right]dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(-i\bar{x}t)\exp(x_0 it - \gamma|t|)\,dt = P(x) \end{aligned}$$

So, we got the same distribution with no change in mean or variance.

3. We derive the right-hand of the following expression

$$\varphi(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}\exp\left(nu + n^2\frac{\sigma^2}{2}\right) \tag{5}$$

applying a Taylor's expansion in order to get the expression in linear way:

$$\varphi(t) = \int_{-\infty}^{+\infty} \exp(itx)\,P(x)\,dx$$

we have that

$$\varphi(t) \simeq \phi(t_0) + \phi'(t_0)\big|\,(t-t_0) + \phi''(t_0)\big|\,(t-t_0)^2 + + \phi'''(t_0)\big|\,(t-t_0)^3 \ldots \tag{6}$$

where

$$\begin{aligned} \phi(t_0) &= \int_{-\infty}^{+\infty} P(x)\,dx = 1 \\ \phi'(t_0) &= i\int_{-\infty}^{+\infty} x\exp(itx)\,P(x)\,dx \\ \phi''(t_0) &= i^2\int_{-\infty}^{+\infty} x^2\exp(itx)\,P(x)\,dx \\ &\vdots \end{aligned}$$

Then, taking into account the expression (6) and considering that $t_0 = 0$,we have

$$
\begin{aligned}
\varphi(t) &\simeq \int_{-\infty}^{+\infty} P(x) + it \int_{-\infty}^{+\infty} x \exp(itx) P(x)\, dx ... \\
&+ (it)^2 \int_{-\infty}^{+\infty} x^2 \exp(itx) P(x)\, dx + ... \\
\varphi(t) &\simeq 1 + it \cdot E(x) + \frac{1}{2}(it)^2 \cdot E(x^2) + \\
&\frac{1}{6}(it)^3 \cdot E(x^3) + ...
\end{aligned}
$$

In order to get the expression (5), we need to replace above expectation in terms of parameters of the probability function, this is mean and variance. First, for sake of simplicity we do a change of variable; $y = \ln(x)$. Thus, we have the following:

$$
\begin{aligned}
E(x^s) &= \int_{-\infty}^{+\infty} \exp(sy) \exp\left(-\frac{(y-u)^2}{2\sigma^2}\right) dy \\
&= \int_{-\infty}^{+\infty} \exp\left(sy - \frac{(y-u)^2}{2\sigma^2}\right) dy \\
&= \int_{-\infty}^{+\infty} \exp\left(\frac{2\sigma^2 sy}{2\sigma^2} - \frac{(y-u)^2}{2\sigma^2}\right) dy \\
&= \int_{-\infty}^{+\infty} \exp\left(\frac{2\sigma^2 sy - (y-u)^2}{2\sigma^2}\right) dy \\
&= \int_{-\infty}^{+\infty} \exp\left(\frac{2\sigma^2 sy - (y^2 + u^2 - 2yu)}{2\sigma^2}\right) dy \\
&= \int_{-\infty}^{+\infty} \exp\left(\frac{2\sigma^2 sy + 2yu - y^2 - u^2}{2\sigma^2}\right) dy \\
&= \int_{-\infty}^{+\infty} \exp\left(\frac{2y(\sigma^2 s + u) - y^2 - u^2}{2\sigma^2}\right) dy \\
&= \int_{-\infty}^{+\infty} \exp\left(\frac{y(\sigma^2 s + u)}{\sigma^2}\right) \exp\left(\frac{-y^2}{2\sigma^2}\right) \exp\left(\frac{-u^2}{2\sigma^2}\right) dy
\end{aligned}
$$

Where $\exp(sy) = (\exp(y))^s$. By adding exp of $-\left(\frac{\sigma^2 s+u}{\text{sqrt}(2\sigma)}\right)^2$ and $\left(\frac{\sigma^2 s+u}{\text{sqrt}(2\sigma)}\right)^2$ we have the following:

$$
\begin{aligned}
E\left(x^{s}\right) &= \int_{-\infty}^{+\infty} \exp\left(\frac{y\left(\sigma^{2}s+u\right)}{\sigma^{2}} - \frac{y^{2}}{2\sigma^{2}} - \left(\frac{\sigma^{2}s+u}{2\sigma^{2}}\right)^{2} + \left(\frac{\sigma^{2}s+u}{2\sigma^{2}}\right)^{2} - \frac{u^{2}}{2\sigma^{2}}\right) dy \\
&= \int_{-\infty}^{+\infty} \exp\left(\frac{y\left(\sigma^{2}s+u\right)}{\sigma^{2}} - \frac{y^{2}}{2\sigma^{2}} - \left(\frac{\sigma^{2}s+u}{\text{sqrt}\left(2\sigma\right)}\right)^{2}\right) \exp\left(\left(\frac{\sigma^{2}s+u}{\text{sqrt}\left(2\sigma\right)}\right)^{2} - \frac{u^{2}}{2\sigma^{2}}\right) dy \\
&= \int_{-\infty}^{+\infty} \exp -\left(\frac{y-\left(\sigma^{2}s+u\right)}{\text{sqrt}\left(2\sigma\right)}\right)^{2} \exp\left(\left(\frac{\sigma^{2}s+u}{\text{sqrt}\left(2\sigma\right)}\right)^{2} - \frac{u^{2}}{2\sigma^{2}}\right) dy \\
&= \exp\left(\left(\frac{\sigma^{2}s+u}{\text{sqrt}\left(2\sigma\right)}\right)^{2} - \frac{u^{2}}{2\sigma^{2}}\right) \int_{-\infty}^{+\infty} \exp -\left(\frac{y-\left(\sigma^{2}s+u\right)}{\text{sqrt}\left(2\sigma\right)}\right)^{2} dy \\
&= \exp\left(\left(\frac{\sigma^{2}s+u}{2\sigma^{2}}\right)^{2} - \frac{u^{2}}{2\sigma^{2}}\right) \\
&= \exp\left(\frac{\sigma^{4}s^{2}+u^{2}+2us\sigma^{2}}{2\sigma^{2}} - \frac{u^{2}}{2\sigma^{2}}\right) \\
&= \exp\left(\frac{\sigma^{4}s^{2}+2\sigma^{2}su}{2\sigma^{2}}\right) \\
&= \exp\left(\frac{\sigma^{2}s^{2}}{2}+su\right)
\end{aligned}
$$

Leaving it out makes this untrue.

Here, we used the fact that $\int_{-\infty}^{+\infty} \exp\left(-\left(\frac{y-\left(\sigma^{2}s+u\right)}{\text{sqrt}(2\sigma)}\right)^{2}\right) dy = 1$.[4]

1. Finally, the last problem calls to verify the following expression:

$$\int_{k}^{+\infty} \frac{1}{x}\exp\left(tx\right)\exp\left(-\frac{\left(\ln x\right)^{2}}{2}\right) dx \geq \int_{k}^{+\infty} \frac{1}{x} dx \qquad (7)$$

The following is the same:

$$\int_{k}^{z} \frac{1}{x}\exp\left(tx\right)\exp\left(-\frac{\left(\ln x\right)^{2}}{2}\right) dx \geq \int_{k}^{z} \frac{1}{x} dx$$

you can't assume what you're proving!

then when $z \to \infty$ we have the expression ??. So, if ?? is true, that will mean that $\Delta z = \varepsilon$ we will have that right side of expression ?? increases faster than the left side, so

[4] We suppresed the term $\frac{1}{\text{sqrt}(2\pi)\sigma}$ in the calculations for sake of simplicity. But you should consider that $\int_{-\infty}^{+\infty} \frac{1}{\text{sqrt}(2\pi)\sigma}\exp\left(-\left(\frac{y-\left(\sigma^{2}s+u\right)}{\text{sqrt}(2\sigma)}\right)^{2}\right) dy = 1$

Then why don't you call $\frac{1}{\sqrt{2\pi}\,\sigma} = K$ or something?

the derivatives should keep the same relation, so:

$$\frac{1}{x}\exp(tx)\exp\left(-\frac{(\ln x)^2}{2}\right) \geq \frac{1}{x}$$

$$\exp(tx)\exp\left(-\frac{(\ln x)^2}{2}\right) \geq 1$$

By taking natural log, and given that $x > 0$, we have:

$$tx - \frac{(\ln x)^2}{2} \geq 0$$

Then we divided the latter expression by tx.

$$\frac{tx - \frac{(\ln x)^2}{2}}{tx} \geq 0$$

In this case we need to know how the above expression behaves when $x \to \infty$. So, we have the following:

$$\lim_{x\to\infty} \frac{tx - \frac{(\ln x)^2}{2}}{tx} = \lim_{x\to\infty} 1 - \frac{(\ln x)^2}{2tx} = \lim_{x\to\infty} 1 - \frac{(\ln x)^2 x^{-1}}{2t}$$

Applying L'Hopital rule:

$$\lim_{x\to\infty} \frac{tx - \frac{(\ln x)^2}{2}}{tx} = \lim_{x\to\infty} 1 - \frac{2\frac{1}{x}}{2t}$$

$$= \lim_{x\to\infty} 1 - \frac{\frac{1}{x}}{t}$$

$$= \lim_{x\to\infty} 1 - \frac{1}{tx} \geq 0$$

So, in the next $x \to \infty$ the inequality holds. Thus when $k > 0$ and $z \to \infty$ we have that

$$\int_k^{z\to\infty} \frac{1}{x}\exp(tx)\exp\left(-\frac{(\ln x)^2}{2}\right)dx \geq c\int_k^{z\to\infty}\frac{1}{x}dx$$

for *some* constant c. But in the case when $t \leq 0$, there exists moments:

$$\int_k^z \frac{1}{x}\exp(-|t|x)\exp\left(-\frac{(\ln x)^2}{2}\right)dx < \infty$$

In the case of $t = 0$ we have

$$\int_k^\infty \frac{1}{x}\exp\left(-\frac{(\ln x)^2}{2}\right)dx = \text{sqrt}(2\pi)[1 - F(k)]$$

$$\int_k^\infty \frac{1}{x}\exp\left(-\frac{(\ln x)^2}{2}\right)dx = \text{sqrt}(2\pi)[1 - F(k)] < \infty$$

why is this true?

So start Here

I'm unclear as to what order you are doing things asymptotically? $t<0 \Rightarrow e^{tx} < 1$ for $x>0$. Then use your previous argument!

In the case of $t < 0$. So, by doing integration by parts:

$$uv|_k^\infty = \int_k^\infty udv + \int_k^\infty vdu \tag{7}$$

u and v are defined as follows

$$\begin{aligned} du &= \frac{1}{x}\exp\left(-\frac{(\ln x)^2}{2}\right)dx \\ v &= \exp(-|t|\,x) \end{aligned}$$

So, $u = \text{sqrt}(2\pi)\int_k^\infty \frac{1}{x\,\text{sqrt}(2\pi)}\exp\left(-\frac{(\ln x)^2}{2}\right)dx \doteq \text{sqrt}(2\pi)(1-F(k))$ when $z \to \infty$. Where $F()$ is the log-normal cumulative probability function. In the case of $dv = -|t|\cdot\exp(-|t|\,x)\,d(x)$. So, replacing terms in expression (7).

$$\begin{aligned} \text{sqrt}(2\pi)(1-F(k))\exp(-|t|\,x)|_k^\infty &= -\int_k^\infty |t|\,\text{sqrt}(2\pi)(1-F(k))\exp(-|t|\,x)\,dx \\ &\quad + \int_k^\infty \frac{\exp(-|t|\,x)}{x}\exp\left(-\frac{(\ln x)^2}{2}\right)dx \end{aligned}$$

In the below calculation we make reference to $\int_k^\infty \frac{\exp(-|t|\,x)}{x}\exp\left(-\frac{(\ln x)^2}{2}\right)dx$ as A.

Thus, we have that:

$$\begin{aligned} A &= \text{sqrt}(2\pi)(1-F(k))\exp(-|t|\,x)|_k^\infty + \int_k^\infty |t|\,\text{sqrt}(2\pi)(1-F(k))\exp(-|t|\,x)\,dx \\ &= \text{sqrt}(2\pi)(1-F(k))\exp(-|t|\,x)|_k^\infty + \frac{|t|}{|t|}\text{sqrt}(2\pi)(1-F(k))\int_k^\infty |t|\exp(-|t|\,x)\,dx \\ A &= \text{sqrt}(2\pi)(1-F(k))\exp(-|t|\,x)|_k^\infty + \text{sqrt}(2\pi)(1-F(k))(1-\Lambda(k)) \end{aligned}$$

where $F(k)$ and $\Lambda(k)$ is the normal and exponential cumulative function respectively. Then, evaluating the above expression:

$$A = -\text{sqrt}(2\pi)(1-F(k))(\exp(-|t|\,k)) + \text{sqrt}(2\pi)(1-F(k))(1-\Lambda(k)) < \infty$$

by re-arranging terms

$$A = \text{sqrt}(2\pi)(1-F(k))\,[1-\Lambda(k) - \exp(-|t|\,k)] < \infty$$

We note that $-\exp(-|t|\,k) = \Lambda(k) - 1$.Thus, we should note that

$$A \sim 0 < \infty$$

Because of the approximation in (7) and nonlinear terms in the integral this terms should be close to zero.

References

[1] Casella, G and R. Berger. 2002. Statistical Inference. Second Edition, Duxbury Advanced Studies.

[2] Mendenhall and Scheaffer. 1973. Mathematical Statistics with Applications. Duxbury Press. North Scituate, Massachusetts.

[3] Mathworld website. http://mathworld.wolfram.com/

[4] GAUSS kernel density library. GAUSS.

[5] M.P Wand & M.C Jones. 1995. Kernel Smoothing. Monographs on Statistics and Applied Probability. Chapman & Hall, 1995.

1 Appendix

1.1 Characteristic Function

We proceed to get the characterictic function from the probability distribution function:

$$
\begin{aligned}
\varphi_x(t) &= E(\exp(itx)) = \int_{-\infty}^{+\infty} \exp(itx)\, p(x)\, dx \\
&= \frac{1}{\text{sqrt}(2\pi)\,\sigma^2} \int_{-\infty}^{+\infty} \exp(itx) \exp\left(\frac{-(x-u)^2}{2\sigma^2}\right) dx \\
&= \frac{1}{\text{sqrt}(2\pi)\,\sigma^2} \int_{-\infty}^{+\infty} \exp\left(\frac{2\sigma^2 itx - (x-u)^2}{2\sigma^2}\right) dx \\
&= \frac{1}{\text{sqrt}(2\pi)\,\sigma^2} \int_{-\infty}^{+\infty} \exp\left(\frac{2\sigma^2 itx - (x^2 - 2xu + u^2)}{2\sigma^2}\right) dx \\
&= \frac{1}{\text{sqrt}(2\pi)\,\sigma^2} \int_{-\infty}^{+\infty} \exp\left(\frac{2\sigma^2 itx - x^2 + 2xu - u^2}{2\sigma^2}\right) dx \\
&= \frac{1}{\text{sqrt}(2\pi)\,\sigma^2} \int_{-\infty}^{+\infty} \exp\left(\frac{-x^2}{2\sigma^2} + \frac{-u^2}{2\sigma^2} + \frac{xu + \sigma^2 itx}{\sigma^2}\right) dx \\
&= \frac{1}{\text{sqrt}(2\pi)\,\sigma^2} \int_{-\infty}^{+\infty} \exp\left(\frac{-x^2}{2\sigma^2} + \frac{-u^2}{2\sigma^2} + \frac{(u + \sigma^2 it)\, x}{\sigma^2}\right) dx
\end{aligned}
$$

By adding and substracting $\frac{\left(u+\sigma^2 it\right)^2}{2\sigma^2}$ we have the following expression:

$$
\begin{aligned}
\int_{-\infty}^{+\infty} \exp(itx)\, p(x)\, dx &= \frac{1}{\text{sqrt}(2\pi)\sigma^2} \int_{-\infty}^{+\infty} \exp\left(\frac{-x^2}{2\sigma^2} + \frac{-u^2}{2\sigma^2} + \frac{\left(u+\sigma^2 it\right)x}{\sigma^2} + \frac{\left(u+\sigma^2 it\right)^2}{2\sigma^2} - \frac{\left(u+\sigma^2 it\right)^2}{2\sigma^2} \right) dx \\
&= \frac{1}{\text{sqrt}(2\pi)\sigma^2} \int_{-\infty}^{+\infty} \exp\left(\frac{-x^2}{2\sigma^2} + \frac{\left(u+\sigma^2 it\right)x}{\sigma^2} - \frac{\left(u+\sigma^2 it\right)^2}{2\sigma^2} - \frac{u^2}{2\sigma^2} + \frac{\left(u+\sigma^2 it\right)^2}{2\sigma^2} \right) dx \\
&= \frac{1}{\text{sqrt}(2\pi)\sigma^2} \int_{-\infty}^{+\infty} \exp\left(\begin{array}{c} -\left(\frac{x^2}{2\sigma^2} - \frac{2\left(u+\sigma^2 it\right)x}{2\sigma^2} + \frac{\left(u+\sigma^2 it\right)^2}{2\sigma^2} \right) \\ -\frac{u^2}{2\sigma^2} + \frac{\left(u+\sigma^2 it\right)^2}{2\sigma^2} \end{array} \right) dx \\
&= \frac{1}{\text{sqrt}(2\pi)\sigma^2} \int_{-\infty}^{+\infty} \exp\left(-\left(\frac{\left(x-\left(u+\sigma^2 it\right)\right)^2}{2\sigma^2} \right) - \frac{u^2}{2\sigma^2} + \frac{\left(u+\sigma^2 it\right)^2}{2\sigma^2} \right) dx \\
&= \frac{1}{\text{sqrt}(2\pi)\sigma^2} \int_{-\infty}^{+\infty} \exp\left(-\left(\frac{\left(x-\left(u+\sigma^2 it\right)\right)}{\text{sqrt}(2)\sigma} \right)^2 - \frac{u^2}{2\sigma^2} + \frac{\left(u+\sigma^2 it\right)^2}{2\sigma^2} \right) dx \\
&= \frac{1}{\text{sqrt}(2\pi)\sigma^2} \int_{-\infty}^{+\infty} \exp\left(-\left(\frac{\left(x-\left(u+\sigma^2 it\right)\right)}{\text{sqrt}(2)\sigma} \right)^2 - \frac{u^2}{2\sigma^2} + \frac{\left(u+\sigma^2 it\right)^2}{2\sigma^2} \right) dx
\end{aligned}
$$

By denoting $\gamma = \left(u+\sigma^2 it\right)$, and by knowing that $\left(\text{sqrt}(i)\right)^2$ i^2 $= -1$ we have that:

$$
\begin{aligned}
\int_{-\infty}^{+\infty} \exp(itx)\, p(x)\, dx &= \frac{1}{\text{sqrt}(2\pi)\sigma^2} \int_{-\infty}^{+\infty} \exp\left(-\left(\frac{x-\gamma}{\text{sqrt}(2)\sigma} \right)^2 - \frac{u^2}{2\sigma^2} + \frac{\gamma^2}{2\sigma^2} \right) dx \\
&= \frac{1}{\text{sqrt}(2\pi)\sigma^2} \exp\left(-\frac{u^2}{2\sigma^2} + \frac{\gamma^2}{2\sigma^2} \right) \int_{-\infty}^{+\infty} \exp\left(-\left(\frac{x-\gamma}{\text{sqrt}(2)\sigma} \right)^2 \right) dx \\
&= \frac{1}{\text{sqrt}(2\pi)\sigma^2} \exp\left(-\frac{u^2}{2\sigma^2} + \frac{\left(u+\sigma^2 it\right)^2}{2\sigma^2} \right) \int_{-\infty}^{+\infty} \exp\left(-\left(\frac{x-\gamma}{\text{sqrt}(2)\sigma} \right)^2 \right) dx \\
&= \exp\left(\frac{\sigma^2 (it)^2 + 2uit}{2} \right) \frac{1}{\text{sqrt}(2\pi)\sigma^2} \int_{-\infty}^{+\infty} \exp\left(-\left(\frac{x-\gamma}{\text{sqrt}(2)\sigma} \right)^2 \right) dx \\
&= \exp\left(-\frac{\sigma^2 t^2}{2} + uit \right)
\end{aligned}
$$

And we used the fact that the cumulative normal distribution is equal to 1:

$$
\frac{1}{\text{sqrt}(2\pi)\sigma^2} \int_{-\infty}^{+\infty} \exp\left(-\left(\frac{x-\gamma}{\text{sqrt}(2)\sigma} \right)^2 \right) dx = 1
$$

2 Gauss Program

2.1 Question 1.(i)

```
/*=================================================================================
   Asignment 03 (code sent by Hiraki Tsurumi)
   Modified by Freddy Rojas
   Last update September 27th
   ================================================================================*/
   /****************************************/
   // Question N 1 (i)
   /****************************************/
   new;
   cls;
   library pgraph;
   pqgwin auto;
   graphset;
   #include kernel.src;
   #include density.src;
   n=3000;
   alpha=4;
   beta=1;
   x=rndbeta(n,1,alpha,beta);
   h=1.06*stdc(x)/(n^.2);
   {x1,den1}=kden(x,h);
   up=1.0;
   low=0;
   step=(up-low)/(n-1);
   xx1=seqa(low,step,n);
   fx1=gamma(alpha+beta)/(gamma(alpha)*gamma(beta))*xx1^(alpha-1).*(1-xx1)^(beta-1);
   n2=1000;
   x2=rndbeta(n2,1,alpha,beta);
   h2=1.06*stdc(x2)/(n2^.2);
   {x2,den2}=kden(x2,h2);
   step=(up-low)/(n2-1);
   xx2=seqa(low,step,n2);
   fx2=gamma(alpha+beta)/(gamma(alpha)*gamma(beta))*xx2^(alpha-1).*(1-xx2)^(beta-1);
    _pcolor = { 9 5 }; /* Colors for series */
    _pmcolor = { 1, 8, 2, 8, 8, 8, 8, 8, 15 };
    /*Colors for axes, title, x and y labels, date, box, and background */
    _plwidth={12 12 }; /*Controls line thickness for main curves*/
    _paxht=0.10; /*Controls size of axes labels*/
    _ptitlht = 0.22; /*Controls main title size */
    _plegctl = { 2 7 2 4.0 };
    _plegstr = " Kernel density \000 Exact ";
   @ ytics(0,4,.5,0);@
```

```
 zv_=t.*((3/4)*(1-(1/5).*(zv.^2))./a);
 g=g|meanc(zv_)./h;
j=j+1;
endo;
 res=sortc(z~g[2:rows(z)+1],1);
 retp(res[.,1],res[.,2]);
endp;
```

2.2 Question 1.ii

```
/*=====================================================================================================
   Asignment 03 (code sent by Hiraki Tsurumi)
   Modified by Freddy Rojas
   Last update September 27th
   ===================================================================================================*/
   /****************************************/
   // Question N 1 (ii)
   /****************************************/
   new;
   library pgraph;
   graphset;
   pqgwin auto;
   #include kernel.src;
   #include density.src;
   n=3000;
   alpha=4;
   beta=4;
   x=rndbeta(n,1,alpha,beta);
   h0=1.06*stdc(x)/(n^.2);
   h1=bandw1(x);
   h2=0.01;
   h3=0.15;
   {x0,den0}=kden(x,h0);
   {x1,den1}=kden(x,h1);
   {x2,den2}=kden(x,h2);
   {x3,den3}=kden(x,h3);
   {x4,den4}=kernele(x,h1);
   up=1.0;
   low=0;
   step=(up-low)/(n-1);
   xx=seqa(low,step,n);
   fx=gamma(alpha+beta)/(gamma(alpha)*gamma(beta))*xx^(alpha-1).*(1-xx)^(beta-1);
   graphset;
    _pcolor = { 9 5 }; /* Colors for series */
    _pmcolor = { 1, 8, 2, 8, 8, 8, 8, 8, 15 };
    /*Colors for axes, title, x and y labels, date, box, and background */
```

```
 _plwidth={12 12 }; /*Controls line thickness for main curves*/
 _paxht=0.10; /*Controls size of axes labels*/
 _ptitlht = 0.22; /*Controls main title size */
 _plegctl = { 2 7 6 4.0 };


begwind;
window(2,2,1);
 title("Different bandwiths");
 ylabel("f(x)");
 xlabel("x");
 _pcolor = { 9 5 5}; /* Colors for series */
 _plegstr = " Exact \000 h=0.01 \000 h=0.15 ";
xy(xx~x2~x3,fx~den2~den3);
nextwind;
 title("Bandwith based on standard deviation");
 ylabel("f(x)");
 xlabel("x");
 _plegstr = " Exact \000 stdc(x)";
xy(xx~x0,fx~den0);
nextwind;
 title("Silverman's Optimal Bandwith");
 ylabel("f(x)");
 xlabel("x");
 _plegstr = " Exact \000 Silverman's h";
 _plegctl = { 2 7 5.2 4.5 };
xy(xx~x1,fx~den1);
nextwind;
 title("Epanechnikov Kernel Density vs Exact");
 ylabel("f(x)");
 xlabel("x");
 _plegstr = " Exact \000 Epanechnikov";
 _plegctl = { 2 7 5.3 4.5 };
xy(xx~x4,fx~den4);
endwind;
/* ================================================*/
/* kernel density estimation; Tsurumi's original code*/
/* but modified by Freddy Rojas */
/* ================================================*/
proc(2)=kden(v,h);
local g,j,nn,res;
nn=rows(v);
" ";
" ";
@print "h ";@
```

```
@h;@
g=0;
j=1;
do while j <= nn;
g=g|meanc(pdfn((v[j]-v)./h))./h;
j=j+1;
endo;
res=sortc(v~g[2:nn+1],1);
retp(res[.,1],res[.,2]);
endp;
/* ================================================*/
/* Epachenikov kernel density estimation; */
/* ================================================*/

proc(2)= kernele(z,h);
local a,res,t,g,z_v,zv,zv_,j;
j=1;
g=0;
do while j <= rows(z);
 zv=(z[j]-z)./h;

 t=(abs(zv).<sqrt(5));
 a=code(t,sqrt(5)|1);
 zv_=t.*((3/4)*(1-(1/5).*(zv.^2))./a);
 g=g|meanc(zv_)./h;
j=j+1;
endo;
 res=sortc(z~g[2:rows(z)+1],1);
 retp(res[.,1],res[.,2]);
endp;
```