The AR(1) Process

Cesar E. Tamayo
ctamayo@econ.rutgers.edu
Department of Economics, Rutgers University

1 Representation and properties

Let $\varepsilon_t \sim N(0,1)$ *iid* shock. Then z_t follows an AR(1) process if we can write it as:

$$z_t = (1-\varphi)\theta + \varphi z_{t-1} + \sigma\varepsilon_t$$

this is the *recursive* formulation of the AR(1) process because it *recurs* in the same form at eact t. To go from the recursive formulation, to the infinite order MA formulation, first replace z_{t-1} in the expression for z_t :

$$z_t = (1-\varphi)\theta + \varphi\left[(1-\varphi)\theta + \varphi z_{t-2} + \sigma\varepsilon_{t-1}\right] + \sigma\varepsilon_t$$

next, repeat this recursive replacing and after $k+1$ times one obtains:

$$z_t = (1-\varphi)\theta\sum_{j=0}^{k-1}\varphi^j + \varphi^k z_{t-k} + \sigma\sum_{j=0}^{k-1}\varphi^j\varepsilon_{t-j}$$

which can be rearranged as follows:

$$\begin{aligned}
z_t &= \theta\sum_{j=0}^{k-1}\varphi^j - \varphi\theta\sum_{j=0}^{k-1}\varphi^j + \varphi^k z_{t-k} + \sigma\sum_{j=0}^{k-1}\varphi^j\varepsilon_{t-j} \\
&= \theta\left[\varphi^0 + \sum_{j=0}^{k-1}\varphi^{j+1}\right] - \varphi\theta\sum_{j=0}^{k-1}\varphi^j + \varphi^k z_{t-k} + \sigma\sum_{j=0}^{k-1}\varphi^j\varepsilon_{t-j} \\
&= \theta + \theta\sum_{j=0}^{k-1}\varphi^{j+1} - \theta\sum_{j=0}^{k-1}\varphi^{j+1} + \varphi^k z_{t-k} + \sigma\sum_{j=0}^{k-1}\varphi^j\varepsilon_{t-j} \\
&= \theta + \varphi^k z_{t-k} + \sigma\sum_{j=0}^{k-1}\varphi^j\varepsilon_{t-j}
\end{aligned}$$

and if we let $k \to \infty$ one gets:

$$z_t = \theta + \sigma\sum_{j=0}^{\infty}\varphi^j\varepsilon_{t-j}$$

that is, the infinite-order MA representation of the AR(1) process, saying that the AR(1) process can be written as an infinite sum of past shocks. If $|\varphi| = 1$ we have a unit root or say that z_t has infinite memory.

2 Conditional Distribution

The distribution of z_t conditional on knowing z_{t-1}. Recall that a linear function of a normal RV is itself a normal RV. Since at t the quantity z_{t-1} is known, it can be treated as a constant and therefore z_t, conditional on z_{t-1} is just a normal RV with its mean shifted by $(1-\varphi)\theta+\varphi z_{t-1}$.To obtain the conditional mean and variance of z_t first note that the variance remains unchanged as σ^2 while the mean:

$$\begin{aligned}\mathbb{E}_{t-1}\left[z_t\right] &= \mathbb{E}_{t-1}\left[(1-\varphi)\theta+\varphi z_{t-1}+\sigma\varepsilon_t\right] \\ &= \mathbb{E}_{t-1}\left[(1-\varphi)\theta+\varphi z_{t-1}\right]+\mathbb{E}_{t-1}\left[\sigma\varepsilon_t\right] \\ &= (1-\varphi)\theta+\varphi z_{t-1}\end{aligned}$$

so the conditional (on $t-1$) distribution of z_t :

$$z_t \sim_{t-1} N((1-\varphi)\theta+\varphi z_{t-1},\sigma^2)$$

3 Unconditional Distribution

The distribution of z_t presuming no knowledge of z_{t-1}, z_{t-2}...This is equivalent to the distribution of z_t conditional on knowing z_{t-k} for a very large k, that is, the distribution of z_{t+k} for a very large k with information on t. This is why the inconditional distribution is also called the long-run distribution. To obtain this, we use the infinite order MA representation:

$$\begin{aligned}\mathbb{E}z_t &= \mathbb{E}\left[\theta+\sigma\sum_{j=0}^{\infty}\varphi^j\varepsilon_{t-j}\right] \\ &= \theta+\mathbb{E}\left[\sigma\sum_{j=0}^{\infty}\varphi^j\varepsilon_{t-j}\right] \\ &= \theta\end{aligned}$$

since $|\varphi|=1$ and each $\varepsilon_{t-j}\sim N(0,1)$. Likewise we could compute the unconditional mean as:

$$\begin{aligned}\mathbb{E}\left[z_t\right] &= \mathbb{E}\left[(1-\varphi)\theta+\varphi z_{t-1}+\sigma\varepsilon_t\right] \\ &= (1-\varphi)\theta+\varphi\mathbb{E}\left[z_{t-1}\right] \\ &= (1-\varphi)\theta+\varphi\mathbb{E}\left[z_t\right] \qquad (\because \mathbb{E}\left[z_t\right]=\mathbb{E}\left[z_{t-1}\right])\end{aligned}$$

so that solving:

$$\begin{aligned}(1-\varphi)\mathbb{E}\left[z_t\right] &= (1-\varphi)\theta \\ \mathbb{E}\left[z_t\right] &= \theta\end{aligned}$$

while the unconditional variance is:

$$\begin{aligned} Var\left[z_t\right] &= Var\left[\theta + \sigma \sum_{j=0}^{\infty} \varphi^j \varepsilon_{t-j}\right] \\ &= Var\left[\sigma \sum_{j=0}^{\infty} \varphi^j \varepsilon_{t-j}\right] \\ &= \left(\sigma^2 \sum_{j=0}^{\infty} \varphi^{2j}\right) Var\left[\varepsilon_{t-j}\right] \\ &= \frac{\sigma^2}{1-\varphi^2} \end{aligned}$$

note that in the last step the following expansion is used:

$$\sum_{j=0}^{\infty} \varphi^{2j} = \begin{cases} \frac{1}{\varphi^2-1}\left(\varphi^{\infty}-1\right) & \text{if} \quad \neg\varphi \in \{-1,1\} \\ \infty & \text{if} \quad \varphi \in \{-1,1\} \end{cases}$$

so that the unconditional distribution of z_t is:

$$z_t \sim N\left(\theta, \frac{\sigma^2}{1-\varphi^2}\right)$$

Ntaurally, as long as $0 < |\varphi| < 1$ the unconditional variance is greater than the conditional variance.