Problem Set 2: Proposed solutions
Econ 505 - Spring 2012

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1 Flex-Price Monetary Model & government spending

1.1 Log-linear Approx:

From question 3 (b) of previous problem set we have:

$$\text{LM} \quad : \quad \hat{m}_t = \log\left(\frac{M_t}{P_t}\right) - \log\left(\frac{\bar{M}}{\bar{P}}\right) = \frac{\sigma}{\eta}\hat{c}_t - \frac{\beta}{\eta(1-\beta)}\hat{\imath}_t + \hat{\phi}_t$$

$$\text{IS} \quad : \quad \hat{c}_t = E_t\hat{c}_{t+1} - \frac{1}{\sigma}\left[\hat{\imath}_t - E_t\pi_{t+1}\right] + \hat{\gamma}_t$$

However, because of government spending, we have a new market clearing condition:

$$\begin{aligned} Y_t &= C_t + G_t \\ &\Rightarrow \hat{y}_t = \frac{\bar{C}}{\bar{Y}}\hat{c}_t + \frac{\bar{G}}{\bar{Y}}\hat{g}_t \\ &\Rightarrow \hat{y}_t = \omega\hat{c}_t + (1-\omega)\hat{g}_t \\ &\Rightarrow \hat{c}_t = \frac{1}{\omega}\hat{y}_t - \frac{(1-\omega)}{\omega}\hat{g}_t \end{aligned}$$

and we only need to replace in the log-liear system $\hat{c}_t$:

$$\text{LM} \quad : \quad \hat{m}_t = \log\left(\frac{M_t}{P_t}\right) - \log\left(\frac{\bar{M}}{\bar{P}}\right) = \frac{\sigma}{\eta}\left(\frac{1}{\omega}\hat{y}_t - \frac{(1-\omega)}{\omega}\hat{g}_t\right) - \frac{\beta}{\eta(1-\beta)}\hat{\imath}_t + \hat{\phi}_t$$

$$\text{IS} \quad : \quad \hat{y}_t = E_t\hat{y}_{t+1} - \frac{\omega}{\sigma}\left[\hat{\imath}_t - E_t\pi_{t+1}\right] + (1-\omega)\left[\hat{g}_t - E_t\hat{g}_{t+1}\right] + \omega\hat{\gamma}_t$$

An increase in $\hat{g}_t$ would shift the IS curve to the right if IS curve were drawn on $\hat{y} - \hat{\imath}$ plane as in the old IS-LM model of any undergraduate textbook.

1.2 Log-linear interest rate

Use the Fisher equation:

$$\hat{r}_t = \hat{\imath}_t - E_t\pi_{t+1}$$

and the log-linear IS to find:

$$\hat{r}_t = \frac{\sigma}{\omega}\left(E_t\hat{y}_{t+1} - \hat{y}_t\right) - \sigma\left(\frac{1-\omega}{\omega}\right)\left[\hat{g}_t - E_t\hat{g}_{t+1}\right] + \sigma\hat{\gamma}_t$$

2 Blanchard-Khan II

First we are given that:

$$z_t = \rho z_{t-1} + \varepsilon_t$$

2.1 Not unique solution

We have:

$$E_t[y_{t+2} - 2ay_{t+1} + by_t] = z_t$$

so define

$$x_t = E_t y_{t+1} \text{ so that} \Rightarrow x_{t+1} = E_t y_{t+2}$$

then we can write the system in B-K form:

$$\mathbb{E}_t \begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 2a & -b \\ 1 & 0 \end{pmatrix}}_{M} \begin{pmatrix} x_t \\ y_t \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} z_t$$

Let the eigenvalues of M be λ_2 and λ_2. The characteristic polynomial is given by:

$$f(\lambda) = \lambda^2 - 2a\lambda + b$$

and notice that:

$$\begin{aligned} f(0) &= b > 0 \\ f(1) &= 1 - 2a + b < 0 \end{aligned}$$

Thus, one eigenvalue is between zero and one and the other is greater than one. Since in this case we have two forward looking variables, the system does not have a unique solution.

2.2 Unique solution

This exercise is included in my notes on difference equations but I reproduce it here for the sake of completeness. We have:

$$E_t[y_{t+1} - 2ay_t + by_{t-1}] = z_t$$

for $0 < b < 1$ and $2a > b + 1$.First, define:

$$x_t = y_{t-1} \text{ so that} \Rightarrow x_{t+1} = y_t$$

this way, we can write the SOSDE as two SFODE:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbb{E}_t y_{t+1} \\ x_{t+1} \end{pmatrix} = \begin{pmatrix} 2a & -b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_t \\ x_t \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} z_t \tag{1}$$

or:

$$W_{t+1} = BW_t + Qz_t$$

because we have exactly the same matrix M, we can use the results from part (a) to conclude that there is one eigenvalue outside the unit circle and one inside it. Since in this case we have only one forward looking variable, we conclude that the SOSDE **has a unique solution**. Hence, we next find the eigenvalues corresponding to the characteristic polynomial:

$$\lambda_i = a \pm \sqrt{2a^2 - b}$$

To actually find the solutions, first obtain the eigenvector associated to the eigenvalue outside the unit circle say, λ_1, by solving:

$$\begin{aligned} e^T (B - \lambda_1 I_2) &= \mathbf{0} \\ e^T \begin{pmatrix} 2a - \lambda_1 & -b \\ 1 & -\lambda_1 \end{pmatrix} &= \begin{pmatrix} 0 & 0 \end{pmatrix} \\ e^T &= \begin{pmatrix} 1 & -b/\lambda_1 \end{pmatrix} \end{aligned}$$

so the system (1) can be written as:

$$e^T W_{t+1} = \lambda_1 e^T W_t + e^T Q z_t$$

define:

$$f_t = y_t + \frac{-bx_t}{\lambda_1} \Rightarrow f_{t+1} = y_{t+1} + \frac{-bx_{t+1}}{\lambda_1}$$

then:

$$f_t = \frac{Ef_{t+1}}{\lambda_1} - \frac{z_t}{\lambda_1}$$

or solving forward:

$$\begin{aligned} f_t &= -\frac{1}{\lambda_1} \sum_{k-0}^{\infty} \left(\frac{1}{\lambda_1} \right)^k \mathbb{E}_t z_{t+k} \\ &= -\frac{1}{\lambda_1 - \rho} z_t \quad \text{(recall } \mathbb{E}_t z_{t+1} = \rho z_t \text{ since } \mathbb{E}_t \varepsilon_{t+1} = 0) \end{aligned}$$

so replacing back $x_t = y_{t-1}$ we arrive at:

$$\begin{aligned} y_t &= \frac{b}{\lambda_1} y_{t-1} + \frac{1}{\lambda_1 - \rho} z_t \\ &= \lambda_2 y_{t-1} - \frac{1}{\lambda_1 - \rho} z_t \end{aligned}$$

or, using our findings from λ_1 :

$$y_t = \left(a - \sqrt{a^2 - b} \right) y_{t-1} - \left(\frac{1}{a + \sqrt{a^2 - b} - \rho} \right) z_t$$

3 Monopolistic competition

3.1 Verify the FOC

The firm's problem is:

$$\begin{aligned} \max_{\{Y(i),N(i)\}} \Pi(i) &= P(i) Y(i) - WN(i) \\ s.t. &: Y(i) = AN(i) \\ Y(i) &= C(i) = \left(\frac{P(i)}{P} \right)^{-\theta} C \end{aligned}$$

so replace the constraints in the objective function and solve the unconstrained maximization problem. Trivial.

3.2 Employment subsidy

Suppose that employment was to be subsidized at a rate τ. Then the firm would face the problem:

$$\begin{aligned}\max_{\{Y(i),N(i)\}} \Pi(i) &= P(i)Y(i) - (1-\tau)WN(i) \\ s.t. &: Y(i) = AN(i) \\ Y(i) &= C(i) = \left(\frac{P(i)}{P}\right)^{-\theta} C\end{aligned}$$

in which case the FOC is given by:

$$\frac{P(i)}{P} = \left(\frac{\theta}{\theta-1}\right)(1-\tau)MC$$

so that:

$$\begin{aligned}\tau &= \frac{1}{\theta} \Rightarrow \left(\frac{\theta}{\theta-1}\right)(1-\tau) = 1 \\ &\Rightarrow \frac{P(i)}{P} = MC\end{aligned}$$

and from here on the matter is just as in the lecture notes. In particular, to get the natural rate of output recall that the symmetric equilibrium implies $P(i) = P$ so that:

$$\frac{P(i)}{P} = MC = \frac{W}{AP} \Rightarrow A = \frac{W}{P}$$

next, from the HH FOC recall:

$$\frac{W}{P} = N^{\varphi}Y^{\sigma}$$

and also, from the symmetric equilibrium we have that $Y(i) = Y$ and $N(i) = N$ so that from the firm's FOC:

$$Y(i) = AN(i) \Rightarrow Y/A = N$$

so combining these three expressions:

$$(Y/A)^{\varphi} Y^{\sigma} = A \Rightarrow Y^{n} = A^{\left(\frac{1+\varphi}{\sigma+\varphi}\right)} = Y^{e}$$

so that the output level is exactly as in the Social Planner's problem (Pareto optimal). This can also be achieved with a sales tax of $(1-\tau)$.

4 Monopolistic competition II

4.1 The firm's FOC

We simply replace the first constraint in the firm's problem:

$$\begin{aligned}\max_{\{P(i),Y(i),N(i)\}} \Pi(i) &= P(i)Y(i) - (1-\tau)WN(i) \\ s.t. &: Y(i) = A(N(i))^{\delta} \\ Y(i) &= C(i) = \left(\frac{P(i)}{P}\right)^{-\theta} C\end{aligned}$$

again, replace and solve the unconstrained problem:

$$\max_{P(i)} \Pi(i) = P(i)\left(\frac{P(i)}{P}\right)^{-\theta} Y - W\left[\frac{\left(\frac{P(i)}{P}\right)^{-\theta} Y}{A}\right]^{\frac{1}{\delta}}$$

so that the FOC is:

$$\frac{P(i)}{P} = \left(\frac{\theta}{\theta-1}\right)\underbrace{\frac{1}{\delta}\frac{W}{P}\frac{Y^{\frac{1-\delta}{\delta}}}{A^{\frac{1}{\delta}}}\left(\frac{P(i)}{P}\right)^{-\theta\left(\frac{1-\delta}{\delta}\right)}}_{MC(i)}$$

to see why, notice that since:

$$N(i) = \left(\frac{Y(i)}{A}\right)^{\frac{1}{\delta}}$$

then the marginal cost can be obtained as:

$$\begin{aligned} MC(i) &= \frac{W}{P}\frac{\partial N(i)}{\partial Y(i)} \\ &= \frac{1}{\delta}\frac{W}{P}\frac{Y^{\frac{1-\delta}{\delta}}}{A^{\frac{1}{\delta}}}\left(\frac{P(i)}{P}\right)^{-\theta\left(\frac{1-\delta}{\delta}\right)} \end{aligned}$$

4.2 Firm-specific MC

From the last equation, notice that $MC(i)$ is increasing in Y and therefore increasing in $Y(i)$, while it is decreasing in $P(i)$ (since $\theta > 0$). Finally, notice that if $\delta = 1$ then:

$$MC(i) = \frac{W}{AP} = MC$$

4.3 Symmetric equilibrium

Simply re-write the firm FOC as:

$$P(i)^{1+\theta\left(\frac{1-\delta}{\delta}\right)} = \left(\frac{\theta}{\theta-1}\right)\frac{1}{\delta}\frac{W}{P}\frac{Y^{\frac{1-\delta}{\delta}}}{A^{\frac{1}{\delta}}}P^{1+\theta\left(\frac{1-\delta}{\delta}\right)}$$

and notice that the RHS does not depend on i. Therefore, $P(i) = P$ again.

4.4 Natural and efficient level of output

Because we have just shown that there is a symmetric equilibrium, we can proceed as in question 3(b) of this PS. That is, set $P(i) = P$, $N(i) = N$ and $Y(i) = Y$. Therefore:

$$\begin{aligned} \frac{P(i)}{P} &= \left(\frac{\theta}{\theta-1}\right)\frac{1}{\delta}\frac{W}{P}\frac{Y^{\frac{1-\delta}{\delta}}}{A^{\frac{1}{\delta}}}\left(\frac{P(i)}{P}\right)^{-\theta\left(\frac{1-\delta}{\delta}\right)} \\ 1 &= \left(\frac{\theta}{\theta-1}\right)\frac{1}{\delta}\frac{W}{P}\frac{Y^{\frac{1-\delta}{\delta}}}{A^{\frac{1}{\delta}}} \end{aligned}$$

and, again, using $W/P = N^{\varphi} Y^{\sigma}$ and $Y/A = N$ we arrive at:

$$Y^n = \left[\frac{\theta - 1}{\theta}\right]^{\frac{\delta}{\varphi+1+\sigma\delta-\delta}} A^{\frac{1+\varphi}{\varphi+1+\sigma\delta-\delta}}$$

However, notice that now the social planner's problem is different; the technological constraint is not:

$$Y = AN$$

as before, but instead is:

$$Y = A\left(N\right)^{\delta}$$

so we solve:

$$\max_{Y} \left\{ \frac{Y^{1-\sigma}}{1-\sigma} - \frac{\left[\left(\frac{Y(i)}{A}\right)^{\frac{1}{\delta}}\right]^{1+\varphi}}{1+\varphi} \right\}$$

so that:

$$Y^e = \delta^{\frac{\delta}{\varphi+1+\sigma\delta-\delta}} A^{\frac{1+\varphi}{\varphi+1+\sigma\delta-\delta}}$$

and we confirm that $Y^n < Y^e$.

5 Appendix to question 1

From question 3(a)-(c) of the PS1 we have the non-linear system:

$$\begin{aligned}
\text{LM} \quad &: \quad \phi_t^{\eta} \frac{\left(\frac{M_t^s}{P_t}\right)^{-\eta}}{Y_t^{-\sigma}} = \frac{i_t}{1+i_t} \\
\text{IS} \quad &: \quad \beta E_t \left(\frac{\gamma_{t+1Y_t}}{\gamma_{tY_{t+1}}}\right)^{\sigma} \left(\frac{P_t}{P_{t+1}}\right) = \frac{1}{1+i_t}
\end{aligned}$$

and non-stochastic steady state:

$$\begin{aligned}
\bar{\imath} &= \beta^{-1} - 1 \\
\bar{P} &= \bar{M} \left(\frac{\bar{\imath}}{1+\bar{\imath}}\right)^{1/\eta} \\
\bar{C} &= 1
\end{aligned}$$

which yields the log-linear system given in question 1(a) of PS2.:

$$\hat{m}_t = \log\left(\frac{M_t}{P_t}\right) - \log\left(\frac{\bar{M}}{\bar{P}}\right) = \frac{\sigma}{\eta}\hat{y}_t - \frac{\beta}{\eta(1-\beta)}\hat{\imath}_t + \hat{\phi}_t$$

and:

$$\hat{y}_t = E_t \hat{y}_{t+1} - \frac{1}{\sigma}\left[\hat{\imath}_t - E_t \underbrace{(\hat{p}_{t+1} - \hat{p}_t)}_{\pi_{t+1}}\right] + \hat{\gamma}_t - \underbrace{E_t \gamma_{t+1}}_{=0}$$