Intermediate Micro Review

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1 The Price System (Smith, Ch 1)

Single good economics. Let p be the price of the single good and y the supply of such good (i.e. total amount sellers want to sell). Let x be demand for the good (i.e. total amount consumers want to buy). Then:

- $y = y(p)$ is the supply function and $y'(p) > 0$.
- $x = x(p)$ is the demand function and $x'(p) < 0$.
- Equilibrium is $q^* = y(p^*) = x(p^*)$

If $p < p^*$ one has excess demand and viceversa for excess supply. Excess demand (supply) drives up (down) price and there's a tendency towards equilibrium.If this were always the case, we would have a stable equilibrium price.

1.1 Stability of equilibrium (cobweb cycle)

Suppose that because of lags in production (e.g., building capacity) supply is $y(p) = bE_{t-1}(p_t) = bp_{t-1}$ and demand is $x(p) = \alpha - \beta p_t$. Then *market clearing* $\neq$ *equilibrium. Market clearing* would be:

$$\begin{aligned} \alpha - \beta p_t &= bp_{t-1} \\ p_t &= \frac{\alpha}{\beta} - \frac{b}{\beta} p_{t-1} \end{aligned} \tag{1}$$

Whereas *equilibrium* implies $p_t = p_{t-1} = p^*$ and:

$$\begin{aligned} \alpha - \beta p^* &= bp^* \\ p^* &= \frac{\alpha}{\beta + b} \end{aligned} \tag{2}$$

Substract (2) from (1) and get:

$$p_t - p^* = -\frac{b}{\beta}(p_{t-1} - p^*)$$

Naturally, there's a tendency towards equilibrium only if $b < \beta$. Otherwise one gets the cobweb cycle or an unstable equilibrium.

1.2 Comparative statics

Sales Tax. Introduce a sales tax so that now, the price faced by consumers is $\pi = p + t$. Since demand depends upon π and supply is a function of p, market clearing is now $x(\pi) = y(p)$.To see what happens if the tax amount changes, differentiate w.r.t. t :

$$\begin{aligned} \frac{d\pi}{dt} &= \frac{dp}{dt} + 1 \\ x'(\pi)\frac{d\pi}{dt} &= y'(p)\frac{dp}{dt} \end{aligned}$$

Solving this two equation system yields:

$$\begin{aligned}
\frac{dp}{dt} &= \frac{x'(\pi)}{y'(p) - x'(\pi)} \\
\frac{d\pi}{dt} &= \frac{y'(p)}{y'(p) - x'(\pi)}
\end{aligned}$$

Since by assumption $y'(p) > 0$ and $x'(p) < 0$ we have that $\frac{dp}{dt} < 0$ and $\frac{d\pi}{dt} > 0$. Finally, since $x(\pi) = q = y(p)$ we have:

$$\frac{dq}{dt} = \frac{x'(\pi)y'(p)}{y'(p) - x'(\pi)}$$

and $\frac{dq}{dt} < 0$. Summarizing, regardless of who pays the sales tax, the result is that if t rises, p falls and π rises so that q falls (deadweight loss). Introducing a sales tax shifts the demand schedule downwards; at each p consumers are willing to buy less than they otherwise would since they now face $\pi = p + t$

Income effects. Suppose now that demand is a function of both prices and income, so market clearing is $y(p) = x(p, m)$ and p is an implicit function of m :

$$\begin{aligned}
y'(p)\frac{dp}{dm} &= \frac{\partial x}{\partial p}\frac{dp}{dm} + \frac{\partial x}{\partial m} \\
\frac{dp}{dm} &= \frac{x_m}{y_p - x_p}
\end{aligned}$$

and since $y_p > 0, x_p < 0$ and assuming $x_m > 0$, we conclude (not surprisingly) that $\frac{dp}{dm} > 0$. Likewise, since $y(p) = q = x(p, m)$, we have:

$$\frac{dq}{dm} = \frac{x_m y_p}{y_p - x_p}$$

and we conclude that, given the assumptions, $\frac{dq}{dm} > 0$. Moreover, to measure the relative size of the income effects, define the *income elasticity of demand* as:

$$e_{xm} = \frac{m}{x}\frac{\partial x}{\partial m}$$

Cross-price effects. Suppose goods $1, 2$ and a subsidy to good 2. Then equilibrium in both markets would be:

$$\begin{aligned}
y^1(p^1) &= x^1(p^1, \pi^2) && \text{(3a)} \\
y^2(p^2) &= x^2(p^1, \pi^2) && \text{(3b)} \\
\pi^2 &= p^2 - s && \text{(3c)}
\end{aligned}$$

where π^2 is the price of good 2 faced by consumers, s is the subsidy and p^2 is the price received by producers of good 2. To see what happens when s changes, differentiate (3a)-(3c) w.r.t. s and replace. Define $y_1^1 = dx^1/dp^1$ and $x_2^1 = \partial x^1/\partial \pi^2$ to obtain:

$$\begin{aligned}
y_1^1\frac{dp^1}{ds} &= x_1^1\frac{dp^1}{ds} + x_2^1\left(\frac{dp^2}{ds} - 1\right) &\implies& \quad \frac{dp^1}{ds} = \frac{-x_2^1 y_2^2}{(y_1^1 - x_1^1)(y_2^2 - x_2^2) - x_2^1 x_1^2} \\
y_2^2\frac{dp^2}{ds} &= x_1^2\frac{dp^1}{ds} + x_2^2\left(\frac{dp^2}{ds} - 1\right) &\implies& \quad \frac{dp^2}{ds} = \frac{-(y_1^1 - x_1^1)x_2^2 - x_2^1 x_1^2}{(y_1^1 - x_1^1)(y_2^2 - x_2^2) - x_2^1 x_1^2}
\end{aligned}$$

Naturally, the final change in prices and quantitties of both goods depend upon one another in a complicated way. The reason is that both income and substitution effects play a role. To guarantee the stability of the new equilibrium we require that $(y_1^1 - x_1^1)(y_2^2 - x_2^2) - x_2^1 x_1^2 > 0$.

Finally, to measure the responsiveness of demand and supply to self and cross-prices, define the price elasticities of demand and supply:

$$e_{xp} = \frac{p}{x}\frac{dx}{dp} \qquad e_{xp} = \frac{p}{y}\frac{dy}{dp} \qquad e^i_{xj} = \frac{p^j}{x^i}\frac{\partial x^i}{\partial p^j}$$

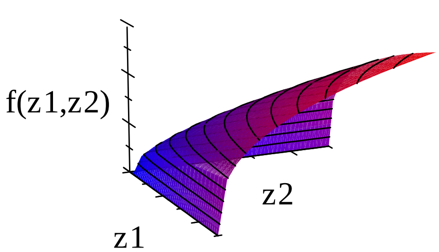
2 Producer theory (Smith, Ch 2-3; Varian, Ch 18-22)

2.1 Technology and returns to scale

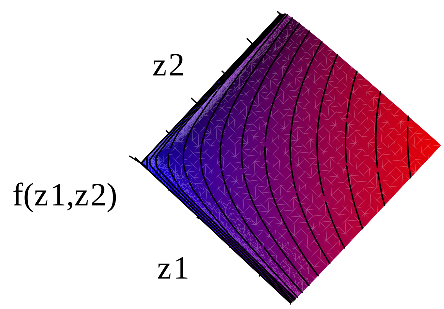
Suppose one is dealing with a typical single-output firmm which produces under a specific technology:

$$y \leq f(z_1, z_2, ..., z_n)$$

in most applications one will require the input set to be convex which in turn rules out convex production functions. However, one usually wants to work with production functions that have convex isoquants (z_1, z_2 contours):



A convex production set



Convex contours

Moreover, one usually deals with different types of returns to scale. For $k > 1$:

- We say that a technology exhibits *decreasing returns to scale* if $f(kz) < kf(z)$. This simply means that if $c(y)$ is the cost of producing y, then:

$$AC(y) = \frac{c(y)}{y} < \frac{c(ky)}{ky} = AC(ky)$$

- On the other hang, *increasing returns* imply that $f(kz) > kf(z)$ so that the average cost $c(y)/y$ decreases as y increases:

$$AC(y) = \frac{c(y)}{y} > \frac{c(ky)}{ky} = AC(ky)$$

- And *constant returns to scale* imply $f(kz) = kf(z)$ Naturally, the Average cost is constant too.

Two important concepts in production technologies are:

- **Technical rate of substitution**. The TRS is simply the rate at which technology permits one to substitute one input by another:

$$TRS = -\frac{MP_1(z_1, z_2)}{MP_2(z_1, z_2)} = \frac{\partial f/\partial z_1}{\partial f/\partial z_2}$$

  that is, the slope of the isoquants in the input space or countours in the input-output space.

- **Diminishing marginal product**. Which is simply a sensible statement about second derivatives:

$$\frac{\partial^2 f}{\partial z_i^2} < 0$$

2.2 Cost minimization

A price-taker firm using inputs $\mathbf{z} = [z_1, z_2, ..., z_n]'$ and facing input prices $\mathbf{w} = [w_1, w_2, ..., w_n]'$ sets out to minimizing the cost of producing some level y:

$$\begin{gathered}\min_z \mathbf{w} \cdot \mathbf{z} \\ s.t. \\ f(\mathbf{z}) = y\end{gathered}$$

with lagrangean:

$$\mathcal{L} = \mathbf{w} \cdot \mathbf{z} + \lambda(\mathbf{y} - f(\mathbf{z}))$$

and associated F.O.C.:

$$\begin{aligned} w_i - \lambda \frac{\partial f}{\partial z_i} &= 0 \qquad (4) \\ f(\mathbf{z}) &= y \qquad (5) \end{aligned}$$

saying, again, that the firm will use inputs up to the point where their price equal the value of their marginal products. The solution to this problem will depend on w_1, w_2 and y and is known as the cost function: $c(\mathbf{w},y)$. As in the profit max problem, one can write the isocost functions for two inputs:

$$\begin{aligned} C &= w_1 z_1 + w_2 z_2 \\ z_1 &= \frac{C}{w_2} - \frac{w_1}{w_2} z_2 \end{aligned}$$

in the two input (z_1, z_2) −space, these will be paralel lines with negative slopes $-\frac{w_1}{w_2}$. Geometrically, the firm wants to be in the lowest (most southwestern) isocost line and if one has convex (z_1, z_2) upper contours as in Figure (), the optimal would be at the tangency point between the isocost (straight line) and the isoquant (convex curve yielding constant y), that is, from the F.O.C. (4)-(5) one obtains that the TRS equals the input price ratio:

$$\frac{\partial f/\partial z_1}{\partial f/\partial z_2} = \frac{w_1}{w_2} \qquad (6)$$

When the optimal choice of inputs $z_i(\mathbf{w}, y)$ is used to calculate the cost of production, one obtains the cost function:

$$c(\mathbf{w}, y) = \mathbf{w} \cdot \mathbf{z}(\mathbf{w}, y)$$

It is straightforward to see that the cost function is homogeneous of degree 1 in $\mathbf{w}$. Furthermore, differentiationg the cost function w.r.t. y yields:

$$\begin{aligned}
\frac{\partial c(\mathbf{w},y)}{\partial y} &= \sum_{i=1}^{n} w_i \frac{\partial z_i(\mathbf{w},y)}{\partial y} \\
&= \lambda \sum_{i=1}^{n} \frac{\partial f}{\partial z_i} \frac{\partial z_i(\mathbf{w},y)}{\partial y} \qquad \text{(use F.O.C. (4))} \\
\frac{\partial c(\mathbf{w},y)}{\partial y} &= \lambda(\mathbf{w},y) \qquad \left(\text{since } \frac{\partial f}{\partial z_i}\frac{\partial z_i(\mathbf{w},y)}{\partial y} = \frac{df}{dy} = 1 \text{ by (5)}\right)
\end{aligned}$$

so that the Lagrange multiplier is precisely the <u>*marginal cost*</u>.

2.2.1 Elasticity of substitution

Finally, the shape of the isoquant $\bar{y} = f(z_1, z_2)$ shows how the input mix changes as the price changes, this can be masured by the *elasticity of substitution*:

$$\sigma = \frac{w_1/w_2}{z_1/z_2} \frac{d\,(z_1/z_2)}{d\,(w_1/w_2)}$$

which in turn tells what happens to the input shares on total production as prices change. When such elasticity of substitution is constant, the production function is called CES.

Example 1 *A CES function. Consider the technology:*

$$y = f(z_1, z_2) = \left(z_1^{1/2} + z_2^{1/2}\right)^2$$

so that:

$$\frac{\partial f/\partial z_1}{\partial f/\partial z_2} = \frac{\sqrt{z_2}}{\sqrt{z_1}}$$

and from the F.O.C. (6) we have that:

$$\frac{\sqrt{z_2}}{\sqrt{z_1}} = \frac{w_1}{w_2} \Longrightarrow \frac{z_1}{z_2} = \left(\frac{w_1}{w_2}\right)^{-2}$$

and the elasticity of substitution is:

$$\begin{aligned}
\sigma &= \frac{w_1/w_2}{z_1/z_2}(-2)\left(\frac{w_1}{w_2}\right)^{-3} \\
\sigma &= -2
\end{aligned}$$

so if the input price ratio changes by 10%, the input mix ratio changes by 20% (in opposite direction)

Example 2 *The Cobb-Douglas aggregate production function. Suppose that an economy operates under CD with two inputs, capital (K) and labor (L):*

$$F(K, L) = K^{\alpha} L^{\beta}$$

first, $\frac{F_k}{F_L} = \left(\frac{\alpha}{\beta}\right)\left(\frac{L}{K}\right)$ and with r being the rental price of K and w the wage, from the F.O.C. one has that:

$$\begin{aligned}
\left(\frac{\alpha}{\beta}\right)\left(\frac{L}{K}\right) &= \frac{r}{w} \\
\frac{K}{L} &= \left(\frac{r}{w}\right)^{-1}\left(\frac{\alpha}{\beta}\right)
\end{aligned}$$

so that the elasticity of substitution is:

$$\begin{aligned} \frac{r/w}{K/L}\frac{d(K/L)}{d(r/w)} &= \frac{r/w}{K/L}\left(-\frac{\alpha}{\beta}\right)\left(\frac{r}{w}\right)^{-2} \\ \sigma &= -1 \end{aligned}$$

that, is, not only a Cobb-Douglas belongs to the family of CES functions, it exhibits elasticity of substitution equal to one; if the input price ratio changes by 10% so does the input mix ratio (in the opposite direction).

2.3 Profit maximization

If, in contrast to the cost min problem, the firm can also choose how much to produce and faces output price p is assumed to maximize profits:

$$\pi = py - \mathbf{w} \cdot \mathbf{z} \tag{7}$$

If one deals with only one input, z_1, (7) can be rewriten as:

$$y = \frac{\pi}{p} - \frac{w_1}{p} z_1$$

This equation describes *isoprofit* lines in the input-output space with positive slope; combinations of (y, z_1) that yield the same profit. Geometrically, the firm wants to attain the highest possible isoprofit line. With a convex input requirement set and a strictly concave production function, the firm would optimize at the point where an isoprofit line is tangent to the production function. Formally, the constrained optimization problem of the firm is:

$$\begin{aligned} & \max_{y,z} py - \mathbf{w} \cdot \mathbf{z} \\ & s.t. \\ y \quad \leq \quad & f(z_1, z_2, ..., z_n) \end{aligned}$$

which is equivalent to the problem:

$$\max_{z} pf(\mathbf{z}) - \mathbf{w} \cdot \mathbf{z}$$

the first order necessary conditions are the n-equations:

$$p\frac{\partial f}{\partial \mathbf{z}} = \mathbf{w} \tag{8}$$

$$y = f(\mathbf{z}) \tag{9}$$

simply stating that, when firms are price takers, they will use inpunt i up to the point to wich the value of its marginal product equals its price. In order for (8) to be sufficient we require the input requirement set to be convex, as above.

Alternatively, if the firm has solved its cost minimization problem and know its cost function $c(\mathbf{w}, y)$ then the problem is a single variable (unconstrained) optimization one:

$$\max_{y} py - c(\mathbf{w}, y)$$

with necessary F.O.C.

$$p = \frac{\partial c(\mathbf{w}, y)}{\partial y} \tag{10}$$

that is, the firm produces up to the point where the marginal cost equals the price of output. The second-order sufficient condition is that $\frac{\partial^2 c(\mathbf{w},y)}{\partial y^2} > 0$, i.e., that the *cost function is convex* or exhibits increasing marginal cost.

Remark 3 *Since* $\frac{\partial c(\mathbf{w},y)}{\partial y} = \lambda(\mathbf{w}, y) = p$ *one can substitute* λ *in the cost min F.O.C. (4):*

$$\begin{aligned} w_i - \lambda\frac{\partial f}{\partial z_i} &= 0 \\ w_i - \frac{\partial c(\mathbf{w}, y)}{\partial y}\frac{\partial f}{\partial z_i} &= 0 \\ w_i - p\frac{\partial f}{\partial z_i} &= 0 \end{aligned}$$

which is just the F.O.C. for the original profit max problem (8)

Comparative Statics. A simple exercise of comparative statics under convex input set and concave production function will yield:

- Demand for input i will rise whenever w_i falls or p rises.

- Demand for input i will fall whenever w_i rises or p falls.

Claim 4 *The only reasonable long-run profit level for a firm with constant returns to scale operating in a competitive market is zero.*

To see this, suppose that the firm maximizes profits at some $\pi^* > 0$ with an optimal combination of inputs $\mathbf{z}^*$. Then $\pi^* = pf(\mathbf{z}^*) - \mathbf{w} \cdot \mathbf{z}^* \geq pf(\mathbf{z}) - \mathbf{w} \cdot \mathbf{z}$ for any possible $\mathbf{z}$. Now, since the firm exhibits constant returns to scale, $\mathbf{z}' = k\mathbf{z}^*$ for $k > 1$ implies $k\mathbf{y}^*$ and $k\boldsymbol{\pi}^*$, that is a higher level of profits, which contradicts the hypothesis that $\mathbf{z}^*$ maximizes profits. Therefore, only if $\pi^* = 0$ we have that for any $k > 1$, $k\boldsymbol{\pi}^* = \boldsymbol{\pi}^* = 0$.

Remark 5 *One can recover the firm's technology when it is not known by looking at the firms revealed profitability. Much like revealed preferences via demands for goods can help recover the shape of the utility function, the choice of inputs by the firm over some other feasible input bundles embed information about feasibility and profitability. If enough of these are observed, one coud recover the firm's technology (enough "tangency points").*

Example 6 *Cobb-Douglas technology and two inputs. Consider the production function* $f(z_1, z_2) = z_1^\alpha z_2^\varphi$. *First, one sets out to minimize cost of producing certain* y*: The Lagrangean:*

$$\mathcal{L} = w_1 z_1 + w_2 z_2 + \lambda(y - z_1^\alpha z_2^\varphi)$$

and F.O.C.

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial z_1} &= w_1 - \lambda\alpha z_1^{\alpha-1} z_2^\varphi = 0 \\ \frac{\partial \mathcal{L}}{\partial z_2} &= w_2 - \lambda\beta z_1^\alpha z_2^{\varphi-1} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= y - z_1^\alpha z_2^\varphi. = 0 \end{aligned}$$

solving this sysem for z_1, z_2 *one obtains:*

$$\begin{aligned} z_1 &= \left[y\left(\frac{\alpha}{\varphi}\frac{w_2}{w_1}\right)^\varphi \right]^{\frac{1}{\alpha+\varphi}} \\ z_2 &= \left[y\left(\frac{\varphi}{\alpha}\frac{w_1}{w_2}\right)^\alpha \right]^{\frac{1}{\alpha+\varphi}} \end{aligned}$$

and const function:

$$c(\mathbf{w}, y) = w_1 \left[y \left(\frac{\alpha}{\varphi} \frac{w_2}{w_1} \right)^{\varphi} \right]^{\frac{1}{\alpha+\varphi}} + w_2 \left[y \left(\frac{\varphi}{\alpha} \frac{w_1}{w_2} \right)^{\alpha} \right]^{\frac{1}{\alpha+\varphi}}$$

Now, one can solve the single variable unconstrained optimization problem:

$$\max_y py - c(\mathbf{w}, y)$$

with F.O.C.

$$\begin{aligned} p &= \frac{\partial c(\mathbf{w}, y)}{\partial y} \\ p &= \left(\frac{1}{\alpha+\varphi} \right) \left\{ w_1 \left[\left(\frac{\alpha}{\varphi} \frac{w_2}{w_1} \right)^{\varphi} \right]^{\frac{1}{\alpha+\varphi}} + w_2 \left[\left(\frac{\varphi}{\alpha} \frac{w_1}{w_2} \right)^{\alpha} \right]^{\frac{1}{\alpha+\varphi}} \right\} y^{\frac{\alpha+\varphi-1}{\alpha+\varphi}} \end{aligned}$$

To complete the solution, one can solve for y as a function of p, w_1, w_2 and replace on the equations for z_1, z_2 to leave them only as functions of p, w_1, w_2.

2.4 Cost Curves

If, in the short-run some input, say z_2 is fixed at $\bar{z}_2$ (fixed cost) while some inputs are choice variables, the production function is:

$$y = f(z_1, \bar{z}_2)$$

so z_1 is an implicit function of y and $\bar{z}_2$:

$$z_1 = z_1(\bar{z}_2, y)$$

the cost function is now:

$$c(\mathbf{w}, y, \bar{z}_2) = \underbrace{w_1 z_1(\bar{z}_2, y)}_{variable\ cost} + \underbrace{w_2 \bar{z}_2}_{fixed\ cost}$$

and the firm has a whole family of cost curves:

1. Short-run marginal cost (SRMC): $\partial c(\mathbf{w}, y, \bar{z}_2)/\partial y$
2. Short-run average cost (SRAC): $c(\mathbf{w}, y, \bar{z}_2)/y$
3. Average variable cost (AVC): $w_1 z_1(\bar{z}_2, y)/y$

Now, short-run profit maximization requires a slight variation of (10):

$$p = \frac{\partial c(\mathbf{w}, y, \bar{z}_2)}{\partial y}$$

and an additional condition is required; that revenue exceeds variable cost or $p > AVC$:

$$py - w_1 z_1(\bar{z}_2, y) > 0$$

for, if this were not the case, fixed costs would make the firm loose money. Thus, the SRMC, when it is above the AVC (so $p > AVC$) represents the firm's *short-run supply function.*

Naturally, since in the long-run both inputs are choice variables, the firm can do at least as good in the L-R as it does in the S-R (it can always choose $z_2 = \bar{z}_2$) so it is always the case that:

$$c(\mathbf{w}, y, \bar{z}_2) \quad \geq \quad c(\mathbf{w}, y) \quad \Longrightarrow \quad \underbrace{\frac{c(\mathbf{w}, y, \bar{z}_2)}{y}}_{SRAC} \geq \underbrace{\frac{c(\mathbf{w}, y)}{y}}_{LRAC}$$

In fact, the LRAC is a *lower envelope* for the SRAC. However, there is not such clear-cut relationship betwenn marginal cost curves since, for some levels of y, one has that $\frac{\partial c(\mathbf{w},y,\bar{z}_2)}{\partial y} > \frac{\partial c(\mathbf{w},y)}{\partial y}$ ($SRMC$>$LRMC$) while for some levels of output the opposite is true.

2.5 Firm and industry suply

Suppose that demand for a certain good is $x(p)$ with $x'(p) < 0$. This is in fact the demand function for an entire industry composed of firm's producing the same good. On the other hand, the supply function of the industry is the horizonthal aggregation of the N firms' marginal cost curves:

$$y(p) = \sum_{i=1}^{N} y^i(p)$$

2.5.1 Free entry and heterogeneous firms:

- **Case 1: Identical U-shaped *LRAC* functions, free entry**. If one starts from a situation with N firms each producing at a level where $LRMC > LRAC$ then $p > LRAC$ and profits are positive. With free entry, new firms will have incentives to enter the industry and will supply additional output, driving the proce down to the point where $LRMC = LRAC$. The industry L-R supply curve is horizonthal.

- **Case 2: Heterogeneous *LRAC* functions, free entry.** If different firms face different cost functions, the industry's LRS will have to be upward sloping since newcomers require higher price to break-even, so to achieve higher level of output (more firms) one requires higher prices.

- **Case 3: Identical flat *LRAC* functions.** This means constant returns to scale so $LRMC = LRAC$. Free entry is irrelevant since firms can always expand indefinitely (flat LRAC). Starting from $p \neq LRAC = LRMC$, S-R supply is *determinate* and expansion/contraction (entry/exit) will drive p towards $LRMC$ where individual firm's output and total numer of firms is *indeterminate* but the industry's output is determined by demand.

- **Case 4: Firms have decreasing returns and there are barries to entry.** Firms make profit but don't want to expand since they face decreasing returns and since there is no new entry, price and output levels are fixed.

- **Case 5: Firms have decreasing returns and free entry.** Since profits are positive (as in case 4), new firms enter and drive p downwards until the LRS is flat. Eventually there would be an indefinitely large number of firms each producing an infinitesimal amount of output.

- **Case 6: All firms face increasing returns.** Output is determinate in the S-R but since all firms face increasing returns ($LRMC$, $LRAC$ slope downwards) there's an incentive for merging so eventually there remains one single firm (Monopoly)

2.5.2 Features of the competitive industry

- The supply curve of a competitive industry will slope upwards in the S-R and will be horizonthal or slope upwards in the L-R.
- The marginal cost of output is the same for all firms and equals the price of output.
- In the L-R all firms produce positive output, make non-negative profits and...
- In fact, firms make zero profits and produce at the point which minimizes average cost of production.

3 Non-competitive markets (Smith, Ch 6; Varian 24-28)

3.1 Monopoly

The monopolistic firm is price-maker instead of price taker. Hence, whenever it increases output, it drives down price. The firm has output y and cost function $c(y)$. Its output price is not the constat p as before but a function of y: $p = p(y)$ (the inverse demand). Since $p'(y) < 0$ the firm faces downward sloping demand. The firm's revenue is:

$$R = p(y)y$$

and its profit-max problem is:

$$\max_y p(y)y - c(y)$$

with F.O.C.:

$$\begin{aligned} \underbrace{p(y) + p'(y)y} &= \underbrace{c'(y)} \\ \textit{marginal revenue} &= \textit{marginal cost} \end{aligned}$$

and S.O.C.:

$$2p'(y) + p''(y)y - c''(y) < 0$$

so that for the monopolistic firm the necessary condition is not $MC = p$ bur $MR = MC$. The reason is that p is not fixed but falling as y rises. Alternatively one can write:

$$\begin{aligned} MR &= p(y) + p'(y)y \\ &= p\left(1 + \frac{y}{p}\frac{dp}{dy}\right) \\ &= p\left(1 + \frac{1}{e_{xp}}\right) \end{aligned}$$

since $\frac{y}{p}\frac{dp}{dy}$ is the inverse of the price elasticity of demand. The monopolistic firm will produce at y^* where $MC = MR$ which lies to the left of the y' where $MC = p$. Thus, in a monopoistic industry total <u>output would be lower and price would be higher</u> than those under perfect competition.

3.1.1 Price discrimination

If output is non-resellable, one could partly amend the inefficiencies generated by monopoly power with price discrimination (air tickets, hotel rooms, medical charges).

- **Perfect price discrimination**: every customer can be charged a different rate (the maximum amount each customer is willing to pay) so that total revenue is:

$$R(y) = \int_0^y p(x)dx$$

and profits are:

$$\pi(y) = \int_0^y p(x)dx - c(y)$$

the necessary FOC is simply:

$$p(y) = c'(y)$$

just as in the competitive industry.

- **Imperfect price discrimination**...

3.2 Oligopoly

Suppose that market demand is linear:

$$p = a - bx$$

there are N firms each with linear cost function:

$$c_i = cy_i \text{ for } i = 1, 2..., N$$

quantity consumed is the sum of firms' outputs:

$$x = \sum_{i=1}^{N} y_i$$

3.2.1 Cournot Equilibrium

In the Cournot model, firms optimize under the assumption that other firms stay put. A typical firm therefore has profits:

$$\pi_i(y_i, x) = (a - bx)y_i - cy_i$$

with associated FOC:

$$\frac{\partial \pi_i(y_i, x)}{\partial y_i} = a - b\left(\frac{dx}{dy_i}y_i + \frac{dy_i}{dy_i}x\right) - c = 0 \tag{11}$$

since the firm optimizes under the assumption that other firms do not change their output, $\frac{dx}{dy_i} = 1$ so that the F.O.C. is:

$$a - by_i - bx - c = 0 \tag{12}$$

The set of F.O.C. for each firm characterize what is known as a *Cournot equilibrium.*

Suppose one deals with $N = 2$, that is, a duopoly. Then $x = y_1 + y_2$ and (12) for firm 1 becomes:

$$\begin{aligned} 0 &= a - by_i - b(y_1 + y_2) - c \\ y_1 &= \frac{a - c - by_2}{2b} \end{aligned} \tag{13}$$

while for fim 2 it is:

$$y_2 = \frac{a - c - by_1}{2b} \tag{14}$$

these are the reaction functions for firms 1,2 in the Cournot equilibrium; that is, the best response of each firm given that the other firms do not change their behavior. Production distribution would be given by the intersection of the two reaction functions:

$$(y_1, y_2) = \left(\frac{a-c}{3b}, \frac{a-c}{3b}\right)$$

and total output for sale would be $y = \frac{2(a-c)}{3b}$. Note that if firm 1 becomes a monopolist, total output for sale is given by:

$$(y_1, y_2) = \left(\frac{a-c}{2b}, 0\right)$$

and conversely if firm 2 is the monopolist. Note also that total output under oligopoly is larger than under monopoly.

3.2.2 Stackelberg equilibrium

Suppose that, instead of assuming that y_2 is fixed, firm 1 realizes firm 2 has reaction function (14) If one assumes firm 1 as the *leader* of the industry, firm 1 max-profits subject to (14) holding:

$$\max_{y_1, y_2} [a - b(y_1 + y_2)]\, y_1 - cy_1$$
$$s.t.$$
$$y_2 = \frac{a - c - by_1}{2b}$$

or equivalently:

$$\max_{y_1, y_2} \frac{1}{2}(a-c)y_1 - \frac{1}{2}by_1^2$$

with solution:

$$y_1 = \frac{a-c}{2b}$$

and using (14) to solve for the folower's output y_2:

$$y_2 = \frac{a-c}{4b}$$

naturally the distribution of output between firms is not symetric anymore with the leader producing more than the follower.

3.2.3 Monopolistic competition

In this setting, there's product differentiation so a typical firm faces a demand curve for its product that is a function of its own price and other firm's output prices:

$$y_i = y_i(p_1, p_2, ..., p_n)$$

with $\frac{\partial y_i}{\partial p_i} < 0$ and $\frac{\partial y_i}{\partial p_j} > 0$. Hence, the firm acts as a monopolist since it is price-maker for its own product, operates under a Cournot-type of enviroment since it assumes that other firms' prices are given, but faces a competitive threat since if it makes profits, it attracts other firms to produce a similar good, driving down p_j and demand for i. The firm's problem is therefore:

$$\max_{p_i} p_i y_i(\mathbf{p}) - c(y_i(\mathbf{p}))$$

with FOC:

$$\begin{aligned} c'(y_i) &= p_i + y_i \frac{\partial p_i}{\partial y_i} \\ &= \left(1 + \frac{1}{e_i}\right) \end{aligned}$$

4 Consumer theory (Smith, Ch 4; Varian, Ch 2-5)

4.1 Preferences

Notation:

- strictly preferred bundle: $(x_1, x_2) \succ (y_1, y_2)$
- weakly preferred bundle: $(x_1, x_2) \succsim (y_1, y_2)$
- Indifferent between bundles: $(x_1, x_2) \sim (y_1, y_2)$

Preference relations are usually assumed to be:

- Complete: at least one of the three things above can be said about (x_1, x_2) and (y_1, y_2)
- Reflexive: $(x_1, x_2) \succsim (x_1, x_2)$
- Transitive: $(x_1, x_2) \succsim (y_1, y_2)$ and $(y_1, y_2) \succsim (z_1, z_2) \Rightarrow (x_1, x_2) \succsim (z_1, z_2)$

Given a preference relation $\succsim$ we define (in two dimensions) the *weakly preferred set* as $\left\{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{x} \succsim (x_1, x_2)\right\}$ and the *indifference curve* as:

$$\left\{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{x} \sim (x_1, x_2)\right\}$$

4.2 Utility

When preference relations satisfy completeness, reflexivity and transitivity, one can usually find a representation for them, utulity functions such that:

$$u(\mathbf{x}) \geq u(\mathbf{y}) \Leftrightarrow \mathbf{x} \succsim \mathbf{y}$$

Note that if $u(x_1, x_2) = x_1 x_2$ and we define the *monotonic transformation* $v(x_1, x_2) = \ln(x_1^a x_2^b) = a \ln x_1 + b \ln x_2 = \ln u(x_1, x_2)$, then v describes the same preferences as u since it orders bundles in the same way and in *ordinal utility theory*, that's all we care about. The shape of the utility function defines the shape of indifference curves.

Table 1. Types of indifference curves:

	$u(\mathbf{x})$-form	slope-intercept form	MRS
Cobb-Douglas	$u(\mathbf{x}) = x_1^\alpha x_2^\beta$	$x_2 = (k/x_1^\alpha)^{1/\beta}$ for $k = 1, 2, ...$	decreasing on x_1
substitutes	$u(\mathbf{x}) = ax_1 + bx_2$	$x_2 = \frac{k}{b} - \frac{a}{b} x_1$ for $k = 1, 2, ...$	either 0 or ∞
complements	$u(\mathbf{x}) = \min\{x_1, x_2\}$		always $= -1$

Well behavied preference relations usually are *monotonic*, which implies that their indifference curves have negative slopes. The slope of the indifference curve is known as the *marginal rate of substitution* (MRS) since it measures the rate at which consumers are just willing to substitute consumption of one good for another.

In many cases well behavied preferences are assumed to be *convex*, so that:

$$\begin{aligned} (tx_1 + (1-t)y_1, tx_2 + (1-t)y_2) &\succsim (x_1, x_2) \text{ or} \\ u(tx_1 + (1-t)y_1, tx_2 + (1-t)y_2) &\geq u(x_1, x_2) \end{aligned}$$

Non-convex indifference curves typically lead to specialization or corner solutions, as in, for instance, concave preferences.

Example 7 *One can always take a monotonic transformation of the Cobb-Douglas utility function that make the exponents sum to 1. Let:*

$$u(x_1, x_2) = x_1^a x_2^b$$

raise u *to the power of* $\frac{1}{a+b}$ *and obtain:*

$$\begin{aligned} v(x_1, x_2) &= x_1^{\frac{a}{a+b}} x_2^{\frac{b}{a+b}} \\ &= x_1^{\alpha} x_2^{1-\alpha} \end{aligned}$$

4.3 Budget Constraint

A *budget line* is the set of bundles that cost exactly m the income of the consumer:

$$\{\mathbf{x} \mid \mathbf{p} \cdot \mathbf{x} = m\}$$

while the *budget set* is defined as all the feasible bundles of goods:

$$\{\mathbf{x} \mid \mathbf{p} \cdot \mathbf{x} \leq m\}$$

(generality of the two dimensional case) We can think about the budget set in two dimensions $\mathbf{x} = (x_1, x_2)$ where x_1 =good 1, and x_2 =the amount of money spent on everything else. The price of this last *composite good* is obviously 1 so that the budget set is given by:

$$p_1 x_1 + x_2 \leq m$$

Another way to think of $p_2 = 1$ is of good 2 being a numeraire, since usually what we care about is relative prices. If, on the other hand, $x_2 =$ *good* 2 and $p_2 \neq 1$ and we write the budget line in *slope-intercept form* as:

$$x_2 = \frac{m}{p_2} - \frac{p_1}{p_2} x_1$$

Note that the larger $\frac{p_1}{p_2}$ (good 1 being relatively more expensive), the steeper the budget line becomes.

A budget constraint with a sales tax (τ) on good 1 and a subsidy (σ) on good 2 would look like:

$$(p_1 + \tau)x_1 + (p_2 - \sigma)x_2 = m$$

so that consumers pay τx_1 to the government and $p_1 x_1$ to firms, and firms receive $p_2 x_2$ from consumers and σx_2 from the government.

4.4 Optimal choice

The tangency argument. The tangency argument tells us that a first order *necessary* condition for an optimal choice is:

$$MRS = -\frac{p_1}{p_2}$$

that is, the point where the indifference curve and the budget line are tangent to each other (same slope). In addition if preferences are convex, this is also a *sufficient* condition.

Demand functions. The optimal choice for goods at some set of prices and income is the demanded bundle $\mathbf{x}^* = (x_1^*, x_2^*)$. The relationship that describes how the optimal choice responds to changes in income and prices is a *demand function:* $\mathbf{x}(\mathbf{p}, m)$. This is the result of the constrained optimization problem:

$$\begin{aligned} &\max_x u(\mathbf{x}) \\ &s.t. \\ &\mathbf{p} \cdot \mathbf{x} = m \end{aligned} \tag{15}$$

Some common examples of demand functions under linear constrainst are:

Table 2. Demand functions

	$u(\mathbf{x})$-form	Demand
Cobb-Douglas	$u(\mathbf{x}) = x_1^\alpha x_2^\beta$	$x_1^* = \frac{\beta}{\alpha+\beta}\frac{m}{p_1}$ and $x_2^* = \frac{\alpha}{\alpha+\beta}\frac{m}{p_2}$
substitutes	$u(\mathbf{x}) = ax_1 + bx_2$	$x_i^* = \begin{cases} m/p_i, & \text{if } p_i < p_j \\ \in (0, m/p_i), & \text{if } p_i = p_j \\ 0, & \text{if } p_i > p_j \end{cases}$
complements	$u(\mathbf{x}) = \min\{x_1, x_2\}$	$x_i^* = x_j^* = \frac{m}{p_i + p_j}$

Example 8 *Cobb-Douglas Demand Functions. To derive demand functions from:*

$$u(\mathbf{x}) = x_1^\alpha x_2^\beta$$

one way is to simply equate $MRS = \frac{p_1}{p_2}$ *and use the resulting expression along with the budget constraint to solve for* x_i^*. *In this case we would have:*

$$\frac{\alpha x_2}{\beta x_1} = \frac{p_1}{p_2} \tag{16a}$$

$$p_1 x_1 + p_2 x_2 = m \tag{16b}$$

solve (16a)-(16b) and replace. Alternatively, set up the Lagrangean:

$$\mathcal{L} = x_1^\alpha x_2^\beta + \lambda(m - p_1 x_1 - p_2 x_2)$$

F.O.C. and eliminating λ *yields:*

$$\frac{\alpha x_1}{p_1} = \frac{\beta x_2}{p_2} \tag{17}$$

$$p_1 x_1 + p_2 x_2 = m \tag{18}$$

we can solve the system (17)-(18) and obtain:

$$x_1^* = \frac{\beta}{\alpha+\beta}\frac{m}{p_1} \quad \text{and} \quad x_2^* = \frac{\alpha}{\alpha+\beta}\frac{m}{p_2}$$

Demand function concepts:

- The *income offer curve* is the relationship between the optimal choice of x_1, x_2 as income changes (and prices remain constant).

- The income offer curve results in the *Engle curve* is the relationship between income and a single good x_i. If the good is normal, the Engle curve has positive slope.

- The *price offer curve* conects the optimal choices under different price ratios (and constant income).

- The price offer curve in turn gives rise to the demand curve conecting optimal choice to different price ratios.

Homothetic preferences. This is a case of preferences in which $(x_1, x_2) \succsim (y_1, y_2) \Rightarrow (tx_1, tx_2) \succsim (ty_1, ty_2)$ for any $t > 0$. This implies that optimal choice bundles will be colinear under different levels of income. Perfect complements, substitutes and Cobb-Douglas are all special cases of homothetic preferences.

4.5 Revealed preferences

- **Principle of revealed preferences**. If (x_1, x_2) is chosen when prices are (p_1, p_2) and $p_1x_1 + p_2x_2 \geq p_1y_1 + p_2y_2$ then $(x_1, x_2) \succsim (y_1, y_2)$ or (x_1, x_2) is directly reveled preferred to (y_1, y_2)

- **Weak (strong) axiom of revealed preferences**. If (x_1, x_2) is directly (or indirectly) reveled preferred to (y_1, y_2) and the two bundles are different, it cannot be the case that (y_1, y_2) is directly (or indirectly) reveled preferred to (x_1, x_2)

4.6 Compensated demands & expenditure functions

In general terms, (Marshalian) demand functions as derived above can be represented by the form: $\mathbf{x}(\mathbf{p}, m)$. An alternative way to state the consumer's problem is:

$$\begin{aligned} &\min_x \mathbf{p} \cdot \mathbf{x} \\ &s.t. \\ &u(\mathbf{x}) = u \end{aligned} \tag{19}$$

The first order necessary and sufficient (given diminishing MRS) conditions for the problem are:

$$p_i = \mu \frac{\partial u(\mathbf{x})}{\partial x_i} \text{ for } i = 1, 2 \tag{20a}$$

$$u(\mathbf{x}) = u \tag{20b}$$

solution to this problem gives rise to compensated demand functions $\mathbf{x}(\mathbf{p}, u)$ (or Hicksian demands). The compensated demand functions are constructed by adjusting income as the price changed so as to keep utility constant. This in turn gives rise to the expenditure function, the minimum income that consumers need to spend to attain a certain level of utility under certain prices:

$$e(\mathbf{p}, u) = \mathbf{p} \cdot \mathbf{x}(\mathbf{p}, u)$$

The compensated demands and the expenditure functions are, respectively, homogeneous of degree zero and one in prices. A remarkable propery of the expenditure function is that its derivative w.r.t. prices is just the compensated demand. To see this differentiate w.r.t p_j (apply product rule):

$$\begin{aligned}
\frac{\partial e(\mathbf{p},u)}{\partial p_j} &= x_i(\mathbf{p},u) + \sum_{j=1}^{n} p_j \frac{\partial x_i(\mathbf{p},u)}{\partial p_j} \\
&= x_i(\mathbf{p},u) + \sum_{j=1}^{n} \mu \frac{\partial u(\mathbf{x})}{\partial x_i}\frac{\partial x_i(\mathbf{p},u)}{\partial p_j} && \text{(use F.O.C. (20a))} \\
&= x_i(\mathbf{p},u) && \text{(since we are setting } \partial u(\mathbf{x}) = 0 \text{)}
\end{aligned}$$

Example 9 *Cobb-Douglas preferences. To obtain compensated demands and expenditure function of:*

$$u(\mathbf{x}) = x_1^{\alpha} x_2^{1-\alpha}$$

Using the F.O.C. (20a)-(20b) we solve the system and obtain the compensated demands:

$$x_1^*(\mathbf{p},u) = \left(\frac{\alpha}{1-\alpha}\frac{p_2}{p_1}\right)^{1-\alpha} u \quad \textit{and} \quad x_1^*(\mathbf{p},u) = \left(\frac{1-\alpha}{\alpha}\frac{p_1}{p_2}\right)^{\alpha} u$$

4.7 The Slutsky Equation

If the consumer is solving the expenditure minimization problem (19) and in so doing is spending exactly m, then the same bundle of goods is also solving the utility maximization problem (15). Hence, $x_i(\mathbf{p},u) = x_i(\mathbf{p},m)$ if $m = e(\mathbf{p},u)$ so that:

$$x_i(\mathbf{p},u) = x_i(\mathbf{p}, e(\mathbf{p},u))$$

differentiationg w.r.t. p_i (use chain rule) gives rise to the *Slutsky Equation*:

$$\begin{aligned}
\frac{x_i(\mathbf{p},u)}{\partial p_i} &= \frac{x_i(\mathbf{p},m)}{\partial p_i} + \frac{x_i(\mathbf{p},m)}{\partial m}\frac{e(\mathbf{p},u)}{\partial p_i} \\
\frac{x_i(\mathbf{p},u)}{\partial p_i} &= \frac{x_i(\mathbf{p},m)}{\partial p_i} + \frac{x_i(\mathbf{p},m)}{\partial m} x_i \quad \left(\text{use } \frac{e(\mathbf{p},u)}{\partial p_i} = x_i \text{ from above}\right) \\
\underbrace{\frac{x_i(\mathbf{p},m)}{\partial p_i}} &= \underbrace{\frac{x_i(\mathbf{p},u)}{\partial p_i}} - \underbrace{\frac{x_i(\mathbf{p},m)}{\partial m} x_i} \quad \text{(rearrange)} \\
\textit{price } fx &= \textit{substitution } fx - \textit{income } fx
\end{aligned}$$

That is, the Slutsky Equation breaks a price effect into substitution and income effects. The substitution effect tells us what would happen if prices change but *real* income was to remain unchanged (i.e., compensated for the change in prices); income would be directed towards substituting the more expensive good with more of the cheaper good. The income effect then shows how the reduced (increased) real income now can buy less (more) of both *goods*. Naturally, the size of the price effect depends on the size of these two separate effects.

5 Welfare economics and GEq

If $x(p)$ is demand for good x and $p(x)$ is inverse demand measuring consumer's marginal willingness to pay for the good, then total benefit which consumers receive for consuming x_1 is:

$$CB(x_1) = \int_0^{x_1} p(x)dx$$

that is, the total area under the demand curve. On the other hand, consumer surplus is the area under the curve once one takes away the amount spent on that good, px_1 :

$$CS(x_1) = \int_0^{x_1} p(x)dx - p_1 x_1$$

a fall in prices will increase consumer surplus by: *i)* a fall in price of the goods already being consumer and *ii)* the additional consumption of the good. If the consumer is spending $e(\mathbf{p}, u)$ in n goods and one of the prices falls from p_i' to p_i'', then the compensating variation in his money income is $e(\mathbf{p}', u) - e(\mathbf{p}'', u)$ where $\mathbf{p}'$ and $\mathbf{p}''$ differ only on the *ith*-coordenate. Hence:

$$\begin{aligned} e(\mathbf{p}, u) - e(\mathbf{p}', u) &= \int_{p_i'}^{p_i''} \frac{\partial e(\mathbf{p}, u)}{\partial p_i} dp_i \\ &= \int_{p_i'}^{p_i''} x_i(\mathbf{p}, u) dp_i \quad \text{(using the property of } e(\mathbf{p}, u) \text{ above)} \end{aligned}$$

6 Choice under uncertainty

6.1 Insurance

Insurance provides a way to change the probability distribution of outcomes under different states of nature. Suppose that an agent has wealth w and faces the probability π of a loss.L He can buy insurance for the amount K at a price proportional to the amount insured: γK. Therefore, his expected income if he buys insurance is:

$$\begin{gathered} w - L + K - \gamma K \text{ with probability } \pi \\ w - \gamma K \text{ with probability } (1 - \pi) \end{gathered}$$

therefore, he is able to forego one unit of wealth in the good state for one unit of wealth in the bad state at the rate:

$$\frac{\Delta C_G}{\Delta C_B} = \frac{\gamma}{1 - \gamma}$$

6.2 Expected utility and risk aversion

If π_1 is the probability of consuming w_1 and π_2 is the probability of consuming w_2 , a utility function with the particular form:

$$u(\pi_1, \pi_2, w_1, w_2) = \pi_1 v(w_1) + \pi_2(w_2)$$

is called a Von Neuman-Morgenstern or expected utility function.

An individual is said to be risk averse if:

$$u(\pi_1 w_1 + \pi_2 w_2) > \pi_1 u(w_1) + \pi_2 u(w_2)$$

that is, if his expected utility of wealth is less than the utility of expected wealth. i.e., the risk averse individual has a concave utility function. ($w \mapsto u(w)$ is concave). If $\pi_1 = \pi$ (probability of loss) and $\pi_2 = 1 - \pi$ (no loss probability), an individual is said to face "fair" premium for insurance if $\pi = \gamma$. This happens when insurance companies just breakeven. In such circumstances, the optimal amount of insurance is determined by:

$$\begin{aligned} \frac{\pi \Delta u(w_1)/\Delta w_1}{(1 - \pi)\Delta u(w_2)/\Delta w_1} &= \frac{\pi}{1 - \pi} \\ \frac{\Delta u(w_1)}{\Delta w_1} &= \frac{\Delta u(w_2)}{\Delta w_2} \end{aligned} \tag{21}$$

so that the marginal utility of an extra dollar of income if the loss occurs should be equal to the marginal utility of an extra dollar if the loss doesn't occur. If the consumer is risk averse, ($u()$ is concave) we have that $w_1 > w_2 \Rightarrow u(w_1) < u(w_2)$ and viceversa. Thus, (21) is satisfied IIF $w_1 = w_2$ which happens only under full insurance. Summarizing, under "fair" premium, the optimal insurance level is always full insurance.

Example 10 *Suppose a risk averse individual considers investing in one asset which return varies accross states of nature: r_g in the good state of nature with probability π and r_g in the bad state of nature with probability $(1-\pi)$. Therefore expected wealth is*

$$W_E = \pi(w + xr_g) + (1-\pi)(w + xr_b)$$

The consumer's problem is:

$$\max_{x \leq w} f(x)$$

and expected utility is:

$$f(x) = \pi u(w + xr_g) + (1-\pi)u(w + xr_b) = EU(x)$$

the F.O.C.

$$f'(x) = \pi u'(w + xr_g)r_g + (1-\pi)u'(w + xr_b)r_g = 0$$

gives the optimal choice x^* to invest in the risky asset. Note that $x^* > 0$ whenever $\pi r_g + (1-\pi)r_b > 0$, (i.e., whenever the expected return of the asset is positive). The second-order condition:

$$f'(x) = \pi u''(w + xr_g)r_g^2 + (1-\pi)u''(w + xr_b)r_g^2$$

is unambiguously negative since $u''(w) < 0$ by the assumption of risk aversion.

6.3 Risky assets and mean-variance utility

Let w take S states of nature with probabilities π_i. The probability distribution of wealth (since it is a discrete RV) can be described by its mean:

$$\mu_w = \sum_{i=1}^{S} \pi_i w_i$$

and variance:

$$\sigma_w^2 = \sum_{i=1}^{S} \pi_i (w_i - \mu_w)^2$$

so that the utility of a probability distribution can be expressed in terms $u(\mu_w, \sigma_w^2)$. In this case risk aversion would mean that $\mu_w \mapsto u(\mu_w, \sigma_w^2)$ is increasing and $\sigma_w^2 \mapsto u(\mu_w, \sigma_w^2)$ decreasing.

Example 11 *Suppose an individual considers investing on a riskless asset with return r_f and a risky asset with expected return r_m and variance σ_m^2. The investor allocates amount x into the risky asset ro its expected reutrn is:*

$$r_x = xr_m + (1-x)r_f$$

while the variance of his portfolio will be:

$$\sigma_x^2 = x^2\sigma_m^2$$

in the two polar cases or corner solutions one has:

$$\begin{aligned} u(r_x, \sigma_x^2) &= u(r_m, \sigma_m^2) \quad \text{if } x = 1 \\ u(r_x, \sigma_x^2) &= u(r_f, 0) \quad \text{if } x = 0 \end{aligned}$$

on a $r_x - \sigma_x^2$ space, indifference curves will be convex and have positive slopes. An optimal portfolio choice between risky and riskless assets will be characterized by:

$$\underbrace{\frac{\Delta u(\cdot)/\Delta\sigma}{\Delta u(\cdot)/\Delta\mu}} = \underbrace{\frac{r_m - r_f}{\sigma_m}}$$

risk-return MRS = price of risk

7 References

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