A dynamic movement of the spot rate such as the interest rate has been modeled by the following stochastic differential equation

\[ dr = (\alpha + \beta r)dt + \sigma(r, t)dW \]  

(1)

where \( r (r > 0) \) is the interest rate and \( W \) is a standard Brownian motion. The diffusion parameter \( \sigma(r, t) \) is often specified by

\[ \sigma(r, t) = \sigma r^\gamma. \]

If \( \gamma = 0 \) then equation (1) becomes Vasicek’s model \(^1\) If \( \gamma = .5 \) then it becomes CIR (Cox, Ingersoll and Ross) model. If \( \alpha > 0 \) and \( \beta < 0 \), equation (1) becomes a continuous time AR(1) process with conditional heteroscedastic errors. An intuitive interpretation of the parameters of (1) is that \( \beta \) captures the mean-reverting effect; \( \sigma^2 \) is the volatility of the diffusion process, and \( \gamma \) measures the elasticity of volatility with respect to the level of the interest rate.

Feller (1951), in his study of a parabolic equation, proved that if \( \alpha > 0 \) and \( \beta < 0 \), the CIR diffusion equation has a unique fundamental solution and displays mean reversion. The CIR diffusion process is said to be positive preserving if \( \alpha \geq \gamma \sigma^2 \), since an upward drift is sufficiently large to make the origin inaccessible.

---

Given equation (7) the likelihood function is given by

\[
\ell(\alpha, \beta, \sigma, \gamma | \text{data}) \propto \sigma^{-n} \prod_{t=1}^{n} r_{t-1}^{-\gamma} \exp \left\{ -\frac{1}{2} \left( \nu s^2(\gamma) + (\theta - \hat{\theta})' X' D^{-1} X (\theta - \hat{\theta}) \right) \right\} \tag{8}
\]

where

\[
\theta = (\alpha, \beta)', \quad \hat{\theta} = (X' D^{-1} X)^{-1} X' D^{-1} y, \quad \nu s^2 = (y - X \hat{\theta})' X' D^{-1} X (y - X \hat{\theta})
\]

\[
y = \begin{bmatrix} r_1 - r_0 \\ \\ \\ r_n - r_{n-1} \end{bmatrix}, \quad X = \begin{bmatrix} 1 & r_0 \\ 1 & r_1 \\ \vdots & \vdots \\ 1 & r_{n-1} \end{bmatrix}, \quad D = \begin{bmatrix} r_0^\gamma & 0 & 0 \\ 0 & \ddots & \vdots \\ 0 & \cdots & r_{n-1}^\gamma \end{bmatrix}
\]

Let the prior probability density function (pdf) be given by

\[
p(\alpha, \beta, \gamma, \sigma) \propto \sigma^{-1}. \tag{9}
\]

Then the posterior pdf of \( \gamma \) is given by

\[
p(\gamma | \text{data}) \propto \prod_{t=1}^{n} r_{t-1}^{-\gamma} |X' D^{-1} X|^{-\frac{1}{2}} (\nu s^2)^{-\frac{n-2}{2}}. \tag{10}
\]

Let us choose the posterior mean \( \hat{\gamma} \) of \( \gamma \) as the point estimate. Then given \( \hat{\gamma} \), the point estimates of \( \theta \) and of \( \sigma \) may be given by

\[
\hat{\theta} = (X' D^{-1} X)^{-1} X' D^{-1} y \quad \text{and} \quad \hat{\sigma}^2 = (y - X \hat{\theta})' X' D^{-1} X (y - X \hat{\theta}).
\]
Short term Interest Rate in Japan

In Japan uncollateralized overnight call rate, or call rate in short, is the representative data for the short term interest rate. The name call market comes from “money at call,” that implies that it is a market for financial institutions to adjust net funds in hand. The uncollateralized overnight call rate was begun on July 31 1985. Since 1999 the Bank of Japan has adopted the so called zero-interest rate policy, we end the sample period at October 31 1998.

Figure 1 presents the uncollateralized overnight call rate and the official discount rate from July 31 1985 to October 31 1998. We see from Figure 1 that the official discount rate was lowered from 1.0% to .5% in September 8 1995, indicating that virtually from this time on the zero interest rate period has begun.

\[ \Delta r_t = \alpha + \beta r_t + \sigma |r_{t-1}|^\gamma u_t \]  
(11)

\[ \Delta r_t = \alpha + \beta r_t + \delta \Delta z_t + \sigma |r_{t-1}|^\gamma u_t \]  
(12)

In equation (2) we include the difference of the discount rate, \( \Delta z_t \), as the observable jump.

We present the estimation results for (i) assuming that the error term is white noise

\[ u_t \sim WN(0,1), \]

and (ii) assuming that it follows an ARMA–GARCH process

\[ u_t = \frac{\Theta(B)}{\Phi(B)} \epsilon_t \]

\[ \sigma_t^2 = \alpha_0 + \sum_{j=1}^{r} \alpha_j \epsilon_{t-j}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2 \]

where \( B \) is the backward shift operator. For the white noise error, we use as the estimation methods the constrained MLE, GMM, and Bayesian procedure. For the ARMA–GARCH process we use the Markov Chain Monte Carlo (MCMC) with Metropolis-Hastings algorithms. The MCMC algorithms are explained in the appendix.

Table 1 presents the estimated results.
Table 1: Estimated Results of the Call Rate Models

<table>
<thead>
<tr>
<th></th>
<th>White Noise</th>
<th>ARMA(2,1)–GARCH(1,2)</th>
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<tbody>
<tr>
<td></td>
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<td>Bayes</td>
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<td>.0014 (0.0012)</td>
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<tr>
<td>β</td>
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<td>.0504</td>
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<td>β₁</td>
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</tr>
<tr>
<td>β₂</td>
<td>.2724 (0.0727)</td>
<td>.2840 (0.0784)</td>
</tr>
</tbody>
</table>
1 MCMC Algorithm for the Parameters of CKLS Model with ARMA($p, q$)-GARCH($r, s$) Error Terms

Let us explain the MCMC algorithm for the CKLS model with ARMA-GARCH error terms. The model is

\[ y_t = x_t \gamma + |r_{t-1}|^\lambda u_t \]  \hspace{1cm} \text{(13)}

\[ u_t = \frac{\Theta(B)}{\Phi(B)} \epsilon_t \]  \hspace{1cm} \text{(14)}

\[ \sigma_t^2 = \alpha_0 + \sum_{j=1}^r \alpha_j \epsilon_{t-j}^2 + \sum_{j=1}^s \beta_j \sigma_{t-j}^2 \]  \hspace{1cm} \text{(15)}

where \( \lambda \geq 0, \alpha_0 > 0, \alpha_j \geq 0, (j = 1, \ldots, r), \beta_j \geq 0, (j = 1, \ldots, s) \)

Substituting (2) into (1) we obtain

\[ \epsilon_t = y_t - g(Z_t) \]

\[ g(Z_t) = x_t \gamma - \sum_{j=1}^p \phi_j \epsilon_{t-j} - \sum_{j=1}^q \theta_j \epsilon_{t-j} \]

\[ \epsilon_t = y_t - x_t \gamma \]

\[ \sigma_t^2 = \alpha_0 + \sum_{j=1}^r \alpha_j \epsilon_{t-j}^2 + \sum_{j=1}^s \beta_j \sigma_{t-j}^2 \]

From equation (5) we obtain the probability density function (pdf) of the doubly truncated \( y_t \) as

\[ f(y_t \mid Z_t) = \frac{1}{\sigma_t |r_{t-1}|^\lambda} \phi \left( \frac{y_t - g(Z_t)}{\sigma_t} \right) \]  \hspace{1cm} \text{(16)}
and the likelihood function becomes

\[ L = \prod_{t=1}^{n} \frac{1}{\sigma_t | r_{t-1}^\lambda} \phi \left( \frac{y_t - g(Z_t)}{\sigma_t} \right) \]  

(17)

Let the prior pdf of the parameters \( \lambda, \gamma', \phi', \theta', \alpha' \) and \( \beta' \) be given by

\[
\pi(\lambda, \gamma, \phi, \theta, \alpha, \beta) = N(\lambda_0, \sigma_\lambda) \times N(\gamma_0, \Sigma_\gamma) \times N(\phi_0, \Sigma_\phi) \times N(\theta_0, \Sigma_\theta) \times N(\alpha_0, \Sigma_\alpha) \times N(\beta_0, \Sigma_\beta) 
\]  

(18)

We do not impose conditions that the roots of \( \Phi(B) \) and of \( \Theta(B) \) are outside of the unit circle and thus allowing for nonstationarity and noninvertibility.

The posterior pdf is then given by

\[
p(\lambda, \gamma, \phi, \theta, \alpha, \beta | \text{data}) = \pi(\lambda, \gamma, \phi, \theta, \alpha, \beta) \prod_{t=1}^{n} \frac{1}{\sigma_t | r_{t-1}^\lambda} \phi \left( \frac{y_t - g(Z_t)}{\sigma_t} \right) 
\]  

(19)

Bollerslev (1986) shows that the model can be written as ARMA(l, s) process. Let \( w_t = \epsilon_t^2 - \sigma_t^2 \). We can rewrite (11) as ARMA process of \( \epsilon_t^2 \):

\[
\epsilon_t^2 = \alpha_0 + \sum_{j=1}^{l} (\alpha_j + \beta_j) \epsilon_{t-j}^2 + w_t - \sum_{j=1}^{s} \beta_j w_{t-j} 
\]  

(20)

where \( l = \max(r, s) \), \( \alpha_j = 0 \) for \( j > r \) and \( \beta_j = 0 \) for \( j > s \). It is easy to show that

\[
w_t/\sigma_t^2 + 1 \sim \chi^2(1)
\]

Therefore expectation and variance of \( w_t \) conditional on information at \( t-1 \) are

\[
E(w_t | \mathcal{I}_{t-1}) = 0 \\
\text{var}(w_t | \mathcal{I}_{t-1}) = 2\sigma_t^4 
\]

Nakatsuma (2000a) approximates \( w_t \) distribution with \( N(0, 2\sigma_t^4) \) and for their MCMC algorithm this approximation works well. We will use the same approximation in construction MCMC algorithm for doubly-truncated model.

Let us construct Metropolis-Hastings (MH) algorithm for the parameters of the doubly truncated ARMA-GARCH regression model. We start from explaining the difference between our algorithm and other algorithms in literature.

First, compared to the MCMC algorithms for untruncated ARMA and ARMA-GARCH models (Chib and Greenberg (1994), Billio et. al. (1999), and Nakatsuma (2000a), among others) where the parameters are drawn
using the means of the proposal densities (let us call this non-random walk draw), the key feature of our MH algorithms is the random walk Markov chain (see Robert and Casella (1999), p.245). This random walk MH algorithm is necessary since if we ignore double truncation and draw parameters by the non-random walk draw, then draws for some of the parameters (such as $\gamma$ and $\phi$) are consistently off from the true values, leading to a very low acceptance rate in the MH algorithm.

The second important difference compared to Chib and Greenberg (1994) and Nakatsuma (2000a) is that we do not impose the stationarity or invertibility constraints. The MH algorithm works well for nonstationary time series, and we can directly test the unit root hypothesis from the posterior pdf of $\rho$, the maximum absolute value of the roots of $\Phi(B)$. See Goldman, Radchenko, Nakatsuma and Tsurumi (2000b) further on this point.

Third, our algorithms compared to those suggested by Chib and Greenberg (1994) for untruncated ARMA model and by Nakatsuma (2000a) for untruncated ARMA-GARCH model are simpler and computational time is reduced since we do not use nonlinear least squares (NLLS) estimation in MA and GARCH blocks.

Our MH algorithm consists of separate blocks for (i) regression parameters, $\gamma$; (ii) AR coefficients $\phi$, (iii) MA coefficients, $\theta$; (iv) ARCH coefficients, $\alpha$, and (v) GARCH coefficients $\beta$. We describe the procedure below.

(i) Choose initial values for $\gamma$, $\phi$, $\theta$, $\alpha$ and $\beta$. Let them be denoted by $\gamma^{(0)}$, $\phi^{(0)}$, $\theta^{(0)}$, $\alpha^{(0)}$ and $\beta^{(0)}$.

(ii) Let the superscript $(i)$ denote the $i$-th draw. We draw parameters block by block.

(a) **Regression coefficient block** Generate the $k$ elements of $\gamma^{(i)}$ from the multivariate normal distribution

$$\mathcal{N}(\gamma^{(i-1)}, \Sigma_{\gamma^{(i-1)}})$$

where $\Sigma_{\gamma^{(i-1)}}$ is the $k \times k$ covariance matrix defined in Appendix. If $\gamma^{(i)}$ satisfy

$$\Phi(b_t) - \Phi(a_t) > 0$$

where $a_t$ and $b_t$ are evaluated by $\gamma^{(i)}$, $\phi^{(i-1)}$, $\theta^{(i-1)}$, $\alpha^{(i-1)}$ and $\beta^{(i-1)}$, then accept $\gamma^{(i)}$ with probability

$$\lambda_1 = \min \left\{ \frac{p(\gamma^{(i)}, \phi^{(i-1)}, \theta^{(i-1)}, \alpha^{(i-1)}, \beta^{(i-1)}|\text{data})}{p(\gamma^{(i-1)}, \phi^{(i-1)}, \theta^{(i-1)}, \alpha^{(i-1)}, \beta^{(i-1)}|\text{data})}, 1 \right\}.$$ 

Otherwise set $\gamma^{(i)} = \gamma^{(i-1)}$. Since we are making the random walk draw, the proposal densities become symmetric, and the ratio of them is unity.
(b) **AR coefficient block** Generate the $p$ elements of $\phi^{(i)}$ from
\[ N(\phi^{(i-1)}, \Sigma_{\phi^{(i-1)}}) \]
where $\Sigma_{\phi^{(i-1)}}$ is defined in Appendix. If $\phi^{(i)}$ satisfy
\[ \Phi(b_{i}) - \Phi(a_{i}) > 0 \]
where $a_{i}$ and $b_{i}$ are evaluated by $\gamma^{(i)}$, $\phi^{(i)}$, $\theta^{(i-1)}$, $\alpha^{(i-1)}$ and $\beta^{(i-1)}$, then accept $\phi^{(i)}$ with probability
\[ \lambda_{2} = \min \left\{ \frac{p(\gamma^{(i)}, \phi^{(i)}, \theta^{(i-1)}, \alpha^{(i-1)}, \beta^{(i-1)}|\text{data})}{p(\gamma^{(i)}, \phi^{(i-1)}, \theta^{(i-1)}, \alpha^{(i-1)}, \beta^{(i-1)}|\text{data})}, 1 \right\}. \]
Otherwise set $\phi^{(i)} = \phi^{(i-1)}$.

For the unit root test we compute
\[ \rho^{(i)} = \max(\text{absolute values of the } p\text{-roots of } \Phi(B)) \]
using $\phi_{1}^{(i)}, \ldots, \phi_{p}^{(i)}$.

(c) **MA coefficient block** Generate the $q$ elements of $\theta^{(i)}$ from
\[ N(\theta^{(i-1)}, \Sigma_{\theta^{(i-1)}}) \]
where $\Sigma_{\theta^{(i-1)}}$ is defined in Appendix. If $\theta^{(i)}$ satisfy
\[ \Phi(b_{t}) - \Phi(a_{t}) > 0 \]
where $a_{t}$ and $b_{t}$ are evaluated by $\gamma^{(i)}$, $\phi^{(i)}$, $\theta^{(i)}$, $\alpha^{(i-1)}$ and $\sigma^{2(i-1)}$, then accept $\theta^{(i)}$ with probability
\[ \lambda_{3} = \min \left\{ \frac{p(\gamma^{(i)}, \phi^{(i)}, \theta^{(i)}, \alpha^{(i-1)}, \beta^{(i-1)}|\text{data})}{p(\gamma^{(i)}, \phi^{(i-1)}, \theta^{(i-1)}, \alpha^{(i-1)}, \beta^{(i-1)}|\text{data})}, 1 \right\}. \]
Otherwise set $\theta^{(i)} = \theta^{(i-1)}$.

(d) **ARCH block** Generate $r + 1$ elements $\alpha^{(i)}$ from
\[ N(\alpha^{(i-1)}, \Sigma_{\alpha^{(i-1)}}) \]
where $\Sigma_{\alpha^{(i-1)}}$ is defined in Appendix. If $\alpha^{(i)}$ satisfy
\[ \alpha^{(i)} > 0 \]
\[ \Phi(b_{t}) - \Phi(a_{t}) > 0 \]
where $a_{t}$ and $b_{t}$ are evaluated by $\gamma^{(i)}$, $\phi^{(i)}$, $\theta^{(i)}$, $\alpha^{(i)}$ and $\beta^{(i-1)}$, then accept $\alpha^{(i)}$ with probability
\[ \lambda_{4} = \min \left\{ \frac{p(\gamma^{(i)}, \phi^{(i)}, \theta^{(i)}, \alpha^{(i)}, \beta^{(i-1)}|\text{data})}{p(\gamma^{(i)}, \phi^{(i)}, \theta^{(i)}, \alpha^{(i-1)}, \beta^{(i-1)}|\text{data})}, 1 \right\}. \]
Otherwise set $\alpha^{(i)} = \alpha^{(i-1)}$. 

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(e) **GARCH block** Generate $s$ elements of $\beta^{(i)}$ from

$$N(\beta^{(i-1)}, \Sigma_{\beta^{(i-1)}})$$

where $\Sigma_{\beta^{(i-1)}}$ is defined in Appendix. If $\beta^{(i)}$ satisfy

$$\beta^{(i)} > 0$$

$$\Phi(b_t) - \Phi(a_t) > 0$$

where $a_t$ and $b_t$ are evaluated by $\gamma^{(i)}$, $\phi^{(i)}$, $\theta^{(i)}$, $\alpha^{(i)}$ and $\beta^{(i)}$, then accept $\beta^{(i)}$ with probability

$$\lambda_5 = \min \left\{ \frac{p(\gamma^{(i)}, \phi^{(i)}, \theta^{(i)}, \alpha^{(i)}, \beta^{(i)}|data)}{p(\gamma^{(i)}, \phi^{(i)}, \theta^{(i)}, \alpha^{(i)}, \beta^{(i-1)}|data)}, 1 \right\}.$$  

Otherwise set $\beta^{(i)} = \beta^{(i-1)}$.

(iii) We make $N$ draws of the parameters in each of the five blocks, and we burn the first $m$ draws. Out of the remaining $N - m$ draws, we keep every $r$-th draw. We check convergence by testing that the draws attain mean and covariance stationarity.

**Remark:** Chib and Greenberg (1994) generate the initial $p$ values of $u_t$ and $q$ values of $\epsilon_t$ using the Kalman smoothing algorithm in state space representation. Nakatsuma (2000a) draws initial error directly from a proposed Gaussian distribution. We set these initial values at zero, since we have found that the parameter draws are insensitive to the initial values of $u_t$ and $\epsilon_t$'s.

### Proposal Densities for ARMA-GARCH Model

The likelihood function for the doubly truncated ARMA($p, q$)-GARCH($r, s$) model is given by:

$$L = \frac{1}{\prod_{t=1}^{n} \sigma_t} \phi \left( \frac{y_t - g(Z_t)}{\sigma_t} \right)$$

where $\sigma_t$ is a function of $\alpha$ and $\beta$ defined in equation (3) in the text. We use the multivariate normal distribution as the proposal density for each block. Let us explain how the proposal densities are given.

If we ignore truncation, the ARMA($p, q$)-GARCH($r, s$) model is given by
\[
y_t = \epsilon_t + g(Z_t) = x_t \gamma - \sum_{j=1}^{p} \phi_j \epsilon_{t-j} + \epsilon_t + \sum_{j=1}^{q} \theta_j \epsilon_{t-j} \tag{22}
\]
\[
\sigma_t^2 = \alpha_0 + \sum_{j=1}^{r} \alpha_j \epsilon_{t-j}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2 \tag{23}
\]

Nakatsuma (2000) draws \( \epsilon_0 \) from the normal density. Following Goldman, Radchenko, Nakatsuma and Tsurumi (2000b) we set initial errors to zero.

**Proposal density for** \( \gamma^{(i)} \): Equation (25) can be re-written as
\[
y_t^* = x_t^* \gamma + \epsilon_t \tag{24}
\]
where
\[
y_t^* = y_t - \sum_{j=1}^{p} \phi_j y_{t-j} - \sum_{j=1}^{q} \theta_j y_{t-j} \\
x_t^* = x_t - \sum_{j=1}^{p} \phi_j x_{t-j} - \sum_{j=1}^{q} \theta_j x_{t-j}
\]
where \( y_t = y_t^* = 0 \) and \( x_t = x_t^* = 0 \) for \( t \leq 0 \). Equations (27) and (26) suggests that the proposal density is
\[
\gamma^{(i)} \sim N \left( \gamma^{(i-1)}, \Sigma_{\gamma^{(i-1)}} \right)
\]
\[
\Sigma_{\gamma^{(i-1)}} = s_1^2 \left( X_{\gamma^{(i-1)}}^\prime \Sigma_{\gamma^{(i-1)}}^{-1} X_{\gamma^{(i-1)}} + \Sigma_{\gamma}^{-1} \right)^{-1}
\]
\( X_{\gamma^{(i-1)}} = (x_1^*, \ldots, x_n^*)^\prime \) and \( \Sigma_{\gamma^{(i-1)}}^{-1} \) is a diagonal variance-covariance matrix of \( \epsilon_t \) with \( \sigma_t^{(i-1)} \) computed from (26) using (i-1)th draws \( \alpha^{(i-1)} \) and \( \beta^{(i-1)} \). Parameter \( s_1 \) affects acceptance rate and correlation among draws. If acceptance rate is above 40% we set \( s_1 = 1.0 \).

**Proposal density for the AR coefficients**, \( \phi^{(i)} \): Given \( \Sigma_{(i-1)} \) the ARMA\((p, q)\) part of the model can be written as
\[
\tilde{y}_t = \tilde{x}_t \phi + \epsilon_t \tag{25}
\]
where
\[
\tilde{y}_t = y_t - x_t \gamma - \sum_{j=1}^{q} \theta_j \tilde{y}_{t-j} \\
\tilde{x}_t = [\tilde{y}_{t-1}, \ldots, \tilde{y}_{t-p}]\]
and $y_t = \tilde{y}_t = 0$ for $t \leq 0$. Equation (28) suggests the following proposal density for $\phi^{(i)}$:

$$
\phi^{(i)} \sim N \left( \phi^{(i-1)}, \Sigma_{\phi^{(i-1)}} \right)
$$

where

$$
\Sigma_{\phi^{(i-1)}} = \frac{s_2}{2} \left( X_{\phi}^{'} \Sigma_{(i-1)}^{-1} X_{\phi} + \Sigma_{\phi}^{-1} \right)^{-1}
$$

and $X_{\phi} = [\tilde{x}_1', \ldots, \tilde{x}_n]'$. We set $s_2=1$.

**Proposal density for the MA coefficients $\theta^{(i)}$:** From equation (25) we have

$$
y_t^\dagger = y_t - x_t\gamma - \sum_{j=1}^{p} \phi_j (y_{t-j} - x_{t-j}\gamma) - \sum_{j=1}^{q} \theta_j y_{t-j}
$$

(26)

where $y_t = y_t^\dagger = 0$ for $t \leq 0$. Hence, we may use the following proposal density for $\theta^{(i)}$

$$
\theta^{(i)} \sim N \left( \theta^{(i-1)}, \Sigma_{\theta^{(i-1)}} \right)
$$

where

$$
\Sigma_{\theta^{(i-1)}} = \frac{s_3}{2} \left( X_{\theta}^{'} \Sigma_{(i-1)}^{-1} X_{\theta} + \Sigma_{\theta}^{-1} \right)^{-1}
$$

$$
X_{\theta} = (x_1^\dagger, \ldots, x_n^\dagger), \quad x_t^\dagger = (y_t^\dagger, \ldots, y_{t-q}^\dagger).
$$

We set $s_3=1$.

**Proposal density for the ARCH coefficients $\alpha^{(i)}$:** We use equation (15) and transformation similar to transformation in AR block. ² We can rewrite equation (15) as

$$
\bar{\epsilon}_t^2 = \zeta_t \alpha + w_t
$$

(27)

where

$$
\bar{\epsilon}_t^2 = \epsilon_t^2 + \sum_{j=1}^{n} \beta_j \bar{\epsilon}_{t-j}^2
$$

$$
\tau_t = 1 + \sum_{j=1}^{n} \beta_j \tau_{t-j}
$$

$$
\bar{\epsilon}_t^2 = \epsilon_t^2 - \sum_{j=1}^{n} \beta_j \bar{\epsilon}_{t-j}^2
$$

$$
\zeta_t = (\tau_t, \bar{\epsilon}_{t-1}^2, \ldots, \bar{\epsilon}_{t-r}^2)
$$

²The difference is in the sign of coefficient in MA part and presence of constant term $\alpha_0$ in equation (15).
and $\bar{e}_t = 0$ for $t \leq 0$. We can write likelihood using approximated $N(0, 2\sigma_t^4)$ distribution for $w_t$. Hence we may use the following proposal density for $\alpha_i$

$$\alpha^{(i)} \sim N\left(\alpha^{(i-1)}, \Sigma_{\alpha^{(i-1)}}\right)$$

where

$$\Sigma_{\alpha^{(i-1)}} = s_4^2 \left( X_{\alpha}^t \Lambda_{(i-1)}^{-1} X_{\alpha} + \Sigma_{\alpha}^{-1} \right)^{-1}$$

$$X_{\alpha} = (\zeta_1, \cdots, \zeta_n).$$

and $\Lambda_{(i-1)}$ is diagonal matrix $\text{diag}(2\sigma_1^4, \cdots, 2\sigma_n^4)$ with $\sigma(.)$’s evaluated using $\alpha^{(i-1)}$ and $\beta^{(i-1)}$. We set $s_4=1$ when acceptance rate is reasonable, and we set $s_4$ when acceptance rate is small.

**Proposal density for the GARCH coefficients $\beta^{(i)}$:** We use equation (10) and transformation similar to transformation in MA block.

$$y_t^\beta = \epsilon_t^2 - \alpha_0 - \sum_{j=1}^{l} (\alpha_j + \beta_j) \epsilon_{t-j}^2 + \sum_{j=1}^{s} \beta_j y_{t-j}^\beta$$

$$x_t^\beta = -(y_{t-1}^\beta, \cdots, y_{t-s}^\beta).$$

where $y_t^\beta = 0$ for $t \leq 0$. Hence we may use the following proposal density for $\beta^i$

$$\beta^{(i)} \sim N\left(\beta^{(i-1)}, \Sigma_{\beta^{(i-1)}}\right)$$

where

$$\Sigma_{\beta^{(i-1)}} = s_5^2 \left( X_{\beta}^t \Lambda_{(i-1)}^{-1} X_{\beta} + \Sigma_{\beta}^{-1} \right)^{-1}$$

$$X_{\beta} = (x_1^\beta, \cdots, x_n^\beta).$$

and $\Lambda_{(i-1)}$ is diagonal matrix $\text{diag}(2\sigma_1^4, \cdots, 2\sigma_n^4)$ with $\sigma(.)$’s evaluated using $\alpha^{(i)}$ and $\beta^{(i-1)}$. We set $s_5=1$. 
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