(10) 1. (An example that $\text{Cov}(X, Y) = 0 \neq X, Y$ independent. Also, $\text{Cov}(X, Y) = 0 \neq \text{E}(Y|X) \neq 0$.) Let $X$ and $Z$ be two independently distributed standard normal random variables. Let $Y = X^2 + Z$.

(a) Show that $\text{E}(Y|X) = X^2$.

(b) Show that $\text{E}(Y) = 1$.

(c) Show that $\text{E}(XY) = 0$

(d) Show that $\text{Cov}(X, Y) = 0$

(10) 2. Show $\text{E}(Y|X) = \text{E}Y \implies \text{Cov}(X, Y) = 0$.

(10) 3. In the Remark on p.3 of the answers to Assignment #2, I pointed out that there are cases where

$$\text{E}\left(\frac{Z}{W}\right) = \frac{\text{EZ}}{\text{EW}}.$$  

One such a case is a ratio of quadratic forms in normal variates. Let

$$R = \frac{\lambda_1 \eta_1^2 + \lambda_2 \eta_2^2 + \cdots + \lambda_m \eta_m^2}{\eta_1^2 + \eta_2^2 + \cdots + \eta_m^2} = \frac{U}{V}, \quad \eta = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_m \end{bmatrix} \sim \text{N}(0, \sigma^2 I).$$

Show that (i) $R$ is a homogeneous degree zero function of independent gamma variables, and (ii) $R$ and $V$ are independent of each other. (See Remark 1 below. Hence

$$\text{E}R = \frac{\text{EU}}{\text{EV}}.$$  

Remark 1: Pitman’s theorem “the sum of independent gamma variables and a homogeneous degree zero function of them are independent.”

(10) 4. Let us have the classical linear regression model

\[ y = X\beta + \epsilon \]

where \( y, X, \beta, \) and \( \epsilon \) are defined on p.51 of the Lecture Notes. Prove

\[
E(\hat{\beta}) = \beta \\
\text{Var}(\hat{\beta}|X) = \sigma^2(X'X)^{-1}
\]

using the assumptions

\[
\begin{align*}
E(\epsilon|X) &= 0 \\
E(\epsilon\epsilon'|X) &= \sigma^2I
\end{align*}
\]

(40) 5. Generate the data as follows:

\[ \ln y_i = \beta_1 + \beta_2 x_{i2} + \beta_3 x_{i3} + \epsilon_i, \quad i = 1, \cdots, n \quad \epsilon_i \sim N(0, \sigma^2), \]

where

\[
n = 50, \quad \sigma = .35, \quad \beta = \begin{bmatrix} 1 \\ -0.5 \\ 1.3 \end{bmatrix}, \quad \text{seed} = 123456789;
\]

\[
x_{i2} \sim U(0, 3) \quad i = 1, \cdots, n + 1 \\
x_{i3} = \text{seqa}(1, 1, n + 1) \ast .01 + \text{rndu}(n + 1, 1);
\]

Using Simpson’s rule, obtain the predictive densities of

(a) \( \ln y_{n+1} \);
(b) \( y_{n+1} \).

Discuss the predictive densities.