Economics of Capital Markets

Bond Duration and Convexity

Major Topics:

Economics of Capital Markets

Introduction
Bond Duration and Convexity
Introduction

- We already know the relationship between bond prices and yields (or required rates of return)
  - Some others may be new

1. The value of a bond is inversely related to changes in the investor’s required rate of return.
   - The higher the required rate of return, the lower the bond value.
1. Prices and returns on bonds are inversely related

\[
P_0^B = \frac{C}{y} \left[ 1 - \frac{1}{(1 + y)^n} + \frac{F}{(1 + y)^n} \right]
\]

2. The market value of a bond will be less than the par value if the investor’s required rate is above the coupon interest rate.
3. Long-term bonds have greater interest rate risk than short term bonds.
3. **Corollary**: Low coupon bonds have greater interest rate sensitivity than high coupon bonds.

The sensitivity of a bond’s value to changing interest rates depends on both the length of time to maturity and on the pattern of cashflows provided by the bond.
We want to know how the price of a bond changes as the yield changes.
To derive the elasticity, use

\[ p_0^B = \sum_{t=1}^{n} \frac{C}{(1 + y)^t} + \frac{F}{(1 + y)^n} \]

Take the first derivative with respect to \( y \) to get

\[
\frac{dp_0^B}{dy} = \frac{(-1)C}{(1+y)^2} + \frac{(-2)C}{(1+y)^3} + \ldots + \frac{(-n)C}{(1+y)^{n+1}} + \frac{(-n)F}{(1+y)^{n+1}}
\]

\[
= -\frac{1}{1+y} \left[ \frac{1C}{(1+y)^1} + \frac{2C}{(1+y)^2} + \ldots + \frac{nC}{(1+y)^n} + \frac{nF}{(1+y)^n} \right]
\]
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Bond Duration and Convexity

Duration (Continued)

- Multiplying through by $1/P_0^B$

$$\frac{dP_0^B}{dy} \frac{1}{P_0^B} = -\frac{1}{1 + y} \left[ \frac{1C}{(1+y)^1} + \frac{2C}{(1+y)^2} + \cdots + \frac{nC}{(1+y)^n} + \frac{nF}{(1+y)^n} \right] \ast \frac{1}{P_0^B}$$

$$= -\frac{1}{1 + y} D$$

where

$$D = \left[ \frac{1C}{(1+y)^1} + \frac{2C}{(1+y)^2} + \cdots + \frac{nC}{(1+y)^n} + \frac{nF}{(1+y)^n} \right] \ast \frac{1}{P_0^B} > 0$$

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Bond Duration and Convexity

Duration (Continued)

- Now we can get

$$\eta = \frac{dP_0^B}{dy} \frac{1 + y}{P_0^B} = -D < 0$$
Bond Duration and Convexity
Duration (Continued)

- Definition
  - **Duration** measures the responsiveness of a bond’s price to interest rate changes:

  \[
  \text{Duration} = D = \sum_{t=1}^{n} \frac{t \cdot C}{P_0^B} \left(1 + \frac{F}{P_0^B}\right) + \frac{nF}{P_0^B} \]

Bond Duration and Convexity
Duration (Continued)

- Despite the name, the duration is not the same as the time to maturity
Rewrite $D$ as

$$D = \sum_{t=1}^{n} t \left( \frac{t \cdot C_t}{\frac{(1+y)^t}{P_0^B}} \right) + \frac{nF}{(1+y)^n} P_B$$

$$= \sum tw_t + \gamma_n F, \quad \gamma_n = \frac{n}{(1+y)^n} P_B$$

The duration shows the “actual” weighted length of time needed to recover the current cost of the bond, $P_0^B$.
For a zero coupon bond, duration equals the maturity:

\[ D = \sum_{t=1}^{n} t \left( \frac{t \cdot C_t}{(1 + y)^t} \right) + \frac{nF}{P_B} (1 + y)^n \]

\[ = n \left[ \frac{F}{(1 + y)^n} \frac{1}{P_0^B} \right] \]

But

\[ P_0^B = \frac{F}{(1 + y)^n} \]

so

\[ D = n \]
For non-zero bonds, \( D < n \)
- Generally, the higher \( C \) and the higher \( y \), the shorter is duration

Also, the longer the maturity, the greater the duration so the greater is the price change for a small yield change

Convexity
We noticed earlier that the price-yield curve is convex to the origin.

Consider this situation:

\[ p^B \]

\[ p^B* \]

\[ y^* \]

\[ y \]

Tangent Line
The tangent line shows the arte of change of $P^B$ to a change in $y$ at the tangency point.

Notice that, starting at $y^*$, if we move away from $y^*$, the price change will always be underestimated by the tangent.
The accuracy of using the tangent depends on the convexity of the price-yield curve.

Expand the price equation using a Taylor Series Expansion

\[ dP^B_0 = \frac{dp}{dy} dy + \frac{1}{2} \frac{d^2P^B_0}{dy^2} (dy)^2 + \text{error} \]
Dividing by $P^B_0$, we get

$$\frac{dP^B_0}{P^B_0} = \frac{dP^B_0}{dy} \frac{dy}{P^B_0} + \frac{1}{2} \frac{d^2P^B_0}{dy^2} (dy)^2 \frac{(dy)^2}{P^B_0} + \text{error} \frac{\text{error}}{P^B_0}$$

- The first term is the duration for a small yield change

The second term corrects for the convexity

- The second derivative is called the dollar convexity of a bond

$$\text{Convexity} = \frac{d^2P^B_0}{dy^2}$$
It is easy to show that the second derivative of the price-yield relationship is

\[
\frac{d^2 P_0^B}{dy^2} = \sum_{t=1}^{n} \frac{t(t+1)C}{(1+y)^{t+2}} + \frac{n(n+1)F}{(1+y)^{n+2}}
\]

The duration and the convexity adjustment term can be summed to get the estimated price change due to duration and convexity.
Example: 25-year, 6% coupon selling to yield 9%
- Duration = 10.62
- Convexity = 182.92
- Assume the yield rises 200 basis points to 11%

Percentage change in price due to duration:
\[ = -\text{duration} \times dy \]
\[ = -(10.62) \times (0.02) \]
\[ = -21.24\% \]

Percentage change in price due to convexity:
\[ = 0.5 \times \text{convexity} \times (dy)^2 \]
\[ = 0.5 \times 182.92 \times 0.02^2 \]
\[ = 3.66\% \]

Percentage change in price = -21.24% + 3.66% = -17.58%
Consider two coupon bonds, $A$ and $B$.

- $B$ has greater convexity.

Bond Duration and Convexity
Application of Duration and Convexity

Consider two coupon bonds, $A$ and $B$ (Continued).

- Bond $B$ will have a greater price no matter what happens to the yield.
- Also,

  \[
  \text{Yield Rises} \Rightarrow \text{Loss on } B \text{ is Less} \\
  \text{Yield Falls} \Rightarrow \text{Gain on } B \text{ is More}
  \]
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Bond Duration and Convexity
Application of Duration and Convexity (Continued)

- The market takes B’s greater convexity into account when pricing B
  - The market prices the convexity since the convexity adds a risk element

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Bond Duration and Convexity
Application of Duration and Convexity (Continued)

- Pricing depends on expectations
  - If investors expect very small yield changes, the advantages of holding B are small – both A and B offer the same approximate price (i.e., return)
    » There is little paid to the convexity
  - If a big change is expected, B will be priced higher than A
    » B has more risk
There are three properties of bonds due to convexity

1. As the required yield increases (decreases), the convexity of a bond decreases (increases).
   » This is referred to as positive convexity

There are three properties (Continued)

1. As the required yield increases: Implication
   - As the yield rises, the price declines but is “slowed” by the decline in the duration
   - As the yield falls, the price rises but is “enhanced” by the increase in the duration
There are three properties (Continued)

1. As the required yield increases: Graphically

2. For a given yield and maturity, the lower the coupon, the greater the convexity of a bond
   • A zero coupon bond has the highest convexity

3. For a given yield and duration, the lower the coupon, the smaller the convexity
   • A zero coupon bond has the lowest convexity
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Bond Duration and Convexity
Application of Duration and Convexity (Continued)

- There are three properties (Continued)

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<th>Bond (Coupon/Maturity)</th>
<th>Convexity</th>
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<tr>
<td>9%/5-year</td>
<td>19.45</td>
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<td>9%/25-year</td>
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<td>6%/5-year</td>
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