1 First order approximation using Taylor expansion

- Basic Formula: \( f(x) \approx f(\bar{x}) + \frac{df}{dx}(x - \bar{x}) \)

- With 2 variables \( x \) and \( y \):

\[
\begin{align*}
  f(x, y) & \approx f(\bar{x}, \bar{y}) + \frac{\partial f(\bar{x}, \bar{y})}{\partial x}(x - \bar{x}) + \frac{\partial f(\bar{x}, \bar{y})}{\partial y}(y - \bar{y}) \\
  f(x, y) - f(\bar{x}, \bar{y}) & \approx f_x(x - \bar{x}) + f_y(y - \bar{y}) \\
  \frac{f(x, y) - f(\bar{x}, \bar{y})}{f(\bar{x}, \bar{y})} & \approx f_x \frac{x - \bar{x}}{f(\bar{x}, \bar{y})} + f_y \frac{y - \bar{y}}{f(\bar{x}, \bar{y})} \\
  \hat{y} & \approx f_x \frac{x - \bar{x}}{f(\bar{x}, \bar{y})} + f_y \frac{y - \bar{y}}{f(\bar{x}, \bar{y})}
\end{align*}
\]

- Example: \( Y = K^\alpha L^{1-\alpha} \)

\[
\begin{align*}
  Y & \approx \underbrace{K^\alpha L^{1-\alpha}}_{\hat{y}} + \alpha \underbrace{K^{\alpha-1} L^{1-\alpha}}_{\bar{Y}} (K - \bar{K}) + (1 - \alpha) \underbrace{\bar{K}^\alpha L^{-\alpha}}_{\bar{Y}} (L - \bar{L}) \\
  Y - \bar{Y} & \approx \alpha \underbrace{K^\alpha L^{1-\alpha}}_{\hat{y}} \frac{K - \bar{K}}{K} + (1 - \alpha) \underbrace{K^\alpha L^{1-\alpha}}_{\bar{Y}} \frac{L - \bar{L}}{L} \\
  Y - \bar{Y} & \approx \alpha \underbrace{\bar{K}^{\alpha-1} L^{1-\alpha}}_{\bar{y}} \frac{K - \bar{K}}{Y} + (1 - \alpha) \underbrace{\bar{K}^\alpha L^{-\alpha}}_{\bar{Y}} \frac{L - \bar{L}}{L} \\
  \hat{Y} & \approx \alpha \bar{K} + (1 - \alpha) \bar{L}
\end{align*}
\]

2 Growth Accounting

- Economic Growth Accounting: Explain how each factor of production contribute to output growth. Consider an economy described by the production function: \( Y = F(K, L) \)

- By definition, MPK (MPL) show the increase in output when capital (labor) increase by 1 unit

\[
\frac{\partial Y}{\partial K} \equiv MPK \quad \text{and} \quad \frac{\partial Y}{\partial L} \equiv MPL
\]

- Total Differentiation

\[
\begin{align*}
  dY & = \frac{\partial Y}{\partial K} dK + \frac{\partial Y}{\partial L} dL \\
  \Rightarrow \frac{dY}{Y} & = \frac{\partial Y}{\partial K} \frac{dK}{Y} + \frac{\partial Y}{\partial L} \frac{dL}{Y} \Rightarrow \frac{dY}{Y} = \frac{\partial Y}{\partial K} \frac{dK}{\bar{Y}} \frac{K}{\bar{K}} + \frac{\partial Y}{\partial L} \frac{dL}{\bar{Y}} \frac{L}{\bar{L}} \\
  \Rightarrow \frac{dY}{Y} & = \left( MPK \cdot \frac{\bar{K}}{\bar{Y}} \right) \frac{dK}{\bar{K}} + \left( MPL \cdot \frac{L}{\bar{Y}} \right) \frac{dL}{\bar{L}}
\end{align*}
\]
• Since we know real rental rate equals MPK, and real wage equals MPL in neoclassical theory of CH. Under CRTS production function,
\[ \frac{dY}{Y} = \alpha \frac{dK}{K} + (1 - \alpha) \frac{dL}{L} \]  
(Euler Theorem)

• Given \( \alpha = 0.3 \) of U.S.
  
  - 10% increase in capital leads to a 3% increase in the output
  - 10% increase in labor leads to a 7% increase in the output

**Economic Growth Accounting with Technology progress**

• Now, \( Y = AF(K, L) \) where A is TFP which is the measure of the current technology including not only purely technological progress but also political stability, regulation, environmental condition etc. TFP captures anything that changes the relation between measured inputs and measured output.

\[
\frac{dY}{dA} = \frac{\partial Y}{\partial A} dA + \frac{\partial Y}{\partial K} dK + \frac{\partial Y}{\partial L} dL
\]

\[
\Rightarrow \frac{dY}{Y} = \frac{dA}{A} + \alpha \frac{dK}{K} + (1 - \alpha) \frac{dL}{L}
\]

Growth in output = Contribution of K + Contribution of L + growth in TFP

• However, since TFP is not observable, it is measured indirectly. (Solow Residual: growth rate of TFP is computed as a residual)

\[
\frac{\Delta A}{A} = \frac{\Delta Y}{Y} - \alpha \frac{\Delta K}{K} - (1 - \alpha) \frac{\Delta L}{L}
\]

• Practice Question

1. When capital income share is 1/3, K grows at 3% and L grows at 2% and output grows at 3%. What is the growth rate of TFP?

\[
\frac{\Delta Y}{Y} = \frac{\Delta A}{A} + \alpha \frac{\Delta K}{K} + (1 - \alpha) \frac{\Delta L}{L} \iff 0.03 = x + (0.3)(0.03) + (0.7)(0.02)
\]

\[
x = 0.007, \text{ Total Productivity grows at } 0.7%.
\]

Thus, output growth (3%) is attributed to growth in capital (0.9%), labor (1.4%), and total factor productivity (0.7%)

2. Define Labor productivity \( \frac{Y}{L} \), show \( \frac{\Delta(Y/L)}{Y/L} = \frac{\Delta A}{A} + \alpha \frac{\Delta(K/L)}{K/L} \)

\[
\frac{\hat{Y}}{\hat{L}} = \hat{Y} - \hat{L} = A\hat{K}^\alpha \hat{L}^{1-\alpha} - \hat{L} = \hat{A} + \alpha \hat{K} + (1 - \alpha)\hat{L} - \hat{L} = \hat{A} + \alpha \left( \hat{K} - \hat{L} \right) = \hat{A} + \alpha \frac{\hat{K}}{\hat{L}}
\]

3. growth rate of labor productivity is \( \frac{\Delta Y/L}{Y/L} \)?

\[
\frac{\Delta Y/L}{Y/L} = \frac{\Delta A}{A} + \alpha \frac{\Delta K/L}{K/L} = 0.007 + (0.3)(0.03 - 0.02) = 0.01
\]