1 Linear First-Order Difference Equations (FODE)

Consider the linear FODE:

\[ x_t - \phi x_{t-1} = b \]  

(1)

can be solved using various techniques.

1.1 Induction & Geometric Series

Given an initial value \( x_0 \) we can iterate backwards to find a solution:

\[ x_t = \phi x_{t-1} + b \Rightarrow x_{t-1} = \phi x_{t-2} + b \]

wicht implies:

\[
\begin{align*}
x_t &= \phi (\phi x_{t-2} + b) + b \\
&= \phi (\phi (\phi x_{t-2} + b) + b) + b \\
&\vdots \\
x_t &= \phi^t x_0 + \phi^{t-1}b + \phi^{t-2}b + \ldots + \phi^2b + \phi b + b \\
&= \phi^t x_0 + b \sum_{k=0}^{t-1} \phi^k
\end{align*}
\]

now if \( |\phi| < 1 \) then using the geometric series arithmetic:

\[
\sum_{k=0}^{t-1} \phi^k = \frac{1 - \phi^t}{1 - \phi}
\]

we obtain:

\[ x_t = \phi^t x_0 + b \left( \frac{1 - \phi^t}{1 - \phi} \right) \]

so the solution to the LFODE takes the form:

\[ x_t = \phi^t \left( x_0 - \frac{b}{1 - \phi} \right) + \frac{b}{1 - \phi} \]

Notice that if we did not have an initial condition, we would need to iterate backwards ad-infinitum:

\[ x_t = \lim_{j \to \infty} \phi^j x_{t-j} + b \sum_{k=0}^{\infty} \phi^k \]
and again, if $|\phi| < 1$ and assuming that the sequence $\{x_t\}$ is bounded from below, then:

$$x_t = \frac{b}{1 - \phi}$$

which we can call the long-run or infinite-horizon solution.

Next, notice that the equation (1) above can also be expressed as:

$$x_t = \psi x_{t+1} + c$$

where $\psi = 1/\phi$ and $c = -b/\phi$. This is a very different view of the world and says that the current value of $x_t$ depends upon its future values (closer to rational expectations theory). Now, if we are given a terminal condition (i.e. the world ends in period $t+j$), $x_{t+j}$, then we use it to iterate forward:

$$x_t = \psi (x_{t+2} + c) + c = \psi (x_{t+3} + c) + c \ldots$$

which, if $|\psi| < 1$ simplifies to:

$$x_t = \psi^j x_{t+j} + c \left( \frac{1 - \psi^j}{1 - \psi} \right)$$

again, if no terminal condition is given:

$$x_t = \lim_{j \to \infty} \psi^j x_{t+j} + c \sum_{k=0}^{\infty} \psi^k$$

and again, if $|\psi| < 1$ and assuming that the sequence $\{x_t\}$ is bounded from above:

$$x_t = \frac{c}{1 - \psi}$$

which is of course the same long-run value of $x_t$. Notice however that the condition required for the existence of such a long-run value are very different; in the first case we require $|\phi| < 1$ while in the second case we require $|\psi| < 1$ or $|\phi| > 1$ (this last case points to the rationale for the B-K condition; in this scalar case one forward looking variable $x_{t+1} = \phi x_t + b$ requires $|\phi| > 1$).

Here’s a useful quote from Enders (2004):

"The backward-looking and forward-looking solutions are two mathematically valid solutions to any n-th order difference equation...For economic analysis, however, the distinction is important since the time paths implied by these alternative solutions are quite different...From a purely mathematical point of view, there is no "most appropriate" solution. However, economic theory may suggest that a sequence be bounded in the sense that the limiting value for any value in the sequence is finite".

### 1.2 Homogeneous part and General solution

An alternative 5-step approach is as follows:
1. **Find the steady state** of \( x \) (i.e., \( x_t = x_{t-1} = x^* \)):

\[
x^* = \frac{b}{1-\phi}
\]

2. **Find the solution to the homogeneous part**:

\[
x_{h,t} - \phi x_{h,t-1} = 0
\]

and let:

\[
x_{h,t} = \xi^t
\]

be a solution to the homogeneous part for each \( t \), so that:

\[
x_{h,t-1} = \xi^{t-1}
\]

and (2) becomes:

\[
\begin{align*}
\xi^t - \phi \xi^{t-1} &= 0 \\
\xi^{t-1} (\xi - \phi) &= 0
\end{align*}
\]

this equation has two solutions, the trivial solution where \( \xi^{t-1} = 0 \) and the non-trivial solution \( (\xi - \phi) = 0 \) so using the latter:

\[
\xi = \phi
\]

and replacing in the hypothesized solution:

\[
x_{h,t} = \xi^t = \phi^t
\]

is the solution to the homogeneous part.

3. **Find the General Solution**: The general solution is a linear combination of the homogeneous solution and the steady state:

\[
x_t = \frac{b}{1-\phi} + a \phi^t
\]

where \( a \) is the lin-comb **undetermined** coefficient.

4. **Find \( a \) for some initial \( x_0 \)**:

\[
x_0 = \frac{b}{1-\phi} + a \phi^0 \implies a = x_0 - \frac{b}{1-\phi}
\]

5. **Replacing in the general solution**:

\[
x_t = \phi^t \left( x_0 - \frac{b}{1-\phi} \right) + \frac{b}{1-\phi}
\]

which is, of course, the same result obtained by use of the induction technique.
1.3 Asymptotic Stability

The above solution can be expressed as:

\[ f(x_0) = \phi \left( x_0 - \frac{b}{1-\phi} \right) + \frac{b}{1-\phi} \]

which has, as fixed point, \( x^* = \frac{b}{1-\phi} \), i.e.:

\[ f(x^*) = f \left( \frac{b}{1-\phi} \right) = \frac{b}{1-\phi} \]

so \( \frac{b}{1-\phi} \) can be interpreted as a stationary point ("steady-state"). We say that the above FODE is asymptotically stable if:

\[ \lim_{t \to \infty} (x_t - x^*) = 0 \]

which will be the case only if:

\[ |\phi| < 1 \]

and note that if \( 0 < \phi < 1 \) we observe a monotonic convergence towards \( x^* \) while if \( -1 < \phi < 0 \) the convergence is via oscillations around \( x^* \).

**Remark 1** Notice that if \( x_t \) were stochastic with \( N(0,1) \), disturbances, the fixed point or L-R value would be the unconditional mean of \( x_t \).

**Example 2** A version of the Cagan (1956) model. Suppose that supply is based on expected prices while demand is based on actual prices:

\[
\begin{align*}
y_{d,t+1} & = \alpha + \beta p_{t+1} \\
y_{s,t+1} & = \gamma + \theta \mathbb{E}_t p_{t+1}
\end{align*}
\]

and equilibrium is therefore given by:

\[ \alpha + \beta p_{t+1} = \gamma + \theta \mathbb{E}_t p_{t+1} \tag{3} \]

expectations are formed according to the adaptive behavior:

\[ \mathbb{E}_t p_{t+1} - \mathbb{E}_{t-1} p_t = \phi(p_t - \mathbb{E}_{t-1} p_t) \]

which can be written using the Lag operator as:

\[
\begin{align*}
\mathbb{E}_t p_{t+1} & = \phi p_t + (1 - \phi) \mathbb{E}_t p_{t+1} \\
[1 - (1 - \phi)L] \mathbb{E}_t p_{t+1} & = \phi p_t \\
\mathbb{E}_t p_{t+1} & = \frac{\phi p_t}{[1 - (1 - \phi)L]}
\end{align*}
\]

now replacing in the equilibrium condition:

\[
\begin{align*}
\alpha + \beta p_{t+1} & = \gamma + \theta \left\{ \frac{\phi p_t}{[1 - (1 - \phi)L]} \right\} \\
\alpha \phi + \beta p_{t+1} - \beta (1 - \phi) p_t & = \phi \gamma + \theta \phi p_t \\
p_{t+1} & = \frac{\phi (\gamma - \alpha)}{\beta} + \left[ 1 - \phi \left( \frac{1 - \theta}{\beta} \right) \right] p_t
\end{align*}
\]
and now we have a FODE without expectation terms which we can solve using the abovementioned techniques. Using the 5 step method. Let \( \frac{\gamma - \alpha}{\beta} = \Omega \) and \( 1 - \phi \left( \frac{1 - \theta}{\beta} \right) = \Psi \) then:

1) : \( \text{Sty - Sate} \implies p^* = \frac{\Omega}{1 - \Psi} \)

2) : \( P_{h,t+1} = \xi^t \implies \xi^{t-1}(\xi - \Psi) = 0 \)
   : \( \implies p_{h,t+1} = \Psi^t \)

3) : \( \text{Grad.Sol.} \implies p_{t+1} = \frac{\Omega}{1 - \Psi} + \alpha \Psi^t \)

4) : \( \text{Find coef} \implies p_0 = \frac{\Omega}{1 - \Psi} + \alpha \Psi^0 \)
   : \( \implies \alpha = \left( p_0 - \frac{\Omega}{1 - \Psi} \right) \)

5) : \( \text{Replace} \implies p_{t+1} = \frac{\Omega}{1 - \Psi} + \left( p_0 - \frac{\Omega}{1 - \Psi} \right) \Psi^t \)

so that the solution to the price equation is:

\[
p_{t+1} = \frac{\gamma - \alpha}{1 - \theta} + \left( p_0 - \frac{\gamma - \alpha}{1 - \theta} \right) \left[ 1 - \phi \left( \frac{1 - \theta}{\beta} \right) \right]^t
\]

with stationary price level equal to \( p^* = \frac{\gamma - \alpha}{1 - \theta} \) and asymptotic stability condition:

\[
\left| \left[ 1 - \phi \left( \frac{1 - \theta}{\beta} \right) \right] \right| < 1
\]

**Example 3 (stochastic LFODE)** Modify the example above to include a stochastic term. In particular, assume now that:

\[
\mathbb{E}_t p_{t+1} = p_t + \epsilon_{t+1}
\]

where \( \epsilon_{t+1} \) is some random shock. Then the equilibrium condition (3) is now:

\[
\alpha + \beta p_{t+1} = \gamma + \theta (p_t + \epsilon_{t+1})
\]

and our difference equation is:

\[
p_{t+1} = \frac{\gamma - \alpha}{\beta} + \frac{\theta}{\beta} p_t + \frac{\theta}{\beta} \epsilon_{t+1}
\]

solving backwards we get:

\[
p_t = \frac{\gamma - \alpha}{\beta - \theta} + \lim_{j \to \infty} \left( \frac{\theta}{\beta} \right)^j p_{t-j} + \sum_{j=1}^{\infty} \left( \frac{\theta}{\beta} \right)^j \epsilon_{t-j+2}
\]

this expression is useful to compute impact multipliers and impulse responses:

\[
\frac{\partial p_{t+1}}{\partial \epsilon_{t+1}} = \frac{\theta}{\beta}
\]
\[
\frac{\partial p_{t+j}}{\partial \epsilon_{t+1}} = \left( \frac{\theta}{\beta} \right)^{j+2}
\]

and we can clearly see that if \( |\theta/\beta| > 1 \) then the IRF would explode.
Example 4 (Cagan’s model under RE) Include money and suppose that instead expectations were formed looking forward as in modern rational expectations models.

\[ p_t = \frac{1}{1 + \eta} m_t + \frac{\eta}{1 + \eta} E_t p_{t+1} \]

so that \( p_t \) is a forward-looking variable, depends on exogenous \( m_t \) ("fundamentals") and expectations about future price levels. This way:

\[ p_{t+1} = \frac{1}{1 + \eta} m_{t+1} + \frac{\eta}{1 + \eta} E_t p_{t+2} \]

and so on. Solving forward:

\[ p_t = \frac{1}{1 + \eta} \lim_{T \to \infty} \sum_{k=0}^{T-1} \left( \frac{\eta}{1 + \eta} \right)^k E_t m_{t+k} + \lim_{T \to \infty} \left( \frac{\eta}{1 + \eta} \right)^T E_t p_{t+T} \]

a bounded solution would entail:

\[ \lim_{T \to \infty} \left( \frac{\eta}{1 + \eta} \right)^T E_t p_{t+T} = 0 \]

\[ \left| \frac{1}{1 + \eta} \lim_{T \to \infty} \sum_{k=0}^{T-1} \left( \frac{\eta}{1 + \eta} \right)^k E_t m_{t+k} \right| < \infty \]

which are both satisfied whenever the sequence \( \{m_t\} \) is bounded given that:

\[ \left| \frac{\eta}{1 + \eta} \right| < 1 \]

the solution therefore is:

\[ p_t = \frac{1}{1 + \eta} \lim_{T \to \infty} \sum_{k=0}^{T-1} \left( \frac{\eta}{1 + \eta} \right)^k E_t m_{t+k} \]

so the log price level is given by discounted expected future log money supply. Consider the simplest example of a money rule, i.e., \( m_t = \overline{m} \forall t \), then:

\[ p_t = \overline{m} \frac{1}{1 + \eta} \sum_{k=0}^{\infty} \left( \frac{\eta}{1 + \eta} \right)^k \]

and, since \( \frac{1}{1 + \eta} \sum_{k=0}^{\infty} \left( \frac{\eta}{1 + \eta} \right)^k = 1 \) we have that \( p_t = p_{t+1} = \overline{m} \forall t \) so that inflation is zero.

Exercise 5 (Gali pp. 47) Find a solution to the DE:

\[ \pi_t = \beta E_t \pi_{t+1} + \lambda \tilde{m} c_t \]

where \( \tilde{m} c_t \) is some exogenous process.
2 Systems of linear FODE (or VDE)

Suppose that $X_t$ is a 2-dimensional vector and:

$$X_t - M X_{t-1} = B$$

where $M$ is a $2 \times 2$ matrix known as the state-transition matrix and $B$ is $2 \times 1$. This system can be solved in a similar fashion as single equations:

1. Solve for the steady state:

$$X^* = (I - M)^{-1}B$$

2. Find the solution to the homogeneous part:

$$X_{h,t} - M X_{h,t-1} = 0$$

and again, suppose it has the shape:

$$X_{h,t} = A \mu^t$$

so that:

$$X_{h,t-1} = A \mu^{t-1}$$

where $A$ is a column vector of unknown parameters and $\mu$ is some scalar. Replacing in (4) implies:

$$A \mu^t - M A \mu^{t-1} = 0$$

$$\mu^{t-1} A (\mu I - M) = 0$$

$$A (M - \mu I) = 0$$

(5)

Again, this equation will have two solutions. However, we are only interested in the non-trivial solution so $A \neq 0$ which in turn implies that $(M - \mu I)$ must be singular, i.e.:

$$\det(M - \mu I) = 0$$

$$\det \begin{pmatrix} m_{11} - \mu & m_{12} \\ m_{21} & m_{22} - \mu \end{pmatrix} = 0$$

which yields a quadratic equation in $\mu$:

$$\mu^2 - (m_{11} + m_{22}) \mu + (m_{11} m_{22} - m_{12} m_{21}) = 0$$

$$\mu^2 - \text{tr}(M) + \det(M) = 0$$

the roots of this equation are called characteristic roots or eigenvalues. The two solutions to this equation are:

$$\mu_1, \mu_2 = \frac{1}{2} \left[ \text{tr}(M) \pm \sqrt{\text{tr}(M)^2 - 4 \det(M)} \right]$$

and substituting $\mu_1, \mu_2$ in (5) results in the two eigenvectors of $A$:

$$A^1 = \begin{pmatrix} 1 \\ \frac{m_{11} - \mu_1}{m_{12}} \end{pmatrix}$$
and:
\[ A^2 = \left( \begin{array}{c} \frac{1}{m_{12}} \\ -m_{11} \end{array} \right) \]
and the solution to the homogeneous system is given by the linear combination:
\[ X_{h,t} = c_1 A^1 \mu_1^t + c_2 A^2 \mu_2^t \]
where \( c_1, c_2 \) are undetermined coefficients.

3. Obtain the General Solution as in the previous section:
\[
\begin{align*}
X_t &= X^* + X_{h,t} \\
X_t &= (I - M)^{-1} B + c_1 A^1 \mu_1^t + c_2 A^2 \mu_2^t
\end{align*}
\]

(6)

4. As before, solve for the undetermined coefficients using some initial value vector \( X_0 \):
\[ X_0 = (I - M)^{-1} B + c_1 A^1 \mu_1^0 + c_2 A^2 \mu_2^0 \]
this is a system of linear equations from which the coefficients \( c_1, c_2 \) can be determined. Then substituting into (6) gives the solution to the VDE.

2.1 Asymptotic stability
It is easily seen that the asymptotic stability of the stationary point \( X^* \) will depend upon the eigenvalues \( \mu_1, \mu_2 \) of the state-transition matrix, \( A \). Consider only the real-valued ones. If they both lie within the unit circle:
\[ |\mu_i| < 1 \text{ for } i = 1, 2 \]
the critical (stationary) point is asymptotically stable. If only one of them is inside the unit circle, this implies that there are only two possible converging trajectories (i.e., a saddle-point).

3 Higher order DE
Suppose that we are interested in solving the second order (stochastic) difference equation:
\[ E_t [y_{t+1} - 2ay_t + by_{t-1}] = z_t \]
for \( 0 < b < 1 \) and \( 2a > b + 1 \). First, define:
\[ x_t = y_{t-1} \text{ so that } x_{t+1} = y_t \]
this way, we can write the SOSDE as two SFODE:
\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
E_t y_{t+1} \\
x_{t+1}
\end{pmatrix} =
\begin{pmatrix}
2a & -b \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
y_t \\
x_t
\end{pmatrix} +
\begin{pmatrix}
1 \\
0
\end{pmatrix} z_t
\]
(7)
or:
\[ W_{t+1} = BW_t + Qz_t \]
It easily seen that the matrix \( B \) has one eigenvalue outside the unit circle and one inside it. Since in this case we have only one forward looking variable, we conclude that the SOSDE has a unique solution. Moreover, the eigenvalues can be found to be:
\[ \lambda_i = a \pm \sqrt{a^2 - b} \]
To actually find the solutions, first obtain the eigenvector associated to the eigenvalue outside the unit circle say, $\lambda_1$, by solving:

$$e^T (B - \lambda_1 I_2) = 0$$
$$e^T \begin{pmatrix} 2a - \lambda_1 & -b \\ 1 & -\lambda_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -b/\lambda_1 \end{pmatrix}$$

so the system (7) can be written as:

$$e^T W_{t+1} = \lambda_1 e^T W_t + e^T Q z_t$$

define:

$$f_t = y_t + \frac{-bx_t}{\lambda_1} \Rightarrow f_{t+1} = y_{t+1} + \frac{-bx_{t+1}}{\lambda_1}$$

then:

$$f_t = \frac{E f_{t+1}}{\lambda_1} - \frac{z_t}{\lambda_1}$$

or solving forward:

$$f_t = -\frac{1}{\lambda_1} \sum_{k=0}^{\infty} \left( \frac{1}{\lambda_1} \right)^k \mathbb{E}_t z_{t+k}$$

$$= -\frac{1}{\lambda_1 - \rho} z_t$$ (recall $\mathbb{E}_t z_{t+1} = \rho z_t$ since $\mathbb{E}_t \varepsilon_{t+1} = 0$)

so replacing back $x_t = y_{t-1}$ we arrive at:

$$y_t = \frac{b}{\lambda_1} y_{t-1} + \frac{1}{\lambda_1 - \rho} z_t$$

$$= \lambda_2 y_{t-1} - \frac{1}{\lambda_1 - \rho} z_t$$

or, using our findings from $\lambda_1$:

$$y_t = (a - \sqrt{a^2 - b}) y_{t-1} - \left( \frac{1}{a + \sqrt{a^2 - b} - \rho} \right) z_t$$

References
