1 Simple optimal growth (Problems 1&2)

Suppose that we modify slightly problem 1 by mixing it with the functional forms assumed in problem 2:

\[ \sum_{t=0}^{\infty} \beta^t \ln C_t, \]

subject to:

\[ K_t^\alpha \geq K_{t+1} + C_t, \]

\[ 0 < \beta < 1 \quad \alpha > 0 \]

\[ C_t \geq 0, \quad K_{t+1} \geq 0 \]

\[ K_0 \text{ given} \]

(a) The first constraint will bind. Which property(ies) of the primitive functions ensure that?

→ Answer: the properties \( F'(K) > 0 \) and \( U'(C) > 0 \) ensure that the first constraint binds. The HH (planner) can always improve any allocation that wates resources either by increasing consumption in the current period or by increasing \( K \) in the next period thereby increasing consumption from that period onwards.

(b) The second constraint will never bind. Which property(ies) of the primitive functions ensure that?

→ Answer: the property \( \lim_{C \to 0} U'(C) = \infty \). The HH (planner) can always improve upon an allocation that gives \( C_t = 0 \) for some \( t \) by reducing consumption at some other date.

(c) Show that the solution to this problem satisfies the transversality condition (TVC).

→ Solution: The value function can be found to be:

\[ V(K) = F + G \log K \]
where:
\[ G = \frac{\alpha}{1 - \alpha^\beta}, \quad \text{and} \quad F = \frac{\log(1 - \alpha \beta) + \frac{\alpha^\beta}{1 - \alpha^\beta} \log(\alpha \beta)}{1 - \alpha^\beta} \]

So, The TVC is given by:
\[ \lim_{t \to \infty} \beta^{t+1} V'(K_{t+1}) K_{t+1} = \lim_{t \to \infty} \beta^{t+1} \left( \frac{\alpha}{1 - \alpha^\beta} \right) = 0 \]

since \( 0 < \alpha, \beta < 1 \). NOTE that is is not enough to argue that \( \lim_{t \to \infty} \beta^{t+1} = 0 \), the TVC holds. You must first show that the term \( V'(K_{t+1}) K_{t+1} \) is bounded and then you can use the discounting argument. Some of you asked about the relationship between this TVC and the one you derived in the sequential formulation
\[ \lim_{t \to \infty} \beta^{t} u_C (C_t) = 0 \]
so replacing in the TVC we have \( \lim_{t \to \infty} \beta^{t+1} V'(K_{t+1}) K_{t+1} = 0 \). Finally, recall the step before reaching the Euler equation in the recursive formulation yields \( u_C (C_t) = \beta V'(K_{t+1}) \). Now replace in the TVC we have so far, yielding
\[ \lim_{t \to \infty} \beta^{t+1} V'(K_{t+1}) K_{t+1} = 0. \]

2  **Sargent’s (1987) two sector model:**

The math of this problem is straightforward but since it is a little different from what you have seen so far, it is worth solving it here. In particular, it is the interpretation of the equilibrium conditions what is worth the while.

First, notice that, optimal intra-period equilibrium allocations imply that the last two constraints must bind, i.e.:
\[ L_{1t} + L_{2t} = L_t \]
\[ K_{1t} + K_{2t} = K_t \]

for, otherwise, the household would be wasting resources. To stress the intuition, suppose that an optimal plan calls for the household to supply \( L_t^* \). Since labor can only be used to produce in the consumption and capital goods sectors, it must be that \( L_t^* = L_{1t} + L_{2t} \). The same logic holds for \( K_t^* \). Thus, we can eliminate \( K_t \) and \( L_t \) from our constraints. Next, we can solve the dynamic optimization problem:
\[
\max_{L_{1t}, L_{2t}, C_t, K_{1t+1}, K_{2t+1}} \sum_{i=0}^{\infty} \{ \beta^i u(C_t, L_{1t} + L_{2t}) + \lambda_t [F^1(K_{1t}, L_{1t}) - C_t] + \mu_t [F^2(K_{1t}, L_{1t}) - K_{1t+1} - K_{2t+1}] \}
\]

and the F.O.C. are now trivially obtained as:
\[
\begin{align*}
[C_t] & : \beta^i u_C(C_t, L_{1t} + L_{2t}) = \lambda_t \\
[L_{1t}] & : \beta^i u_L(C_t, L_{1t} + L_{2t}) = -\lambda_t F^1_{L}(K_{1t}, L_{1t}) \\
[L_{2t}] & : \beta^i u_L(C_t, L_{1t} + L_{2t}) = -\mu_t F^2_{L}(K_{2t}, L_{2t}) \\
[K_{1t+1}] & : \lambda_{t+1} F^1_{K}(K_{1t+1}, L_{1t+1}) = \mu_t \\
[K_{2t+1}] & : \mu_{t+1} F^2_{K}(K_{2t+1}, L_{2t+1}) = \mu_t
\end{align*}
\]
So we can eliminate the multipliers and find the Euler equations:

\[-u_L(C_t, L_{1t} + L_{2t}) = u_C(C_t, L_{1t} + L_{2t}) F^1_L(K_{1t}, L_{1t})\]

Marg disutility of labor

\[-u_L(C_t, L_{1t} + L_{2t}) = \beta u_C(C_{t+1}, L_{1t+1} + L_{2t+1}) F^1_K(K_{1t+1}, L_{1t+1}) F^2_L(K_{2t}, L_{2t})\]

Marg disutility of labor

The interpretation of the first equation is obvious: a consumption-leisure trade-off. The intuition behind the second equation is also straightforward: working an additional hour in the capital sector yields \(F^2_L(K_{2t}, L_{2t})\) additional units of capital, which can be used next period to yield \(F^1_K(K_{1t+1}, L_{1t+1})\) extra units of the consumption good, in turn yielding \(u_C(C_{t+1}, L_{1t+1} + L_{2t+1})\) extra utility in \(t+1\), all of which is expressed in present value terms by using \(\beta\). Finally:

\[
\frac{F^2_L(K_{2t+1}, L_{2t+1})}{F^2_K(K_{2t+1}, L_{2t+1})} = \frac{F^1_K(K_{1t+1}, L_{1t+1})}{F^1_L(K_{1t+1}, L_{1t+1})}
\]

so that, at the optimum, the HH equalizes the ratios of marginal products of \(K\) and \(L\) across sectors.

3 Optimal Linear Regulator problem and the Riccati Equation (Ex 5.1 in LS)

Problem 5.1 in LS asks you to solve the linear regulator problem:

\[
\max_{u_t} \left\{ -\sum_{t=0}^{\infty} \beta^t \left[ x'_t Q x_t + u'_t Q u_t + 2u'_t H x_t \right] \right\} \quad (ALQ1)
\]

\[s.t. : \quad x_{t+1} = Ax_t + Bu_t \quad (ALQ2)\]

Now, I will NOT solve this for you. There are two reasons. First, I want you to actually try it. Second if you ever find yourself in trouble with this exercise, you can always look at the solutions manual of LS.

Instead, I will give a detailed solution for the problem (5.2.8) on page 103 of LS:

\[
\max_{\{u_t\}_{t=0}^{\infty}} -\sum_{t=0}^{\infty} \beta^t \left[ x'_t Q x_t + u'_t Q u_t \right] \quad (LQ1)
\]

\[s.t. : \quad x_{t+1} = Ax_t + Bu_t, \quad (LQ2)\]

again for two reasons. First, because the details of this solution are not in LS and second, because if you know how to solve (LQ1)-(LQ2), you can solve (ALQ1)-(ALQ2). Start by writing the problem in recursive form:

\[
V(x_t) = \max_{u_t} \left\{ -[x'_t R x_t + u'_t Q u_t] + \beta V(x_{t+1}) \right\}
\]

\[s.t. : \quad x_{t+1} = Ax_t + Bu_t \]
and notice the minus in front of the brackets. This is because, in LS, the matrix $Q$, in addition to being symmetric, is positive definite (PD) and therefore $-x'Qx$ is a concave function (since $x'Qx$ is a quadratic form and therefore convex). Next, guess that the solution takes the form $V(x) = -x'Px$ where $P$ is a PD symmetric matrix that we don’t know. Replace our guess and the transition equation into the value function:

$$-x'Px = \max_u \{-x'Rx - u'Qu - \beta (Ax + Bu)'P(Ax + Bu)\}$$

Now, if in fact $V(x) = -x'Px$, then $P$ must satisfy the F.O.C. w.r.t. $u$:

$$\frac{\partial}{\partial u} \left[-x'Rx - u'Qu - \beta (Ax + Bu)'P(Ax + Bu)\right] = 0$$

at this point we need the derivation rules:

$$\frac{\partial c'Kc}{\partial c} = (K + K')c, \quad \frac{\partial n'Kc}{\partial n} = Kc, \quad \frac{\partial n'Kc}{\partial c} = K'n$$

so notice that:

$$\frac{\partial}{\partial u} \left(-u'Qu\right) = -(Q + Q')u = -2Qu \quad (\because Q = Q') \quad (1)$$

Next:

$$\beta (Ax + Bu)'P(Ax + Bu) = \beta (Ax + Bu)'(PAx + PBu)$$

$$= \beta (x'A' + u'B') (PAx + PBu)$$

$$= \beta (x'A'PAx + x'A'PBu + u'B'PAx + u'B'PBu)$$

and so using the matrix derivative rules:

$$\frac{\partial}{\partial u} \left(-\beta (Ax + Bu)'P(Ax + Bu)\right) = -\beta \left( \frac{\partial x'A'PBu}{\partial u} + \frac{\partial u'B'PAx}{\partial u} + \frac{\partial u'B'PBu}{\partial u} \right)$$

$$= -\beta \left( (A'PB)'x + B'PAx + 2B'PBu \right) \quad (2)$$

$$= -\beta \left( B'P'Ax + B'PAx + 2B'PBu \right) \quad (3)$$

$$= -\beta \left( 2B'PAx + 2B'PBu \right) \quad (\because P = P') \quad (4)$$

so assembling together (1) and (2):

$$-2Qu = \beta \left\{ 2B'PAx + 2B'PBu \right\}$$

$$\Rightarrow u = -(Q + \beta B'PB)^{-1} \beta B'PAx \quad (5)$$

Now if we substitute the maximizer (5) back in (LQ1):

$$-x'Px = \left\{ x'Rx + (\left(Q + \beta B'PB\right)^{-1} \beta B'PAx) Q (Q + \beta B'PB)^{-1} \beta B'PAx \right.\right.$$

$$+ \beta \left( Ax - B (Q + \beta B'PB)^{-1} \beta B'PAx \right) \left(P \left(Ax + B(Q + \beta B'PB)^{-1} \beta B'PAx \right) \right.$$
and rearranging:

\[ P = R + \beta A'PA - \beta^2 A'PB \left( Q + \beta B'PB \right)^{-1} B'PA \]

So now you can solve exercise 5.1. The only hint you need is: use the same guess for the value function.

4 Acemoglu 6.8

(a) The problem in recursive form can be formulated as:

\[
V(k) = \max_{k' \in \Gamma(k)} \left\{ Ak - k' - \frac{a}{2} \left[ Ak - k' \right]^2 + \beta V(k') \right\}
\]

\[
\Gamma(k) = [0, Ak] \quad (\because C \geq 0)
\]

I will not give all the details of this problem either; just some intuition

(b) The key to arguing the existence of a solution without actually finding one is to show that Theorem 6.3 (page 189 of Acemoglu’s book) applies, which in turn requires assumptions 6.1 and 6.2 to hold. Theorem 6.3 is an existence theorem, which ensures that a continuous, bounded function that is a solution to the Bellman equation. In turn, Assumptions 6.1 and 6.2 concern the constraint set \( \Gamma(k) \) and the convergence of \( \sum_{t=0}^{\infty} \beta^t u(C_t) \) to a finite number. Assumption 6.2 trivially holds since \( \Gamma(k) \) has every property you can possibly want from it (non-empty, compact-valued, LHC, UHC, convex-valued, etc). The convergence property is ensured by the upper bound on \( k, \dot{k} \) and \( 0 < \beta < 1 \).

(c) Notice that the F.O.C. and EC yield the equilibrium condition:

\[
1 = \beta A + aAk - a \left( 1 + A^2 \beta \right) \psi(k) + \beta Aa \psi(k')
\]

\[
= \beta A + aAk - a \left( 1 + A^2 \beta \right) \psi(k) + \beta Aa \psi(\psi(k))
\]

and we see that there is a constant and a linear trend so a good guess for the policy function is:

\[
\psi(k) = \Theta k + \Xi
\]

Replace in the equilibrium condition:

\[
1 = \beta A + aAk - a \left( 1 + A^2 \beta \right) \left[ \Theta k + \Xi \right] + \beta Aa \left[ \Theta (\Theta k + \Xi) + \Xi \right]
\]

and you will find two possible solutions for \( \Theta \). Only the one satisfying the strict monotonicity property is the unique solution to the policy rule.