Golden rule in Solow and OGM

Some of you asked about question 2 in the 2009 midterm.

1. We first define a BGP and derive the golden rule. A BGP is a situation in which all variables grow at a constant rate. In particular, if they all grow at the same rate, \( \dot{K} = \dot{L} \) and therefore \( \dot{k} = 0 \). Using the Solow equation (and assuming \( g = 0 \)):

\[
s f(k) = k[n+\delta]
\]

and:

\[
\bar{c} = (1 - s) f(\bar{k}) = f(\bar{k}) - \bar{k}[n+\delta]
\]

so as you did in class, simply obtain:

\[
\frac{\partial \bar{c}}{\partial s} = [f'(\bar{k}) - (n + \delta)] \cdot \frac{\partial \bar{k}}{\partial s}
\]

and when \( \frac{\partial \bar{c}}{\partial s} = 0 \) this yields the golden rule for capital accumulation:

\[
f'(\bar{k}_{Gold}) = (n + \delta)
\]

2. Next we show that the Solow model can in fact deliver capital overaccumulation. We must show that \( f'(\bar{k}) - (n + g + \delta) < 0 \) so that \( \frac{\partial \bar{c}}{\partial s} < 0 \). Suppose for instance that \( f(k) = k^\alpha \). Then we need:

\[
\alpha \bar{k}^{\alpha-1} - (n + \delta) < 0
\]

and to find the parameter values that deliver this, we use (1) i.e., the Solow equation at BGP, to find:

\[
\bar{k} = \left( \frac{s}{n + g + \delta} \right)^{1/\alpha}
\]

and replacing \( \bar{k} \):

\[
\alpha \left( \frac{s}{n + g + \delta} \right)^{-1} - (n + \delta) < 0 \Leftrightarrow \alpha < s
\]

so \( \alpha < s \Rightarrow \bar{k} > \bar{k}_{Gold} \) and the Solow model predicts overaccumulation of capital
3. Finally we compare the GR of the Solow model with the SS capital delivered by the OGM. The equilibrium conditions for the OGM in continuous time, with population growth are:

\[
\dot{\pi}(t) = -\frac{\partial H}{\partial k(t)} = -\pi(t) [f'(k(t)) - \delta - n]
\]

\[
\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \delta - \rho - n}{\sigma}
\]

and since along the BGP, \(\dot{c}(t) = 0\) we have that the modified GR in the OGM is:

\[
f'(\bar{k}) = \rho + \delta + n > \delta + g + n = \bar{k}_{Gold}
\]

which in turn means that \(\bar{k} < \bar{k}_{Gold}\).

**Stochastic Solow model**

Some of you asked about the details of the log-linearization of the fundamental Solow equation when there is uncertainty. The Solow equation in this case is:

\[
k_{t+1} = \frac{(1 - \delta) k_t + s\nu f(k_t)}{(1 + \gamma) (1 + n)}
\]

(2)

and in SS:

\[
(1 + \gamma) (1 + n) - (1 - \delta) = \frac{s\nu f(\bar{k})}{\bar{k}}
\]

(3)

To log-linearize (2) proceed as usual:

\[
\frac{\dot{k}_{t+1}}{s\nu f(\bar{k})} = \frac{(1 - \delta) \bar{k}}{(1 + \gamma) (1 + n)} \dot{k}_t + \frac{s\nu f'(\bar{k}) \bar{k}}{(1 + \gamma) (1 + n)} \dot{k}_t + \frac{s\nu f(\bar{k})}{(1 + \gamma) (1 + n)} \hat{v}_t
\]

Now, divide both sides by \(s\nu f(\bar{k})\):

\[
\frac{\dot{k}_{t+1}}{\bar{k}} = \frac{(1 - \delta) \bar{k}}{s\nu f(\bar{k}) (1 + \gamma) (1 + n)} \dot{k}_t + \frac{f'(\bar{k}) \bar{k}}{(1 + \gamma) (1 + n) f(\bar{k})} \dot{k}_t + \frac{1}{(1 + \gamma) (1 + n)} \hat{v}_t
\]

next, notice from (3) that \([(1 + \gamma) (1 + n) - (1 - \delta)] \bar{k} = s\nu f(\bar{k})\) so that:

\[
\frac{\dot{k}_{t+1}}{[(1 + \gamma) (1 + n) - (1 - \delta)] \bar{k}} = \frac{(1 - \delta) \bar{k}}{[(1 + \gamma) (1 + n) - (1 - \delta)] \bar{k} (1 + \gamma) (1 + n)} \dot{k}_t + \frac{\eta f}{(1 + \gamma) (1 + n)} \dot{k}_t + \frac{1}{(1 + \gamma) (1 + n)} \hat{v}_t
\]
or:

\[
\hat{k}_{t+1} = \frac{(1 - \delta) \hat{k}}{(1 + \gamma)(1 + n)} \hat{k}_t + \left[(1 + \gamma)(1 + n) - (1 - \delta)\right]\left\{\frac{\eta F}{(1 + \gamma)(1 + n)} \hat{k}_t + \frac{1}{(1 + \gamma)(1 + n)} \hat{v}_t\right\}
\]

\[
= \frac{(1 - \delta) \hat{k}}{(1 + \gamma)(1 + n)} \hat{k}_t + \left\{(1 + \gamma)(1 + n) - (1 - \delta)\right\} \left[\eta F \hat{k}_t + \hat{v}_t\right]
\]

which is the expression in the lecture notes.

### 0.1 Eigenvalues of the matrix M:

At the end of the lecture notes on stochastic Solow there are a few unanswered questions. Here are the answers for two of them. Recall the matrix \(M\) in the lecture notes on the stochastic Solow model:

\[
M = \begin{pmatrix} \theta & \phi \rho \\ 0 & \rho \end{pmatrix}
\]

let the two eigenvalues of \(M\) be denoted \(\lambda_1, \lambda_2\) then \(\lambda_i\) must solve:

\[
\lambda_i^2 - \text{tr}(M) \lambda_i + \text{det}(M) = 0
\]

\[
\lambda_i^2 - (\theta + \rho) \lambda_i + \theta \rho = 0
\]

so we require:

\[
|\lambda_i| = \left|\frac{(\theta + \rho) \pm \sqrt{(\theta + \rho)^2 - 4\theta \rho}}{2}\right| < 1 \text{ for } i = 1, 2
\]

or, in other words, \(\theta\) and \(\rho\) are the eigenvalues and we require:

\[
|\theta| < 1 \text{ and } |\rho| < 1
\]

but the only relevant condition is the former since the latter is given by the stationarity of \(v_t\).