"A Theory of Financing Constraints and Firm Dynamics"
G.L. Clementi and H.A. Hopenhayn (QJE, 2006)

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Econ612- Economics - Rutgers

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Program

- Summary
- Physical environment
- The contract design problem
- Characterization of the optimal contract
- Firm growth and survival
- Contract maturity and debt limits
What they do in a nutshell

- The paper develops a theory of endogenous financing constraints.
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- The optimal contract (under asymmetric info) determines non-trivial stochastic processes for firm size, equity and debt.
What they do in a nutshell

- The paper develops a theory of endogenous financing constraints.
- Repeated moral hazard problem
- The optimal contract (under asymmetric info) determines non-trivial stochastic processes for firm size, equity and debt.
- This in turn implies non-trivial firm dynamics even under simple i.i.d. shocks.
Physical environment:

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- At the beginning of each \( t \), project can be liquidated (\( \alpha_t = 1 \)) yielding \( S \geq 0 \) and resulting in payoffs \( Q \) to \( \mathcal{E} \) and \( S - Q \) to \( \mathcal{L} \).
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- Returns are stochastic and equal to $R(k_t)$ if state of nature is $H$ (with prob. $p$) and zero if state is $L$ (prob. $1 - p$).
- At the beginning of each $t$, project can be liquidated ($\alpha_t = 1$) yielding $S \geq 0$ and resulting in payoffs $Q$ to $E$ and $S - Q$ to $L$.
- If project is not liquidated, $E$ repays $\tau$ to $L$. 
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- **Assumption:** $E$ consumes all proceeds $R(k_t) - \tau$ (i.e. no storage)
Physical environment:

Definition (reporting strategy)
A reporting strategy for $\mathcal{E}$ is $\hat{\theta} = \{\hat{\theta}_t (\theta^t)\}_{t=1}^{\infty}$ where $\theta^t = (\theta_1, \theta_2, ..., \theta_t)$
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Definition (contract)
A contract is a vector $\sigma = \{\alpha_t(h^{t-1}), Q_t(h^{t-1}), k_t(h^{t-1}), \tau_t(h^t)\}$ where $h^t = \{\hat{\theta}_1, ..., \hat{\theta}_t\}$
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Definition (feasible contract)
A contract $\sigma$ is feasible if $\alpha_t \in [0, 1], Q_t \geq 0, \tau_t (h^{t-1}, L) \leq 0, \tau_t (h^{t-1}, H) \leq R (k_t)$. 

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Definition (equity and debt)
Expected discounted cash flows for $E$ is called equity, $V_t(\sigma, \hat{\theta}, h^{t-1})$ and for $L$ is called debt, $B_t(\sigma, \hat{\theta}, h^{t-1})$
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Definition (incentive compatibility)
A contract $\sigma$ is incentive compatible if $\forall \hat{\theta}, \ V_1 (\sigma, \theta, h^0) \geq V_1 (\sigma, \hat{\theta}, h^0)$
The first-best (symmetric info)

- Since both are risk neutral and share $\delta$, the optimal contract maximizes total expected discounted profits of the match $(\mathcal{E}, \mathcal{L})$. 

\[ \max k \left[ pR(k) - k \right] \]  

So that $R(k)$ is strictly concave and has a unique $k > 0$ that solves (1).

Assume $k > 0$ so that per-period total surplus is:

\[ \pi = \max k \left[ pR(k) - k \right] = pR(k) - k \]

And PDV of total surplus is

\[ W = \pi_1 \delta > S \] by assumption.

Project is undertaken if $W > I_0$ and once project is started, the firm does not grow, shrink or exit.
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- Since both are risk neutral and share $\delta$, the optimal contract maximizes total exp. discounted profits of the match $(\mathcal{E}, \mathcal{L})$.
- In equilibrium $\mathcal{L}$ provides $\mathcal{E}$ with the unconst. efficient $k$ in every $t$:

$$\max_k [pR(k) - k]$$

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- Since both are risk neutral and share $\delta$, the optimal contract maximizes total exp. discounted profits of the match $(E, L)$.
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Since both are risk neutral and share \( \delta \), the optimal contract maximizes total exp. discounted profits of the match \((E, L)\).

In equilibrium \( L \) provides \( E \) with the unconst. efficient \( k \) in every \( t \):

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\max_k \left[ pR(k) - k \right] \tag{1}
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- Upon continuation, the evolution of equity is given by:

\[
V = \rho (R(k) - \tau) + \delta \left[ \rho V^H + (1 - \rho) V^L \right]
\]  

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$$ V = p(R(k) - \tau) + \delta \left[ pV^H + (1 - p) V^L \right] \quad (2) $$

- While the evolution of debt (not in the paper):

$$ B(V) = p\tau - k + \delta \left[ pB(V^H) + (1 - p) B(V^L) \right] $$
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- The value of equity effectively summarizes all the information provided by the history itself (Spear and Srivastava, 1987; Green, 1987) so it’s the appropriate state variable in a recursive formulation of the repeated contracting problem.
The optimal contract upon continuation maximizes the value for the match $\hat{W}(V_c)$, subject to LL, IC and PK constraints.
Recursive formulation *upon* continuation

- The optimal contract upon continuation maximizes the value for the match $\hat{W}(V_c)$, subject to LL, IC and PK constraints.
- In recursive form, the program to be solved upon continuation is:

$$\hat{W}(V) = \max_{k, \tau, V^H, V^L} pR(k) - k + \delta \left[ pW(V^H) + (1 - p) W(V^L) \right]$$

s.t.

$$V = p(R(k) - \tau) + \delta \left[ pV^H + (1 - p) V^L \right]$$

( PK)

$$\tau \leq \delta \left( V^H - V^L \right)$$

( ICC)

$$\tau \leq R(k), \ V^H \geq 0, \ V^L \geq 0$$

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- Authors show that $V \mapsto \hat{W}(V)$ is increasing and concave.
The optimal contract upon continuation maximizes the value for the match \( \hat{\mathcal{W}} (V_C) \), subject to LL, IC and PK constraints.

In recursive form, the program to be solved upon continuation is:

\[
\hat{\mathcal{W}} (V) = \max_{k, \tau, V^H, V^L} \left[ pR(k) - k + \delta \left( p\mathcal{W}(V^H) + (1 - p)\mathcal{W}(V^L) \right) \right] \\
\text{s.t.} \\
V = p(R(k) - \tau) + \delta \left[ pV^H + (1 - p)V^L \right] \quad \text{(PK)} \\
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\tau \leq R(k), \ V^H \geq 0, \ V^L \geq 0 \quad \text{(LL)}
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Solving this problem yields policy functions \( k (V) \), \( \tau (V) \), \( V^H (V) \) and \( V^L (V) \).
Recursive formulation *before* liquidation decision

- If project is liquidated, \( E \) receives \( Q \) while \( L \) receives \( S - Q \). If project is not liquidated, they get \( V_c, B(V_c) \).
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- Pure strategies may not be optimal for some values of $V$ so $\alpha \in [0, 1]$ and $\mathcal{L}$ offers a "lottery" to $\mathcal{E}$. 

\[ W(V) = \max_{\alpha \in [0, 1]} \left( \alpha Q + (1 - \alpha) V_c \right) \]
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- Thus, in recursive form, the program to be solved prior to liquidation:

$$W(V) = \max_{\alpha \in [0, 1], Q, V_c} \left\{ \alpha S + (1 - \alpha) \hat{W}(V_c) \right\}$$

s.t. : \begin{align*}
\alpha Q + (1 - \alpha) V_c &= V \quad \text{(PK)} \\
V_c &\geq 0, \quad Q \geq 0 \quad \text{(LL)}
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\text{ s.t. } : \alpha Q + (1 - \alpha) V_c = V \quad \text{(PK)}

: \quad V_c \geq 0, \quad Q \geq 0 \quad \text{(LL)}

- Notice that $W(\cdot)$ preserves the properties of $\hat{W}(\cdot)$. 
Regions for \( V \) (Propositions 1 & 2)

The domain of \( V \) can be partitioned in three regions:

- **Region I:** When \( 0 \leq V \leq V_r \), liquidation is possible and randomizing is optimal with \( \alpha(V) = (V_r - V) / V_r \)

**Sketch of argument:**

\[ \alpha = 1 \quad \text{while} \quad \alpha = 0 \]

Now \( W(V) = S \) while \( \alpha = 2(0, 1) \) s.t. \( V \leq V_r \) implies that \( \alpha (V) + (1 - \alpha) \hat{W}(V_r) > \max(S, \hat{W}(V)) \).

**Intuition:** As \( V \) expected value \( \hat{W}(V) \) rises above \( S \) and \( L \) liquidates with low probability (draw graph).

**Region III:** When \( V = pR(k / (1 / \delta)) \) the total surplus is the same as under symmetric information (first-best), i.e., \( W(V) = W \).

**Intuition:** equivalent to \( E \) having a balance of \( k / (1 / \delta) \) in the bank at interest rate \( (1 / \delta) / \delta \) that is exactly enough to finance the project at its optimum scale. Then \( L \) advances \( k \) and collects \( \tau = 0 \) every period.
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  - Sketch of argument: \( \alpha = 1 \Rightarrow W(V) = S \) while \( \alpha = 0 \Rightarrow W(V) = \hat{W}(V) \). Now \( W^* > S \Rightarrow \exists! V_r \) and \( \alpha \in (0, 1) \) s.t. \( V \leq V_r \) implies that \( \alpha S + (1 - \alpha) \hat{W}(V_r) > \max \{ S, \hat{W}(V) \} \).

- **Region III:** When \( V = p_R(k) / (1 - \delta) \) the total surplus is the same as under symmetric information (first-best), i.e., \( W(V) = W \).
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  - Intuition: As $V \rightarrow V_r$ expected value $\hat{W}(V)$ rises above $S$ and $L$ liquidates with low probability (draw graph).

- **Region II:** When $V > V_r$, liquidation is impossible and randomizing is optimal with $\alpha(V) = 1$.
  - Sketch of argument: $W(V) = S$ while $W(V) = \hat{W}(V)$. Now $W^* > S \Rightarrow \exists! V_r$ and $\alpha \in (0, 1)$ s.t. $V \leq V_r$ implies that $\alpha S + (1 - \alpha) \hat{W}(V_r) > \max \{S, \hat{W}(V)\}$.
  - Intuition: Equivalent to $E$ having a balance of $k \left( \frac{1}{1 + \delta} \right)$ in the bank at interest rate $\left( \frac{1}{1 + \delta} \right)$ that is exactly enough to finance the project at its optimum scale. Then $L$ advances $k$ and collects $\tau = 0$ every period.

- **Region III:** When $V \leq 0$, liquidation is impossible and randomizing is optimal with $\alpha(V) = 0$.
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  - Sketch of argument: $\alpha = 1 \Rightarrow W(V) = S$ while $\alpha = 0 \Rightarrow W(V) = \hat{W}(V)$. Now $W^* > S \Rightarrow \exists! V_r$ and $\alpha \in (0, 1)$ s.t. $V \leq V_r$ implies that $\alpha S + (1 - \alpha) \hat{W}(V_r) > \max\{S, \hat{W}(V)\}$.

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  - Intuition: equivalent to $E$ having a balance of $k^*/(1 - \delta)$ in the bank at interest rate $(1 - \delta)/\delta$ that is exactly enough to finance the project at its optimum scale. Then $L$ advances $k^*$ and collects $\tau = 0$ every period.
Region II: When $V_r \leq V < V^*$:

- There is no liquidation in the current period and $V_r$ is strictly increasing.
- The optimal capital advancement policy is single-valued and such that $k(V) < k^*$ (the firm is debt-constrained).

Sketch of argument: suppose that the optimal repayment policy for region II was $\tau = R(k)$ implying that the ICC binds (see below the proofs for both of these results). Then:

$$R(k) = \delta(V_H - V_L)$$

which implies that increasing $k$ is only incentive compatible if $V_H$ also increases. But $W(V)$ concave implies that doing so is costly! (draw graph)
The borrowing constraint region (Propositions 1 & 2 cont.)

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Optimal repayment policy (proposition 3)

- The optimal repayment function satisfies $\tau = R(k)$ for $V < V^*$ and $\tau = 0$ for $V \geq V^*$. 
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- Limited liability then implies $\tau = R(k)$ until $V = V^*$.
Evolution of equity when \( V < V^* \)

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- And we obtain the policy functions:

$$V^L(V) = \frac{V - pR(k)}{\delta}, \quad V^H(V) = \frac{V + (1 - p)R(k)}{\delta}$$
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Simulations (\( R(k) = k^{2/5} \), \( p = 0.5 \), \( \delta = 0.99 \), \( S = 1.5 \)): 

![Graph showing stock price dynamics over time]
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Firm growth and survival

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- The advantage of this model is that requires little structure on the stochastic process driving firm productivity; a simple \( i.i.d. \) process is enough to generate the rich dynamics described above.
Firm growth and survival

Average Size

Exit Hazard Rate

Average Growth

Variance of Growth
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