Chapter 2

Theory of Demand and Consumer Price Index

2.1 Theory of Demand

This is review of Intro Microeconomics. Please refer to intro-micro or intermediate micro texts more for detail.

The rational consumer has a preference function called utility function and the consumer maximizes it subject to the budget constraint. In mathematical notation we have

\[
\max_{x_1, x_2} U(x_1, x_2)
\]

subject to \( P_1 x_1 + P_2 x_2 \leq Y \)

where

\[
\begin{align*}
U(x_1, x_2) &= \text{utility function;} \\
x_i &= \text{quantity of commodity } X_i \ (i = 1, 2) \text{ the consumer purchases;} \\
P_i &= \text{price of commodity } i, \ i = 1, 2; \\
Y &= \text{income of the consumer.}
\end{align*}
\]

Mathematical Solution to the Constrained Maximization Problem

Let us assume that the utility function \( U(x_1, x_2) \) is at least twice differentiable. Forming the Lagrangian

\[
\Lambda = U(x_1, x_2) - \lambda (P_1 x_1 + P_2 x_2 - Y)
\]
we take the first derivatives and set them to zero (the first order conditions):

\[
\begin{align*}
\frac{\partial \Lambda}{\partial X_1} &= \frac{\partial U}{\partial X_1} - \lambda P_1 = 0 \\
\frac{\partial \Lambda}{\partial X_2} &= \frac{\partial U}{\partial X_2} - \lambda P_2 = 0 \\
\frac{\partial \Lambda}{\partial \lambda} &= -P_1 X_1 - P_2 X_2 + Y = 0
\end{align*}
\]

The first order conditions consist of three equations in three unknowns \((X_1, X_2, \lambda)\), and we may obtain the optimal or equilibrium quantities of \(X_1, X_2\) and \(\lambda\). Implicit equilibrium conditions are given by

\[
\frac{\text{MU}_1}{\text{MU}_2} = \frac{P_1}{P_2} \quad \text{or} \quad \frac{\text{MU}_1}{P_1} = \frac{\text{MU}_2}{P_2},
\]

where

\[
\begin{align*}
\text{MU}_1 &= \frac{\partial U}{\partial X_1} = \text{marginal utility of } X_1 \\
\text{MU}_2 &= \frac{\partial U}{\partial X_2} = \text{marginal utility of } X_2.
\end{align*}
\]

**A Numerical Example** Suppose that income is $200 \((Y = 200)\) and the utility function is given by

\[U(X_1, X_2) = X_1 X_2 + 10 X_1.\]

The budget constraint is

\[P_1 X_1 + P_2 X_2 = 200.\]

The marginal utilities are

\[
\begin{align*}
\text{MU}_1 &= \frac{\partial U}{\partial X_1} = X_1 + 10; \quad \text{MU}_2 = \frac{\partial U}{\partial X_2} = X_1.
\end{align*}
\]
Hence, from the first order conditions we have

\[
\frac{MU_1}{MU_2} = \frac{X_2 + 10}{X_1} = \frac{P_1}{P_2},
\]

From this we derive the demand function for \(X_2\):

\[
X_2 = \frac{100}{P_2} - 5.
\]

and the demand function for \(X_1\) is

\[
X_1 = \frac{100 + 5P_2}{P_1}.
\]

Suppose the prices of \(X_1\) and \(X_2\) are

\[
P_1 = 25, \quad P_2 = 5.
\]

Then the equilibrium quantities of \(X_1\) and \(X_2\) that maximize the utility function are

\[
X_1 = 5, \quad X_2 = 15.
\]

The total utility is

\[
U = 5 \times 15 + 10 \times 5 = 125.
\]

From this numerical illustration we have found that given income of \$200, and the prices of \(P_1 = 25\) and \(P_2 = 5\) the consumer will purchase 5 units of \(X_1\) and 15 units of \(X_2\). The total level of satisfaction the consumer attains is 125.

**Graphical Illustration** Let us show the mathematical solution we have arrived at above using a graph of \((X_1, X_2)\). Suppose that the total utility is given by

\[
U = 125, \quad \text{or} \quad X_1X_2 + 10X_1 = 125.
\]

The **indifference curve** (i.e. combinations of \((X_1, X_2)\) that give rise to \(U = 125\)) is

\[
X_2 = \frac{125}{X_1} - 10, \quad 0 \leq X_1 \leq 12.5
\]

since \(X_2\) cannot be negative. This indifference curve is drawn in Figure 1 with lable \(U = 125\), or \(X_2 = 125/X_1 - 10\).
The equilibrium point is indicated in Figure 1 as E \((X_1 = 5, X_2 = 15)\). It is *tangential* to the budget line:

\[
X_2 = \frac{200}{P_2} - \frac{P_1}{P_2}X_1, \quad \text{or} \quad X_2 = 40 - 5X_1.
\]

*Figure 1 Here.*

So far we have illustrated the mathematical solution on a graph. Now let us solve the constrained optimization problem graphically. In Figure 2 four indifference curves are drawn for different utility values \(U=75, U=125, U=175, U=250\). To draw the indifference curve for \(U=75\), we start from the utility function:

\[
U = 75 = X_1X_2 + 10X_1.
\]

Expressing \(X_2\) in term of \(X_1\) we have

\[
X_2 = \frac{75}{X_1} - 10.
\]

For \(U=125, 175, \) and \(250\), we proceed in the similar manner:

\[
U = 125 : \quad X_2 = \frac{125}{X_1} - 10
\]

\[
U = 175 : \quad X_2 = \frac{175}{X_1} - 10
\]

\[
U = 250 : \quad X_2 = \frac{250}{X_2} - 10
\]

In Figure 2 the budget line \(X_2 = 40 - 5X_1\) is drawn in a broken straight line. The area to the left of the budget line (i.e. the triangle) is the feasible region that any point within the triangle satisfies the budget constraint: \(25X_1 + 5X_2 \leq 200\). Since the consumer wants to maximize his utility, he wants to climb up the indifference curves as much as he can. The indifference curve with \(U=125\) is the highest indifference curve the consumer can climb to since it is *tangential* to the budget line. The indifference curve \(U=175\) is *infeasible* since the consumer cannot attain it with income of $200.

*Figure 2 Here.*
A Price Change Causes a Change in Equilibrium. Suppose that the price of \( X_1 \) goes down from \( P_1 = \$25 \) to \( P_1 = \$10 \), while the price of \( X_2 \) and income stay unchanged (\( P_2 = \$5 \), \( Y = \$200 \)). This change in \( P_1 \) causes the budget line to change:

\[
10X_1 + 5X_2 = 200, \quad \text{or} \quad X_2 = 40 - 2X_1
\]
as shown in Figure 3. The new equilibrium point associated with this new budget line is \( E_2 \) with \( X_1 = 12.5 \) and \( X_2 = 15 \).

*Figure 3 Here.*
The Movement from the Old Equilibrium Point $E_1$ to the New Equilibrium Point $E_2$ can be separated into the substitution and income effects. This is shown in Figure 4, and Figure 4 will be explained in class.

Figure 4 Here.

The definitions of the substitution and income effects are

**Substitution effect** The rate at which the consumer substitutes $X_2$ for $X_1$ when the price of $X_1$ changes and the consumer stays on the same indifference curve. In other words the substitution effect is *the change in the amount of a good that would be consumed as the price of that good changes, holding constant all other prices and the level of utility.* In Figure 4 the substitution effect is the movement from $E_1$ with $X_1 = 5, X_2 = 15$ to $S$ with $X_1 = 7.960, X_2 = 5.811$ along the indifference curve $U=125$.

**Income effect** The rate at which the consumer purchases $X_1$ when his income changes, prices remaining constant. In other words the income effect is *the change in the amount of a good that a consumer would buy as purchasing power changes, holding all prices constant.* In Figure 4 the income effect is the movement from $S$ with $X_1 = 7.90, X_2 = 5.811$ to $E_2$ with $X_1 = 12.5, X_2 = 15$.

**Consumer Price Index (CPI)** What does the CPI intend to measure? Normally CPI is understood to measure inflation or the rise in the cost of living. (In the article I downloaded from a website, Kathleen O’Toole says “the index (CPI) was never intended to be a cost-of-living measure but is used as if it was.” The article is attached here.)

Let us assume that the CPI is to measure the cost of living (of an average American family,) and let us say that "living" means *maintaining the same level of satisfaction.* The same level of satisfaction is to stay on the same indifference curve. In Figure 4 the initial equilibrium point was $E_1$ with $X_1 = 5, X_2 = 15$. To maintain this level of satisfaction ($U=125$) the consumer spends $200$. Now the price of $X_1$ changes from $P_1 = 25$ to $P_1 = 10$ To maintain the same level of satisfaction the consumer purchases $X_1 = 7.960$ and $X_2 = 5.811$. The total expenditures is $108.66 (=7.960 \times$
10 + 5.811 \times 5). Hence, the cost of living has gone down from $200 to $108.66. If we take $200 as 100, then the new CPI is 54.33.

In short, the CPI is to measure the substitution effect.

As we see from the example above, to compute the CPI we need to know
1. the utility function of the consumer \((U = X_1X_2 + 10X_1)\);
2. the goods and services the consumer purchases \((i.e. \, X_1 \, \text{and} \, X_2)\);
3. the prices of \(X_1\) and \(X_2\).

To derive the fundamental properties of the demand theory, we do not have to know the utility function of the consumer. All we need are certain general properties the utility function need to have (such as convexity and differentiability). To calculate the ideal CPI, however, we need to know the utility function.

Since it is impossible to measure a change in the cost of living, the consumer price index (CPI) that is published by the government should be treated as a proxy of the cost of living.

There are many index formulae. Among them the Laspeyers index and Paasche index are the basic indices. When they are applied to compute the CPI, we have the following formulae:

The **Laspeyers CPI** is given by

\[
L_t = \frac{p_{t1}q_01 + p_{t2}q_02 + \cdots + p_{tn}q_{0n}}{p_{01}q_{01} + p_{02}q_{02} + \cdots + p_{0n}q_{0n}}
\]

where \(q_{0i} = \) the amount of commodity \(i\) the consumer buys at the base year 0; \(p_{0i} = \) the price of commodity \(i\) at the base year 0, and \(p_{ti} = \) the price of commodity \(i\) at year \(t\).

The **Paasche CPI** is given by

\[
P_t = \frac{p_{t1}q_{t1} + p_{t2}q_{t2} + \cdots + p_{tn}q_{tn}}{p_{01}q_{t1} + p_{02}q_{t2} + \cdots + p_{0n}q_{tn}}
\]

where \(q_{ti} = \) the amount of commodity \(i\) the consumer buys at the current year \(t\).
It is known that the Laspeyers CPI overestimates the true CPI while the Paasche CPI underestimates it.

For two commodities \((n = 2)\) let us demonstrate the italicized sentence in the box above. With \(n = 2\) the Laspeyers CPI becomes

\[
L_t = \frac{p_{t1}q_{01} + p_{t2}q_{02}}{p_{01}q_{01} + p_{02}q_{02}}
\]

while the Paasche CPI is

\[
P_t = \frac{p_{t1}q_{t1} + p_{t2}q_{t2}}{p_{01}q_{t1} + p_{02}q_{t2}}
\]

We use the numerical examples of Figure 3 (or 4). We need to change notations \(P_1, P_2, X_1,\) and \(X_2\) to conform to the notations of \(L_t\) and \(P_t\). We will present them in the table below:

<table>
<thead>
<tr>
<th>time ((t))</th>
<th>price (t=0)</th>
<th>quantity (q_{01})</th>
<th>price (t=t)</th>
<th>quantity (q_{t1})</th>
<th>price (t=0)</th>
<th>quantity (q_{02})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(t = 0)</td>
<td>(p_{01} = 25)</td>
<td>(q_{01} = 5)</td>
<td>(p_{02} = 5)</td>
<td>(q_{02} = 15)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(t = t)</td>
<td>(p_{t1} = 10)</td>
<td>(q_{t1} = 12.5)</td>
<td>(p_{t2} = 10)</td>
<td>(q_{t2} = 15)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| substitution effect | \(q_{01}^* = 7.906\) | substitution effect | \(q_{02}^* = 5.811\) |

Notes: \(q_{01}^*\) = quantity of \(X_1\) to stay on the indifference curve \(U = 125\).
\(q_{02}^*\) = quantity of \(X_2\) to stay on the indifference curve \(U = 125\).

The Laspeyers CPI, \(L\) becomes

\[
L = \frac{p_{t1}q_{01} + p_{t2}q_{02}}{p_{01}q_{01} + p_{02}q_{02}} = \frac{10 \times 5 + 5 \times 15}{25 \times 5 + 5 \times 15} = \frac{125}{200} = .625
\]

while the Paasche CPI becomes

\[
P = \frac{p_{t1}q_{t1} + p_{t2}q_{t2}}{p_{01}q_{t1} + p_{02}q_{t2}} = \frac{25 \times 12.5 + 5 \times 15}{10 \times 12.5 + 5 \times 15} = \frac{200}{387.5} = .516
\]

The ideal CPI, \(I_0\), that is to measure the substitution effect on the indifference curve \(U \equiv u_0 = 125\) is

\[
I_0 = \frac{p_{t1}q_{01}^* + p_{t2}q_{02}^*}{p_{01}q_{01} + p_{02}q_{02}} = \frac{10 \times 7.906 + 5 \times 5.811}{25 \times 5 + 5 \times 15} = \frac{108.115}{200} = .541.
\]
From the numerical illustration above we see

\[ P < I_0 < L. \]

This means that the Laspeyers CPI overestimates the cost of living (\textit{i.e.} \textit{the cost to maintain the same level of satisfaction}), whereas the Paasche CPI underestimates it.

So far we have shown the relationship \( P < I_0 < L \) by a numerical illustration. Let us show that this relationship holds in general.

We first show \( I_0 < L \): Let \( u_0 \) be the level of satisfaction the consumer attains at initial time \( t = 0 \), when the price levels are \( p_{01}, p_{02}, \ldots, p_{0n} \). The equilibrium level of expenditures \( E_0(u_0) \) is given by

\[ E_0(u_0) = \sum_{i=1}^{n} p_{0i} q_{0i}. \]

Now the price levels change to \( p_{t1}, p_{t2}, \ldots, p_{tn} \) at time \( t \). If the consumer were to maintain the same level of satisfaction \( u_0 \) at the new price levels, then he will spend

\[ E_t(u_0) \equiv \sum_{i=1}^{n} p_{ti} q_{0i}^*. \]

(this is the substitution effect), and the ideal CI, \( I_0 \) is

\[ I_0 = \frac{E_t(u_0)}{E_0(u_0)} = \frac{\sum_{i=1}^{n} p_{ti} q_{0i}^*}{\sum_{i=1}^{n} p_{0i} q_{0i}} \leq \frac{\sum_{i=1}^{n} p_{ti} q_{0i}}{\sum_{i=1}^{n} p_{0i} q_{0i}} = L, \]

since

\[ \sum_{i=1}^{n} p_{ti} q_{ti}^* \leq \sum_{i=1}^{n} p_{ti} q_{0i}. \]

(This inequality is shown graphically in Figure 5 for \( n = 2 \). However, can you explain this inequality assuming that the consumer is rational?) I’ll explain Figure 5 in class.

\textit{Figure 5 Here.}

We show \( P < I_t \) where \( I_t \) is the ideal CPI at the utility level \( u_t \). At time \( t \), the consumer faces the price levels \( p_{t1}, p_{t2}, \ldots, p_{tn} \), and attains the
equilibrium level of satisfaction $u_t$. The expenditures at $u_t$ are

$$E_t(u_t) = \sum_{i=1}^{n} p_i q_{ti}.$$  

If the price levels were $p_{01}, p_{02}, \ldots, p_{0n}$, then the consumer will spend

$$E_0(u_t) = \sum_{i=1}^{n} p_{0i} q^*_ti$$

to stay on the satisfaction level $u_t$.

We have

$$E_0(u_t) < \sum_{i=1}^{n} p_{0i} q_{ti}$$

as demonstrated in Figure 6 graphically. The ideal CPI measured at $u_t$, $I_t$ is

$$I_t = \frac{E_t(u_t)}{U_0(u_t)} = \frac{\sum_{i=1}^{n} p_{ti} q_{ti}}{\sum_{i=1}^{n} p_{0i} q^*_ti} \geq \frac{\sum_{i=1}^{n} p_{ti} q_{ti}}{\sum_{i=1}^{n} p_{0i} q_{ti}} = P.$$

If consumer’s standard of living at time 0 and that at time $t$ does not change then

$$I_0 \sim I_t \equiv I$$

and thus we have

$$P < I < L.$$

*Figure 6 Here.*

The following paragraphs are translation from Yuzo Morita (1974) *Shin Tokei Gairon* (Introduction to Statistics), Nippon Hyoron-Sha, Tokyo, Japan

Consumer behavior changes over time. Laspeyres’ CPI uses the basket of goods and services at the base year ($t = 0$). As time goes by the basket of goods and services the consumer purchases will deviate further and further away from the basket fixed at the base year. Laspeyres’ CPI, thus, will not reflect the cost of living as time passes further away from the base year. If we compute Paasche’s CPI, we may be able to find a ball park figure of the discrepancies between the Laspeyres’ CPI and the cost of living. It is a wise policy to compute...
Paasche CPI from time to time and see the discrepancy between the Laspeyers and Paasche CPI. This way of testing the CPI is called the Paasche test.

Exercise 1

Suppose that the consumer buys two goods at time t, $q_{t1}$ and $q_{t2}$. His utility function is given by

$$U(q_{t1}, q_{t2}) = q_{t1}q_{t2} + 10q_{t1}.$$  

His income is $200 and it stays the same year after year. The following table shows the prices of $q_{t1}$ and $q_{t2}$ at time $t = 0$ and $t = 1$:

<table>
<thead>
<tr>
<th>time</th>
<th>price</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$t = 0$</td>
<td>$p_{01} = 10$</td>
<td>$p_{02} = 5$</td>
<td></td>
</tr>
<tr>
<td>$t = 1$</td>
<td>$p_{11} = 25$</td>
<td>$p_{12} = 5$</td>
<td></td>
</tr>
</tbody>
</table>

1. Find the equilibrium quantities the consumer buys at $t = 0$ and at $t = 1$.
2. Find the quantities $q_{01}^{*}$ and $q_{02}^{*}$ that show the substitution effect.
3. On a graph show the equilibrium positions at time $t = 0$ and $t = 1$. Also show the position of the substitution effect.
4. Compute the Lasypers CPI, the Paasche CPI, and the ideal CPI.

Although it is widely known that the CPI does not measure the cost of living, the CPI has been used in many countries in a wide variety of occasions. For example, in wage negotiations the CPI is used to decide on
a reasonable rate to adjust wages. Many pensions are indexed to the CPI, and so are income taxes.

Here are some news that may interest you.

**Case 1:** “Government is going to resume indexing public pensions starting in 2003” (*Asahi Shimbun 08/05/02*)

Starting in Fiscal Year 2003 (a new fiscal year starts in April in Japan) the Ministry of Welfare, Health and Labor and the Ministry of Treasury have agreed to adjust public pensions downward to account for the deflation. The two ministers, however, are not in agreement on what percentage should be used to reduce public pensions, and they hope to come to an agreement by the end of this year when the fiscal year 2003 budget is drafted.

Over the last three years when the CPI kept declining year after year the government has suspended indexing public pensions to the CPI so that consumer spending will not be reduced by indexing public pensions. According to government’s estimate, the moratorium on indexing public pensions has accumulated to 1.7% of pensions. This year, the government forecasts that the consumer price index will decline by 0.6%, and thus in the three years the moratorium will reach 2.3%. The Ministry of Welfare, Health and Labor is arguing that the deduction in public pensions next fiscal year should be by 0.6% rather than 2.3%. The Ministry of Treasury, on the other hand, wants the deduction rate to be 2.3%.

Both ministries have agreed to continue negotiation on the rate of reduction of public pensions and to come to an agreement by the end of this year taking into consideration of the movement of the CPI and of civil servants’ pay.
**Case 2:** “Economist Boskin sees slow changes in Consumer Price Index” by Kathleen Otoole (article attached here)

Boskin headed a five-member blue-ribbon panel on evaluating the consumer price index and the commission issued a final report on December 1996. Major recommendations are

1. The Bureau or Labor Statistics should adopt new CPI formulas and, with added financial resources, move faster to keep up with changes in the economy.

2. Congress and the president stop automatically adjusting the tax code and federal spending programs by the CPI. If Democrats and Republicans could agree to a downward adjustment of 1.1 percent, the commission said the federal government would save $1 trillion in a dozen years simply by stopping its practice of over-compensating Americans for inflation.