1. The following is **Question #3 from the first midterm examination:**

The random variable $X$ is from the exponential distribution having the probability density function (pdf):

$$f(x) = \begin{cases} \frac{1}{b} \exp \left( -\frac{x}{b} \right) & x \geq 0 \\ 0 & \text{otherwise} \end{cases}, \quad b > 0 \quad (1)$$

It can be shown that the moment generating function, $M(t)$, is given by

$$M(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx = \frac{1}{1 - bt}, \quad t < \frac{1}{b}. \quad (10)$$

**(a)** Using $M(t)$ show that the $r$-th moment about the origin, $E X^r$, is given by

$$E X^r = r! b^r.$$ 

Hence,

$$EX = b,$$

$$EX^2 = 2! b^2 = 2 b^2,$$

$$\sigma^2 \equiv \text{Var}(X) = EX^2 - (EX)^2 = b^2,$$

$$EX^3 = 3! b^3 = 6 b^3,$$

$$EX^4 = 4! b^4 = 24 b^3.$$

**(b)** Let $x_1, x_2, \cdots, x_n$ be a random sample from the exponential distribution (3). Suppose that the estimator of parameter $b$, $\hat{b}$, is given by the sample mean:

$$\hat{b} = \frac{1}{n} \sum_{i=1}^{n} x_i.$$

Show that $\hat{b}$ is unbiased, that is

$$E(\hat{b}) = b.$$
(10) (c) Since $\text{Var}(X) = b^2$, we may estimate the variance by $\hat{b}^2$:

$$\hat{\sigma}^2 = \left( \frac{1}{n} \sum_{i=1}^{n} x_i \right)^2$$

Show that $\hat{\sigma}^2$ is biased:

$$E(\hat{\sigma}^2) = b^2 + \frac{b^2}{n}.$$ 

But the bias disappears as $n \to \infty$.

**Hint:**

(i) $\sum x_i = \sum (x_i - b) + nb$. Hence

$$E\left(\sum_{i=1}^{n} x_i^2\right) = E\left\{\sum_{i=1}^{n} (x_i - b) + nb\right\}^2.$$ 

(ii) Since $(x_1, \cdots, x_n)$ is a random sample we have

$$\text{Cov}(x_i, x_j) = E(x_i - b)(x_j - b) = 0 \text{ for all } i \neq j.$$ 

Questions:

(10) (i) Show that $\bar{x}$ is the MLE.

(10) (ii) Obtain the variance of $\bar{x}$.

(10) (iii) Obtain the Cramér-Rao bound, and show that the variance of $\bar{x}$ attains the Cramér-Rao bound.

(10) (iv) Show that the estimator of the variance of the exponential distribution:

$$\hat{w} = (\bar{x})^2$$

is the MLE of the variance of the exponential distribution.

Estimate
• Linear Probability Model:

\[ y_i = \alpha + \beta x_i + u_i \]

where

\[ y_i = \begin{cases} 1.0 \quad & \text{if disease is absent} \\ 0 \quad & \text{if disease is present at a site three years after treatment} \end{cases} \]

and \( x_i = \) Days (The data is given in the handout.)

• Logit Model:

\[ P_i = \frac{\exp(\alpha + \beta x_i)}{1 + \exp(\alpha + \beta x_i)}. \]

• Probit Model:

\[ P_i = \int_{-\infty}^{\alpha + \beta x_i} \frac{1}{\sqrt{2\pi}} \exp \left( \frac{1}{2} t^2 \right) \, dt. \]

using Eviews. (Data will be sent to you.) In this Eviews data set \( y_i = \) resp, and \( x_i = \) days.

The Eviews command for Logit and Probit models are

- \text{binary(d=l) resp c days}
- \text{binary(d=n) resp c days}

or

- \text{equation eq\_name.binary(d=n) resp c days}
- \text{equation eq\_name.binary(d=l) resp c days}

Note: In “d=l”, “l” is “ℓ” not one. “ℓ” for Logit, and “n” for Probit (normal distribution). To learn more on the probit and logit estimation using Eviews, do

\text{Help} \rightarrow \text{Eviews Help Topics} \rightarrow \text{Index} \rightarrow \text{type “binary” and click the highlighted “binary.”}