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Motivation

- Main question: why do governments repay their debts?
- The practical importance of the question became apparent with actual default episodes, starting with Mexico 1982.
- A lot of research in the 80s and early 1990s (see OR ch. 6).
- Recent revival of the literature: Aguiar-Gopinath, Arellano, Mendoza, and others.
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2. Review dynamic programming and recursive equilibrium concepts
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Plan

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2. Review dynamic programming and recursive equilibrium concepts
3. Discuss computational issues (numerical approximation techniques, numerical DP, functional equations)
4. Return to literature and review main findings, with emphasis on recent developments
Basic Sovereign Debt Problem

- \( t = 0, 1, 2, ... \)
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- One nonstorable good
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- Small country with a representative agent with preferences

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- Agent receives a stochastic, nonstorable endowment $y_t$, $t = 0, 1, 2, \ldots$
- Assume the endowment is a Markov process
Assumptions about international borrowing

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- The agent exits from the default with some probability $\lambda$
Discussion of assumptions so far

1 Costs of default are controversial: EG assumed that the only punishment from default was permanent exclusion from the world capital market. Bulow and Rogoff showed that this alone would not support any positive level of debt if a country in default could save in the world market at the rate $R^*$. 

2 It is assumed that the representative agent is no different from the government. This simplifies the problem a lot but it is obviously quite restrictive.

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Solving the Country’s Problem

Let \( d_t = \) debt at the beginning of period \( t \).
The state at \( t \) is given by \((y_t, d_t)\). The value function is denoted by \( V(y_t, d_t) \).
Let \( V^B(y_t) \) be the value of ending the period in default. Then it must be that:

\[
V^B(y_t) = u((1 - \delta)y_t) + \beta E_t \left\{ \lambda V(y_{t+1}, 0) + (1 - \lambda) V^B(y_{t+1}) \right\}
\]

Let \( V^G(y_t, d_t) \) be the value of ending the period in good standing, so:

\[
V(y_t, d_t) = \text{Max}\{V^G(y_t, d_t), V^B(y_t)\}
\]

and

\[
V^G(y_t, d_t) = \text{Max} \ u(c_t) + \beta E_t V(y_{t+1}, d_{t+1}) \\
\text{s.t.} \quad c_t = y_t + q_t d_{t+1} - d_t
\]

where \( q_t \) is the price at which the country can sell debt in period \( t \).
The Price of the Debt

- Let $\chi(y_t, d_t) = 1$ if the country defaults in period $t$ (this is part of the policy function)
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Risk neutrality then implies:

$$q_t(1 + r^*) = E_t[1 - \chi(y_{t+1}, d_{t+1})]$$
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Hence $q_t = q(y_t, d_{t+1})$
Recursive Equilibrium: Definition

A recursive equilibrium: value functions $V, V^G, V^B$, policy functions $c, d, \chi$, and a debt price function $q$ such that:

1. Given the debt price function $q$, the value functions and policy functions solve the country’s problem.
2. The debt price function satisfies

$$q(y_t, d_{t+1}) = \int [1 - \chi(y_{t+1}, d_{t+1})] \Gamma(y_t, dy_{t+1})$$

where $\Gamma(y, B)$ is the transition function of $y_t$. 
Make an initial guess about the function $q(y_t, d_{t+1})$
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Solve the DP problem of the country. Obtain, among others, the decision rule $\chi$
Solving the Model, in Practice

- Make an initial guess about the function $q(y_t, d_{t+1})$
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- Iterate to convergence
Best reference: Lucas and Stokey, with Prescott

*Exogenous* state: \( z \in Z \)

\( z_t \) is a Markov Process with transition \( Q(z, B) \) (prob \( z_{t+1} \in B \) if \( z_t = z \))

*Endogenous* state \( k \in K \)
Every period an action $a \in A$ is taken

Feasible actions depends on the state: let $\Gamma(k, z)$ denote the feasible correspondence

LS: Assume that

$$k' = \phi(k, a, z')$$
The Bellman Equation

- Let $u(k, z, a)$ denote the current payoff.
- The value function is:

$$v(k, z) = \max_a u(k, z, a) + \beta \int v(k', z') Q(z, dz')$$

s.t. $a \in \Gamma(k, z)$

$$k' = \phi(k, a, z')$$

- The optimal choice of $a = g(k, z)$ is the policy function.
- A solution is a function $v: K \times Z \to \mathbb{R}$ that satisfies the Bellman equation.
Issues in Defining the DP Problem correctly

Often you can define $Z$ however you want, but the definition of $K$ is usually more delicate, since it has to respect more fundamental assumptions on the problem.
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2. Also, note that $z$ captures all the variables that are exogenous to the decision maker. But these variables may be *endogenous* to the model (as in the sovereign debt problem). The law of motion of $z$, given by $Q$, is then taken as given by the individual but then needs to be solved for in equilibrium.
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3. Obvious solution method: value iteration (which we discuss shortly)
Let $Z = [z_L, z_H]$ be the productivity shock, and $Q(z, dz')$ a transition function.

Note: we can usually approximate a typical AR model with a Markov Chain (Tauchen).

$K = [k_L, k_H]$. Often we take $k_L = 0$ and $k_H$ be the maximum sustainable level of capital:

$$k_H = (1 - \delta)k_H + z_Hf(k_H)$$

$a = (c, k')$ constrained by $c \geq 0, k' \in K$ and the feasibility correspondence:

$$c + k' \leq (1 - \delta)k + zf(k)$$

The transition function is simply $k' = \phi(k, a, z') = k'$ and the period utility function $u(k, z, a) = U(c)$.
A Variation with Occasionally Binding Constraints

Suppose that investment cannot be negative:

\[ i = k' - (1 - \delta)k \geq 0 \]

This can be added as part of the feasibility correspondence,

The key aspect of this example is that the constraint will probably bind only occasionally.
Another Famous Example: Lucas Tree Model

\[ z \in Z = [z_L, z_H] : \text{tree dividend} \]
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- \( z \in Z = [z_L, z_H] \): tree dividend
- \( \theta \): agent's holding of shares in the tree. Take \( K = [1 - \varepsilon, 1 + \varepsilon] \).
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- The agent assumes that $p = p(z)$ is the price of the tree
- Bellman equation:

\[
v(\theta, z) = \text{Max } U(c) + \beta \int v(\theta', z')Q(z, dz')
\]

s.t. $c + p(z)\theta' \leq \theta(z + p(z))$

$c \geq 0, \theta' \in K$
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- The solution, of course, depends on the conjecture for \( p(z) \)
- In equilibrium \( p(z) \) must be such that \( \theta \) is always equal to one and \( c = z \)
The FOC for the maximization is the Euler condition:

\[
p(z) u'(c(\theta, z)) = \beta \int u'(c(\theta'(\theta, z), z')) [z' + p(z')] Q(z, dz')
\]

which in equilibrium reduces to

\[
p(z) u'(z) = \beta \int u'(z') [z' + p(z')] Q(z, dz')
\]

a functional equation to be solved for \(p(z)\)
Numerical DP: Discrete Case

- Assume all relevant sets are finite: \( Z = \{z_1, ..., z_n\} \), etc.
- Then the value function is a matrix: \( v_{ij} = v(z_i, k_j) \)
- Let \( V^{(m)} = \{v_{ij}^{(m)}\} \) be the \( m^{th} \) iteration of the value function.
- For each \((i, j)\), one then solves:

\[
v_{ij}^{(m+1)} = \max_a u(z_i, k_j, a) + \sum_{i'} v_{i'j'}^{(m)} \pi_{i,i'}
\]

s.t. \( a \in \Gamma(z_i, k_j) \)

\( k_{j'} = \phi(k_j, a, z_{i'}) \)
Remarks on the discrete case

Sometimes this is a natural procedure, but often the state space is continuous (e.g. optimal growth). Then one expects that a good approximation will require a fine grid.

Curse of dimensionality appears: the size of grids increase exponentially with the number of state variables.

Also, convergence is only linear.

Iterations can be accelerated in various ways.

As mentioned, often a process for $z$ can be well approximated by a Markov chain.

Advantages: there is always a solution; maximization step is a matrix operation; straightforward.

See Ljungqvist and Sargent for many applications.
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Iteration upon iteration: make a guess for the price function $q(y_i, z_j)$; solve the DP problem via value function iteration; update the price function guess; iterate to convergence.
Approximation and Interpolation of Functions
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DP application
Beyond Brute Force: To Review Next Time

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