Sovereign Debt and Recursive Methods

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- A lot of research in the 80s and early 1990s (see OR ch. 6)
- Recent revival of the literature: Aguiar-Gopinath, Arellano, Mendoza, and others

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4. Return to literature and review main findings, with emphasis on recent developments

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- Assume the endowment is a Markov process

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- The agent exits from the default with some probability λ

Discussion of assumptions so far

1. Costs of default are controversial: EG assumed that the only punishment from default was permanent exclusion from the world capital market. Bulow and Rogoff showed that this alone would not support any positive level of debt if a country in default could save in the world market at the rate R^*.

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3. Clearly the assumption of one period non contingent debt is quite ad hoc and restrictive.

Solving the Country's Problem

Let d_t = debt at the beginning of period t.
The state at t is given by (y_t, d_t). The value function is denoted by $V(y_t, d_t)$
Let $V^B(y_t)$ be the value of ending the period in default. Then it must be that:

$$V^B(y_t) = u((1-\delta)y_t) + \beta E_t \left\{ \lambda V(y_{t+1}, 0) + (1-\lambda) V^B(y_{t+1}) \right\}$$

Let $V^G(y_t, d_t)$ be the value of ending the period in good standing, so:

$$V(y_t, d_t) = Max\{V^G(y_t, d_t), V^B(y_t)\}$$

and

$$\begin{aligned} V^G(y_t, d_t) &= Max \quad u(c_t) + \beta E_t V(y_{t+1}, d_{t+1}) \\ s.t. \quad c_t &= y_t + q_t d_{t+1} - d_t \end{aligned}$$

where q_t is the price at which the country can sell debt in period t.

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- Hence $q_t = q(y_t, d_{t+1})$

Recursive Equilibrium: Definition

A recursive equilibrium: value functions V, V^G, V^B, policy functions c, d, χ, and a debt price function q such that:

1. Given the debt price function q, the value functions and policy functions solve the country's problem
2. The debt price function satisfies

$$q(y_t, d_{t+1}) = \int [1 - \chi(y_{t+1}, d_{t+1})] \Gamma(y_t, dy_{t+1})$$

where $\Gamma(y, B)$ is the transition function of y_t.

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- Iterate to convergence

Dynamic Programming Theory: A review

- Best reference: Lucas and Stokey, with Prescott
- *Exogenous* state: $z \in Z$
- z_t is a Markov Process with transition $Q(z, B)$ (prob $z_{t+1} \in B$ if $z_t = z$)
- *Endogenous* state $k \in K$

Feasible Decisions

- Every period an *action* $a \in A$ is taken
- Feasible actions depends on the state: let $\Gamma(k, z)$ denote the *feasible correspondence*
- LS: Assume that

$$k' = \phi(k, a, z')$$

The Bellman Equation

- Let $u(k, z, a)$ denote the *current payoff*
- The *value function* is:

$$\begin{aligned} v(k, z) &= Max_a \; u(k, z, a) + \beta \int v(k', z') Q(z, dz') \\ \text{s.t. } a &\in \Gamma(k, z) \\ k' &= \phi(k, a, z') \end{aligned}$$

- The optimal choice of $a = g(k, z)$ is the *policy function*
- A solution is a function $v : K \times Z \rightarrow \mathbb{R}$ that satisfies the Bellman equation

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2. Also, note that z captures all the variables that are exogenous to the decision maker. But these variables may be *endogenous* to the model (as in the sovereign debt problem). The law of motion of z, given by Q, is then taken as given by the individual but then needs to be solved for in equilibrium.

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3. Obvious solution method: value iteration (which we discuss shortly)

Canonical Example: The Stochastic Growth Model

- Let $Z = [z_L, z_H]$ be the productivity shock, and $Q(z, dz')$ a transition function
- Note: we can usually approximate a typical AR model with a Markov Chain (Tauchen)
- $K = [k_L, k_H]$. Often we take $k_L = 0$ and k_H be the maximum sustainable level of capital:

$$k_H = (1-\delta)k_H + z_H f(k_H)$$

- $a = (c, k')$ constrained by $c \geq 0, k' \in K$ and the feasibility correspondence:

$$c + k' \leq (1-\delta)k + zf(k)$$

- The transition function is simply $k' = \phi(k, a, z') = k'$ and the period utility function $u(k, z, a) = U(c)$

A Variation with Ocassionally Binding Constraints

Suppose that investment cannot be negative:

$$i = k' - (1-\delta)k \geq 0$$

This can be added as part of the feasibility correspondence,

The key aspect of this example is that the constraint will probably bind only *occasionally*.

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- Bellman equation:

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\begin{aligned}
v(\theta, z) &= Max\ U(c) + \beta \int v(\theta', z')Q(z, dz') \\
s.t.\quad c + p(z)\theta' &\leq \theta(z + p(z)) \\
c &\geq 0, \theta' \in K
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- The solution, of course, depends on the conjecture for $p(z)$
- In equilibrium $p(z)$ must be such that θ is always equal to one and $c = z$

The FOC for the maximization is the Euler condition:

$$p(z)u'(c(\theta, z)) = \beta \int u'(c(\theta'(\theta, z), z'))[z' + p(z')]Q(z, dz')$$

which in equilibrium reduces to

$$p(z)u'(z) = \beta \int u'(z')[z' + p(z')]Q(z, dz')$$

a *functional equation* to be solved for $p(z)$

Numerical DP: Discrete Case

- Assume all relevant sets are finite: $Z = \{z_1, ..., z_n\}$, etc.
- Then the value function is a matrix: $v_{ij} = v(z_i, k_j)$
- Let $V^{(m)} = \{v_{ij}^{(m)}\}$ be the m^{th} iteration of the value function.
- For each (i, j), one then solves:

$$
\begin{aligned}
v_{ij}^{(m+1)} &= Max_a \; u(z_i, k_j, a) + \sum_{i'} v_{i'j'}^{(m)} \pi_{i,i'} \\
s.t. \; a &\in \Gamma(z_i, k_j) \\
k_{j'} &= \phi(k_j, a, z_{i'})
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7. See Ljunqvist and Sargent for many applications.

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2. The space for debt is discretized into 400 values. "We ensured that the limits of our asset space never bind along the simulated equilibrium paths."
3. Iteration upon iteration: make a guess for the price function $q(y_i, z_j)$; solve the DP problem via value function iteration; update the price function guess; iterate to convergence.

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