# The Solow Growth Model

## Econ 504

Roberto Chang

Rutgers University

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# The Model

The first ingredient of a dynamic model is the description of the *time horizon* and the set of *commodities*.

In the original Solow model, time is continuous and the horizon is infinite. Without loss of generality assume that time is indexed by $t$ in $[0, \infty)$.

At each point in time, there is only *one* final good

The good is produced via the *aggregate production function*:

$$Y = F(K, AL)$$

Here $Y$, $A$, $K$, and $L$ denote output, labor productivity, capital, and labor, and are functions of time (i.e. we should really write $Y(t)$ and so on, but we omit time arguments when not needed). The product $AL$ is called *effective labor*.

The function $F$ has *neoclassical properties*:

- $F$ is continuously differentiable, strictly increasing, strictly concave
- $F$ has constant returns to scale

A key example is the *Cobb Douglas* function:

$$F(K, AL) = K^{\alpha}(AL)^{1-\alpha}, 0 < \alpha < 1$$

*Exercise:* Show that the Cobb Douglas function enjoys the neoclassical properties listed above.

Once you assume the existence of the aggregate production function $Y = F(K, AL)$, it is clear that the growth of output can be due to growth of $A$, $K$, or $L$.

Solow assumed that $A$ and $L$ grow at *exogenous* rates:

$$\begin{aligned} \dot{L} &= \frac{dL}{dt} = nL \\ \dot{A} &= \gamma A \end{aligned}$$

At time $t = 0$, $A$, and $L$ are given by history. Then one can describe the value of $L$ and $A$ at each point in time:

$$L(t) = L(0)e^{nt}$$

$$A(t) = A(0)e^{\gamma t}$$

So it remains to describe the motion of capital.

Assume $K(0)$ is given by history. The accumulation of capital must follow:

$$\dot{K} = I - \delta K$$

where $I$ denotes investment and $\delta$ the depreciation rate of capital.

As usual, investment is assumed to equal savings. The key behavioral equation of the Solow model is that savings equal a constant fraction of output, so:

$$I = sY$$

This completes the description of the model.

# The Solution

A *solution* is a description of the values of $L, A, Y$, and $K$ *at each point in time*.

Now, from the capital accumulation equation:

$$\begin{aligned}\dot{K} &= sY - \delta K \\ &= sF(K, AL) - \delta K \\ &= ALsF(K/AL, 1) - \delta K \\ &= AL[sF(k, 1) - \delta k]\end{aligned}$$

Here we have defined $k = K/AL$.

Define $f(k) = F(k,1)$, the *intensive* production function. Now we can rewrite the previous equation as:

$$\left(\dot{K}/K\right) k = sf(k) - \delta k$$

But $k = K/AL$ implies

$$\dot{k}/k = \dot{K}/K - (n + \gamma).$$

Combine the two to obtain:

$$\dot{k} = sf(k) - (n + \gamma + \delta)k$$

This is the *fundamental equation of growth*.

$$\dot{k} = sf(k) - (n + \gamma + \delta)k$$

What is the intuition? Capital per effective labor grows faster when there is more investment, hence the role of the savings coefficient $s$. On the other hand, population growth, productivity growth, and faster capital depreciation tend to reduce capital per unit of effective labor.

Obviously once we have solved for the path of $k$ we can obtain $K = kAL$, and $Y$. So we can focus on the solution for $k$.

# The Solution

## The Steady State

A *steady state* is a constant value of $k$, say $k^*$, such that $k(t) = k^*$ is a solution of the growth equation.

Hence $k^*$ must solve:

$$sf(k^*) = (n + \gamma + \delta)k^*$$

*Example:* if the production function is CD, $f(k) = k^\alpha$, so:

$$s(k^*)^\alpha = (n + \gamma + \delta)k^*$$

that is

$$k^* = \left[\frac{s}{n + \gamma + \delta}\right]^{1/1-\alpha}$$

More generally, under the *Inada conditions* $f(0) = 0, f'(0) = \infty, f'(\infty) = 0$ there is a unique nontrivial steady state. This can be shown using the famous Solow diagram.

# The Solution
## Convergence

The same diagram can be employed to show that $k$ must grow when it is between 0 and $k^*$, and falls if it is greater than $k^*$.

Hence the model implies *global* convergence towards the steady state.

# Interpreting the Solow Model

What kind of questions can we answer with this model? In general, we can trace the adjustment path to changes in the parameters of the economy $(s, \delta, n, \gamma)$, or in initial conditions $(K(0), L(0), A(0))$. We can also modify the model in different ways. In this sense, it is an excellent example of a truly dynamic model.

*Example:* Starting from steady state, analyze the consequences of a permanent increase in the savings propensity $s$.

# The Solow Growth Model and Growth

In the steady state, however, $k$ is *constant*. Hence the *extensive* variables, such as $K$ and $Y$, grow at the rate $(n+\gamma)$. Per capita variables, such as $Y/L$, grow at rate $\gamma$. But $n$ and $\gamma$ are *exogenous* variables in the model. So, in this sense, the model does *not* explain long run growth.

See Romer, chapter 1, for a discussion.

To illustrate, as Romer also discusses: a permanent increase in the savings rate $s$ has only transitory effects on the growth rate (e.g. a *level effect*, but not a *growth effect*).

# The Golden Rule

Another classic question leads to the *Golden Rule*. Suppose that one can choose $s$. What would an optimal value of $s$ be?

In growth theory, it was proposed to choose $s$ so as to maximize *steady state consumption*. Now, in steady state,

$$c = (1 - s)f(k)$$

so that the optimal rule would require

$$\frac{dc}{ds} = -f(k) + (1 - s)f'(k)\frac{dk}{ds} = 0$$

In steady state, the Solow equation is

$$sf(k) = (n + \gamma + \delta)k$$

so

$$\frac{dk}{ds} = \frac{f(k)}{(n + \gamma + \delta) - sf'(k)}$$

Combining the preceding, we get

$$(1-s)f'(k) = (n+\gamma+\delta) - sf'(k)$$

or

$$f'(k) - \delta = n + \gamma$$

The Golden Rule level of capital is such that the marginal product of capital, net of depreciation, is equal to the growth rate. If $k$ is larger than this level in steady state there is overaccumulation.

# Mathematical aside

The Solow growth equation is an example of a *first order ordinary differential equation.* Such equations are expressed as:

$$\dot{x}(t) = \frac{dx}{dt} = g(x(t)), t \in T$$

for some function $g$. Here $x(t) \in X = R_+$ (the *state space*) and $T$ is the time interval ($[0, \infty)$ in our Solow discussion)
A *solution* is a function $x : T \to X$ such that $x(t)$ that satisfies the ODE for each $t \in T$. A *steady state* is a constant solution, i.e. $x^* \in X$ such that

$$0 = g(x^*)$$

If $x^*$ is a steady state, it is locally a.s. if $g'(x^*) < 0$, and unstable if the inequality is reversed. (Draw *phase diagram* to see why.)

*Convergence*: suppose $g$ is *linear*:

$$\dot{x} = g(x) = ax$$

Any solution is of the form

$$x(t) = x_0 e^{at}$$

which converges to $x^* = 0$ if $a < 0$. We then say that $x^*$ is (*globally)* stable.

For *nonlinear* $g$, global convergence is much more involved, but *local* convergence is simple: If $x^*$ is a steady state, it is locally a.s. if $g'(x^*) < 0$, and unstable if the inequality is reversed. (Draw *phase diagram* to see why.)