The Solow Growth Model II

Econ 504

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The Solow model is an example of a command economy, but we usually think of it as giving the outcomes of a related market economy.

The market economy is populated by households and firms.

There is a continuum of identical households of measure one. In each period $t$, the representative household can work $L(t)$ hours at wage $W(t)$, in terms of an arbitrary numeraire. The household also own capital and sells its services at a rental rate $R(t)$. Finally, it receives any profits from firms, $\Pi(t)$. 

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$L(t)$ grows at the exogenous rate $n$.

As for capital accumulation, assume that the household saves a fraction $s$ of income in the form of new capital. Hence, if $P(t)$ is the price of output at time $t$,

$$P \dot{K} = s[\Pi + WL + RK] - P \delta K$$

or

$$\dot{K} = s[\pi + wL + rK] - \delta K$$
The representative firm chooses $Y(t)$, $K(t)$, and $L(t)$ to maximize profits:

$$\pi = Y - rK - wL$$

subject to

$$Y = F(K, AL)$$
A *competitive equilibrium* is a collection of paths for prices, $r(t)$ and $w(t)$, $t \geq 0$, output, capital accumulation and profits, such that at each $t$:

1. Given prices, $Y(t)$, $K(t)$ and $L(t)$ are optimal for the firm
2. The household’s budget constraint is satisfied.
3. All markets clear (already implicit in the notation)
In any competitive equilibrium, \( K(t) \) is optimal for the firm if:

\[
r(t) = F_1(K(t), A(t)L(t))
\]

and \( L(t) \) if:

\[
w(t) = A(t)F_2(K(t), A(t)L(t))
\]

Also, substituting \( \pi \) into the household budget constraint, any CE must satisfy:

\[
\dot{K} = sY - \delta K = sF(K, AL) - \delta K
\]

- But the latter is just the Solow equation!
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- It follows that any competitive equilibrium path must satisfy the Solow equation.
- Likewise, given a solution of the Solow equation, one can construct a CE by defining $r(t)$ and $w(t)$ as above, and setting $\pi = 0$ (prove the last statement).
The Solow growth equation is an example of a first order ordinary differential equation. Such equations are expressed as:

\[ \dot{x}(t) = \frac{dx}{dt} = g(x(t)), \quad t \in T \]

for some function \( g \). Here \( T \) is the time interval \([0, \infty)\) in our Solow discussion.

A solution is a function \( x(t) \) that satisfies the ODE for each \( t \). A steady state is a constant solution, i.e. \( x^* \) such that

\[ 0 = g(x^*) \]
Convergence: suppose $g$ is *linear*:

$$\dot{x} = g(x) = ax$$

The solution is

$$x(t) = x_0 e^{at}$$

which converges to $x^* = 0$ if $a < 0$.

For *nonlinear* $g$, global convergence is much more involved, but *local* convergence is simple: If $x^*$ is a steady state, it is locally a.s. if $g'(x^*) < 0$, and unstable if the inequality is reversed. (Draw *phase diagram* to see why.)
Time is indexed by $t = 0, 1, 2, ...$
The model equations are:

\[
\begin{align*}
Y_t &= F(K_t, A_t L_t) \\
L_t &= (1 + n)L_{t-1} \\
A_t &= (1 + \gamma)A_{t-1} \\
K_{t+1} &= (1 - \delta)K_t + I_t \\
I_t &= sY_t
\end{align*}
\]
Now the fundamental equation of growth is:

\[(1 + \gamma)(1 + n)k_{t+1} = (1 - \delta)k_t + sf(k_t)\]

A *solution* is a path for \(k_t\) that satisfies this equation for any \(t\).

A *steady state* is given by the solution of:

\[(1 + \gamma)(1 + n)k^* = (1 - \delta)k^* + sf(k^*)\]

To study convergence, one often looks at a representation of the growth equation in \((k_t, k_{t+1})\) space.
The fundamental equation is an example of a difference equation of the form

\[ x_{t+1} = g(x_t) \]

If \( x^* \) is a steady state, it is locally stable if \(|g'(x^*)| < 1\), unstable if the inequality is reversed. (Again, think about the linear case first for intuition.)
Following Farmer, assume that the production function is

\[ Y_t = v_t F(K_t, A_t L_t) \]

where \( v_t \) is a shock with small, bounded support. The growth equation now becomes

\[ (1 + \gamma)(1 + n)k_{t+1} = (1 - \delta)k_t + v_t sf(k_t) \]
The nonstochastic steady state is given by:

$$(1 + \gamma)(1 + n)\bar{k} = (1 - \delta)\bar{k} + \bar{vsf}(\bar{k})$$

where $\bar{v}$ is the mean of $v_t$. 
The growth equation is nonlinear, but in a neighborhood of the steady state it can be approximated with linear methods.

In particular, defining \( \hat{k}_t = k_t - \bar{k} / \bar{k} \approx \log k_t - \log \bar{k} \), etc. we obtain:

\[
\hat{k}_{t+1} = \theta \hat{k}_t + \phi \hat{v}_t
\]  

(1)

(The RHS of the Solow equation is:

\[
\frac{1 - \delta}{(1 + \gamma)(1 + n)} \hat{k}_t + \left[ 1 - \frac{1 - \delta}{(1 + \gamma)(1 + n)} \right] (\hat{v}_t + \eta_f \hat{k}_t)
\]

where \( \eta_f \) is the elasticity of \( f \) at the ss. Hence \( \phi \) is the term in brackets, and \( \theta = (1 - \phi) + \phi \eta_f \)
Just to make sure: start with the Solow equation:

$$LHS_t \equiv (1 + \gamma)(1 + n)k_{t+1} = (1 - \delta)k_t + v_t sf(k_t) \equiv RHS_t$$

We approximate both sides of the equation:

$$\overline{LHS}_t = \hat{k}_{t+1}$$

$$\overline{RHS}_t = \frac{(1 - \delta)\bar{k}}{(1 - \delta)\bar{k} + \bar{v}sf(\bar{k})} \hat{k}_t + \frac{\bar{v}sf(\bar{k})}{(1 - \delta)\bar{k} + \bar{v}sf(\bar{k})} (\hat{v}_t + \hat{f}(\hat{k}_t))$$

$$= \frac{1 - \delta}{(1 + \gamma)(1 + n)} \hat{k}_t + \left[1 - \frac{1 - \delta}{(1 + \gamma)(1 + n)}\right] (\hat{v}_t + \eta_f \hat{k}_t)$$
The linearized law of motion for capital must be complemented by some assumption about the shocks. So, for instance, if:

\[ \log v_t = (1 - \rho) \log \bar{v} + \rho \log v_{t-1} + \varepsilon_t \]

where \( \varepsilon_t \) is white noise, then

\[ \hat{v}_t = \rho \hat{v}_{t-1} + \varepsilon_t \quad (2) \]
How do we know if the dynamic system 1-2 converges to a well behaved stable *distribution*? The presence of uncertainty complicates matters, but intuitively it is at least necessary that the *deterministic* system:

\[
E_0 \hat{k}_{t+1} = \theta E_0 \hat{k}_t + \phi E_0 \hat{v}_t \\
E_0 \hat{v}_t = \rho E_0 \hat{v}_{t-1}
\]  

must converge, for any initial \((\hat{k}_0, \hat{v}_0)\). It turns out that stability of the deterministic system is sufficient as well.
Defining $X_t = (E_0 \hat{k}_{t+1}, E_0 \hat{\nu}_t)$, the system can be rewritten as

$$X_{t+1} = MX_t$$

(4)

where

$$M = \begin{bmatrix}
\theta & \phi \rho \\
0 & \rho
\end{bmatrix}$$

This leads to the question of when a system of the form 4 is stable. A sufficient condition is that the eigenvalues of $M$ be less than one in absolute value. (Prove this!)

In this case, the eigenvalues are $\rho$ and $\theta$ (why?) So the stability condition is that $|\theta| < 1$. (Show this.)
Computing Empirical Implications of the Model

Consider the system characterized by 1 and 2:

\[
\hat{k}_{t+1} = \theta \hat{k}_t + \phi \hat{v}_t \\
= \theta \hat{k}_t + \phi (\rho \hat{v}_{t-1} + \epsilon_t) \\
= \theta \hat{k}_t + \phi \rho \hat{v}_{t-1} + \phi \epsilon_t \\
\hat{v}_t = \rho \hat{v}_{t-1} + \epsilon_t
\]

Define \( s_t = (\hat{k}_{t+1}, \hat{v}_t)' \), so that the system is:

\[
s_t = Ms_{t-1} + Ne_t
\]

where, as before,

\[
M = \begin{bmatrix} \theta & \phi \rho \\ 0 & \rho \end{bmatrix}
\]

and

\[
N = \begin{bmatrix} \phi \\ 1 \end{bmatrix}
\]

This representation is convenient to compute variances, etc.
One is usually interested in the *impulse responses*: the expected response to a one time shock (i.e. an innovation to $\varepsilon_0$).

To compute them, starting from the nonstochastic steady state, just set $\hat{v}_0 = E_0 \hat{v}_0 = 1$, and $\hat{k}_0 = 0$, compute the expected path of capital from 3, etc.
The variance of $s_t$ is the solution $\Sigma_0$ of:

$$\Sigma_0 = Es_t s_t' = M\Sigma_0 M' + NN'\sigma^2_\varepsilon$$

We can also compute, say, the autocovariance matrix at first lag:

$$\Sigma_1 = Es_t s_{t-1}' = MSE_{s_{t-1}} s_{t-1}' = M\Sigma_0$$

etc.

**Exercise**: (RBCs of ABCs) If $s = 0.2$, $\delta = 0.1$, $n = 0.02$, $\alpha = 0.36$, $\gamma = 0$, and technology shocks are i.i.d., the variance of capital is 0.0955 times the variance of $\varepsilon$. Check this.
Characterizing the behavior of other variables

Once we have the motion of $k_t$ we can find the motion of other variables. For example, from

\[ y_t = v_t f(k_t) \]

one gets

\[ \hat{y}_t = \hat{v}_t + \eta_f \hat{k}_t \]

or $\eta_f \hat{k}_t = \hat{y}_t - \hat{v}_t$. Inserting in the law of motion for $\hat{k}$,

\[ \hat{y}_{t+1} - \hat{v}_{t+1} = \theta(\hat{y}_t - \hat{v}_t) + \eta_f \phi \hat{v}_t \]

or

\[ \hat{y}_{t+1} = \theta \hat{y}_t + u_{t+1} \]

where

\[ u_t = \hat{v}_t + (\eta_f \phi - \theta) \hat{v}_{t-1} \]

If $\rho = 0$, then $\hat{y}_t$ follows an ARMA(1,1) process.
Alternatively,

\[ \hat{y}_t = \hat{v}_t + \eta_f \hat{k}_t = H_0 s_t + H_1 s_{t-1} \]

with \(H_0 = [0 1], H_1 = [\eta_f 0]\), one can compute the variance of \(\hat{y}_t\):

\[
E \hat{y}_t \hat{y}_t' = E \left( H_0 s_t s_t' H_0' + H_0 s_t s_{t-1}' H_1' + H_1 s_{t-1} s_t' H_0' + H_1 s_{t-1} s_{t-1}' H_1 \right) \\
= H_0 \Sigma_0 H_0' + H_1 \Sigma_0 H_1' + H_0 \Sigma_1 H_1' + H_1 \Sigma_{-1} H_0' 
\]
Finally, note that

\[ \hat{y}_t = \log y_t - \log \bar{y} \]

\[ = \log \left( \frac{Y_t}{A_t L_t} \right) - \log \bar{y} \]

\[ = \log Y_t - \log A_t - \log L_t - \log \bar{y} \]

\[ = \log Y_t - t \log [(1 + \gamma)(1 + n)] + \text{constant} \]

Inserting in \( \hat{y}_{t+1} = \theta \hat{y}_t + u_{t+1} \), this implies that, in levels,

\[ \log Y_{t+1} - (t + 1) \log [(1 + \gamma)(1 + n)] + \text{constant} \]

\[ = \theta (\log Y_t - t \log [(1 + \gamma)(1 + n)] + \text{constant}) + u_{t+1} \]

That is,

\[ \log Y_{t+1} = \text{constant} + \theta \log Y_t + (\text{another constant})(t + 1) + u_{t+1} \]