Solving the Optimal Growth Model

Optimal Growth: Solution

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We saw that the solution of the Ramsey problem is given by:

\[
\begin{align*}
  u'(C_t) &= \beta u'(C_{t+1}) f'(K_{t+1}) \\
  K_{t+1} &= f(K_t) - C_t
\end{align*}
\]

plus the two boundary conditions: \( K_0 > 0 \) given and transversality.

The next step is to use this information to solve for the resulting optimal path. Let us do that, starting with a local analysis.
The steady state is given by

\[ u'(C^*) = \beta u'(C^*) f'(K^*) \]

e.g.

\[ 1 = \beta f'(K^*) \]

and

\[ C^* = f(K^*) - K^* \]

Note that the first equation gives the steady state level of capital and is independent of preferences. This is called a *modified Golden Rule*. One often uses it to calibrate the value of \( \beta \).
Use lowercase letters for log (or percentage) deviations from the ss, so:

\[ c_t = \log C_t - \log C^* \approx \frac{C_t - C^*}{C^*} \]

Starting with:

\[ u'(C_t) = \beta u'(C_{t+1}) f'(K_{t+1}) \]

leads to:

\[ -\sigma c_t = -\sigma c_{t+1} - \gamma k_{t+1} \]

where \( \gamma = -K^* f''(K^*) / f'(K^*) \) is (minus) the elasticity of \( f' \), \( \sigma = -C^* u''(C^*) / u'(C^*) \) is (minus) the elasticity of \( u' \).
Likewise, from

$$C_t + K_{t+1} = f(K_t)$$

we get

$$\omega c_t + (1 - \omega) k_{t+1} = \theta k_t$$

where $\omega = C^*/(C^* + K^*) = C^*/Y^*$ is the consumption/output ratio and $\theta = K^* f'(K^*)/f(K)$ is the elasticity of the production function.
So we have two linear difference equations:

\[ \sigma c_t = \sigma c_{t+1} + \gamma k_{t+1} \]
\[ \omega c_t + (1 - \omega)k_{t+1} = \theta k_t \]

Since \( K_0 \) is given, \( k_0 = \log K_0 - \log K^* \) is also given. We need another boundary condition, which intuitively corresponds to the transversality condition.
Recall that the dynamic programming view says that the solution should be of the form $C_t = g(K_t), K_{t+1} = h(K_{t+1})$, that is, that the policy function should be a time invariant function of the state at $t$.

So, let us conjecture that this is true of the linearized system as well! Then, a candidate for the solution is of the form:

\[
\begin{align*}
    c_t &= \nu_c k_t \\
    k_{t+1} &= \nu_k k_t
\end{align*}
\]

where $\nu_c, \nu_k$ are coefficients to be determined.
The key observation is that, if our conjecture is true, the linearized equations should always hold no matter what the value of $k_t$ is. This (hopefully) gives conditions on $\nu_c$ and $\nu_k$ that should suffice to determine them completely.

So, insert the guess into:

$$\omega c_t + (1 - \omega)k_{t+1} = \theta k_t$$

one gets

$$\omega \nu_c k_t + (1 - \omega)\nu_k k_t = \theta k_t$$

and if this is true for any $k_t$,

$$\omega \nu_c + (1 - \omega)\nu_k = \theta$$

This furnishes one restriction on the two unknowns $\nu_c, \nu_k$. 
The other equation:

\[ \sigma c_t = \sigma c_{t+1} + \gamma k_{t+1} \]

presents a slight difficulty, since inserting the guess into it (and noting that \( c_{t+1} = \nu c k_{t+1} \) according to the guess) yields

\[ \sigma \nu c k_t = \sigma \nu c k_{t+1} + \gamma \nu k k_t \]

We cannot get rid of the \( k_t \) terms yet, because \( k_{t+1} \) appears in the RHS. But this is no problem, since using the guess again,

\[ \sigma \nu c k_t = \sigma \nu c \nu k k_t + \gamma \nu k k_t \]

so that we get a second restriction on \( \nu c, \nu k : \)

\[ \sigma \nu c = \sigma \nu c \nu k + \gamma \nu k \]
Summarizing, if our guess is true, the coefficients $\nu_c, \nu_k$ must satisfy

$$\omega \nu_c + (1 - \omega) \nu_k = \theta$$
$$\sigma \nu_c = \sigma \nu_c \nu_k + \gamma \nu_k$$

This is a nonlinear (quadratic) system of two equations in the two unknowns $\nu_c, \nu_k$.

**Question:** This system will typically lead to two solutions. Which one is the right one?

**Answer:** To ensure convergence to the steady state (and hence that the transversality condition holds), the correct solution must respect

$$|\nu_k| < 1$$

In the optimal growth problem, this is satisfied only by one of the solutions.
In fact, one can show that

\[ v_k = \frac{1 + (1/\beta) + \chi \theta \pm \sqrt{(1 + (1/\beta) + \chi \theta)^2 - 4/\beta}}{2} \]

where \( \chi = C^*/K^* \).

Using this expression, it is apparent that only the root with the minus sign gives the correct solution.
Generalizing the Argument

The undetermined coefficients method just outlined generalizes to larger systems and is, in fact, the starting point of existing software.

Many models reduce to a system that can be written as

\[ AX_{t+1} = BX_t \]  

(1)

where \( X_t = (J'_t, Z'_t) \) is a (column) vector of \( nj \) jumping variables \( (J_t) \) and \( np \) state variables \( (Z_t) \).

To solve for a convergent solution, one then postulates that the solution has the form:

\[ J_t = MZ_t \]

\[ Z_{t+1} = NZ_t \]
The method of undetermined coefficients is essentially the same as before: one uses the guesses just made into 1 to eliminate all variables but $Z_t$, arriving to a system of equations that need to be solved for the unknown $M$ and $N$. 
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Such a system will typically be nonlinear, so one needs to use the restriction that the matrix $N$ must be stable (i.e. must have eigenvalues with absolute value less than one).
Suppose that $A$ in $AX_{t+1} = BX_t$ has an inverse. Then the system can be written as

$$X_{t+1} = \Pi X_t$$

for some matrix $\Pi$.

This looks like a standard vector difference equation but, what are the boundary conditions?
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- This looks like a standard vector difference equation but, what are the boundary conditions?
- The initial value of the predetermined variables provide $np$ initial restrictions on $X_0$.
- Other restrictions are derived from the requirement that $X_t$ must converge to zero as $t$ goes to infinity.
Blanchard and Kahn (1982): a unique convergent path is guaranteed if and only if the number of unstable eigenvalues of $\Pi$ (with absolute value greater than one) is exactly $nj$. That implies that there are $nj$ additional restrictions on $X_0$, which guarantees that the system jumps to the saddlepath. If the number of unstable eigenvalues of $\Pi$ is less than $nj$, the solution is indeterminate (there is an infinite number of convergent paths). The latter cannot happen in the optimal growth model (think about why!) but can happen in other models.
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