

Some Notes on the Optimal Growth Model

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These are remarks on issues related to the optimal growth problem. They are intended for my 504 class, so use at your own peril!

1. Method of Undetermined Coefficients

The solution of the system

$$\begin{aligned}u'(C_t) &= \beta u'(C_{t+1}) f'(K_{t+1}) \\ K_{t+1} &= f(K_t) - C_t\end{aligned}$$

will be approximated around the steady state by a solution of the linearized equations

$$\begin{aligned}\sigma c_t &= \sigma c_{t+1} + \gamma k_{t+1} \\ \omega c_t + (1 - \omega) k_{t+1} &= \phi k_t\end{aligned}$$

where lowercase letters are percentage or log deviations of the steady state, $\gamma = -k^* f''(k^*)/f'(k^*)$ is (minus) the elasticity of f' , $\sigma = -c^* u''(c^*)/u'(c^*)$ is (minus) the elasticity of u' , $\omega = C^*/Y^*$, and ϕ the elasticity of f . Noting that, at the ss, $\phi/(1 - \omega) = 1/\beta$, it is convenient to rewrite the equations as

$$c_t = c_{t+1} + \theta k_{t+1}$$

$$k_{t+1} = \frac{1}{\beta} k_t - \chi c_t$$

where $\theta = \gamma/\sigma$ and $\chi = C^*/K^*$.

We posit that the solution has the form

$$c_t = v_c k_t$$

$$k_{t+1} = v_k k_t$$

The previous two equations furnish two new equations in the unknown coefficients. For example,

$$k_{t+1} = \frac{1}{\beta} k_t - \chi c_t \implies v_k k_t = \frac{1}{\beta} k_t - \chi v_c k_t$$

that is,

$$v_k = \frac{1}{\beta} - \chi v_c \tag{1}$$

Also,

$$c_t = c_{t+1} + \theta k_{t+1} \implies v_c k_t = v_c k_{t+1} + v_k k_t$$

Note that we need to eliminate k_{t+1} from the RHS, but this is straightforward:

$$v_c k_t = v_c k_{t+1} + v_k k_t \implies v_c k_t = v_c v_k k_t + v_k k_t$$

so,

$$v_c = v_c v_k + v_k \tag{2}$$

The two equations 1-2 can now be solved for v_c and v_k . Solving for v_k yields

$$v_k = \frac{1 + (1/\beta) + \chi\theta \pm \sqrt{(1 + (1/\beta) + \chi\theta)^2 - 4/\beta}}{2}$$

Which root to take? For stability, we have to take a root that is less than one in absolute value. There is at most one such root.

1.1. Larger Systems

The method just outlined generalizes to larger systems. In fact, often one will have to deal with a system that can be written as

$$AX_{t+1} = BX_t \tag{3}$$

where $X_t = (J_t', Z_t')$ is a (column) vector of nj jumping variables (J_t) and np state variables (Z_t). To solve for a convergent solution, one postulates that the solution has the form:

$$\begin{aligned} J_t &= MZ_t \\ Z_{t+1} &= NZ_t \end{aligned}$$

The method of undetermined coefficients is essentially the same as before: one uses the guesses just made in 3 to eliminate all variables but Z_t , arriving to a system of equations that need to be solved for the unknown M and N . Such a system will typically be nonlinear, so one needs to use the restriction that the matrix N must be stable (i.e. must have eigenvalues with absolute value less than one).

1.2. Blanchard-Kahn

Suppose that A in 3 has an inverse. Then the system can be written as

$$X_{t+1} = \Pi X_t$$

for some matrix Π . This looks like a standard vector difference equation but, what are the boundary conditions? The initial value of the predetermined variables provide np initial restrictions on X_0 . Other restrictions are derived from the requirement that X_t must converge to zero as t goes to infinity. Blanchard and Kahn established that this is guaranteed if and only if the number of unstable eigenvalues of Π (with absolute value greater than one) is exactly nj . That implies that there are nj additional restrictions on X_0 , in order to guarantee that the system jumps to the saddlepath.

If the number of unstable eigenvalues of Π is less than nj , the solution is *indeterminate*.

1.3. Uncertainty

Suppose now that the production function is

$$Y_t = A_t f(K_t)$$

where A_t is a Markov process.

The solution is easiest to derive if we take a dynamic programming viewpoint. The state is given by (K, A) , and the Bellman equation should be:

$$\begin{aligned} v(K, A) &= \text{Max } u(C) + \beta E\{v(K', A')|A\} \\ \text{s.t. } C + K' &= Af(K) \end{aligned}$$

One way to solve this is the dynamic programming way, i.e. solve for the function v using the Bellman equation. It is more usual to look at the first order conditions:

$$u'(C) = \beta E\{v_1(K', A')|A\}$$

noting that

$$v_1(K, A) = u'(C)Af'(K)$$

so that we arrive at the Euler condition

$$u'(C) = \beta E\{u'(C')Af'(K')|A\}$$

This is usually written as

$$u'(C_t) = \beta E_t\{u'(C_{t+1})A_{t+1}f'(K_{t+1})\}$$

Also,

$$C_t + K_{t+1} = A_t f(K_t)$$

The two preceding equations plus the exogenously assumed behavior of A_t describe the dynamics of the system.

The *nonstochastic* steady state is then defined as the solution of

$$u'(C) = \beta u'(C)Af'(K)$$

$$C + K = Af(K)$$

for A equal to the expected value of A_t . That is, it is the steady state of the associated deterministic economy in which A_t is always equal to its expected value. Note that if $A = 1$ this is exactly the same as in the deterministic case.

Then one proceeds almost as before, linearizing the dynamic system around the ss. One gets, for the Euler equation,

$$-\sigma c_t = E_t\{-\sigma c_{t+1} + a_{t+1} - \gamma k_{t+1}\}$$

and for the resource constraint

$$\omega c_t + (1 - \omega)k_{t+1} = a_t + \phi k_t$$

The behavior of a_t is typically approximated by

$$a_{t+1} = \rho a_t + \varepsilon_{t+1}$$

where ε_{t+1} is white noise. Note that this implies

$$E_t a_{t+1} = \rho a_t$$

To solve this system, one defines

$$X_t = \begin{bmatrix} k_t \\ a_t \\ c_t \end{bmatrix}$$

and arranges the system as

$$AE_t X_{t+1} = BX_t$$

The method of undetermined coefficients involves solving for

$$c_t = Ms_t$$

$$k_{t+1} = Ns_t$$

where

$$s_t = \begin{bmatrix} k_t \\ a_t \end{bmatrix}$$

is the state at t .

Once M and N have been computed, one can calculate *impulse responses* and *population moments* in a straightforward way. One can also simulate the system. To do this, note that we can now write:

$$\begin{aligned} s_{t+1} &= \begin{bmatrix} k_{t+1} \\ a_{t+1} \end{bmatrix} = \begin{bmatrix} k_{t+1} \\ \rho a_t \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \varepsilon_{t+1} \\ &= Ps_t + Q\varepsilon_{t+1} \end{aligned}$$

where P is the matrix with N as the top row and $(0 \ \rho)$ as the bottom row, and Q is the column vector $(0 \ 1)'$.

Suppose we want to compute $\Sigma_0 = E s_t s_t'$. From the above,

$$\Sigma_0 = E s_t s_t' = E(P s_{t-1} + Q \varepsilon_t)(P s_{t-1} + Q \varepsilon_t)'$$

or

$$\Sigma_0 = P \Sigma_0 P' + Q Q' \sigma_\varepsilon^2$$

That is, Σ_0 must solve the preceding equation.

It is now easy to find other covariance matrices. For instance, given any $j = 1, 2, \dots$

$$\Sigma_j = E s_t s_{t-j}' = \rho^j E s_{t-j} s_{t-j}' = \rho^j \Sigma_0$$

Exercises: Farmer, Chapter 3, 3-4 (pp 60-61).

2. Continuous Time

As in the Solow model, assume:

$$c_t + i_t = f(k_t)$$

and

$$\dot{k}_t = i_t - \delta k_t$$

(note that I am reintroducing depreciation) so that we get

$$\frac{dk}{dt} = \dot{k}_t = f(k_t) - \delta k_t - c_t \tag{4}$$

The representative agent's utility function is given by

$$\int_0^{\infty} e^{-\rho t} u(c_t) dt$$

where $\rho > 0$ is the *rate of time preference*.

At time t , k_t is the *state* and c_t the *control*. Given a path for c_t , the motion of the state is given by (4).

The standard way to tackle this problem is to introduce a *costate* variable λ_t which acts as a kind of Lagrange multiplier. Then one forms the *Hamiltonian*:

$$H_t = e^{-\delta t} u(c_t) + \lambda_t (f(k_t) - \delta k_t - c_t)$$

The *Maximum Principle* now says that if c_t is optimal,

- at each t , c_t must maximize the Hamiltonian H_t
- the law of motion of the costate is given by

$$\dot{\lambda}_t = -\frac{\partial H_t}{\partial k_t}$$

Let see what are the implications. If c_t maximizes H_t , it follows that

$$\lambda_t = e^{-\rho t} u'(c_t) \quad (5)$$

Note that the costate has the interpretation of the shadow value of output at t .

The second part of the Maximum Principle gives:

$$\dot{\lambda}_t = -\lambda_t(f'(k_t) - \delta) \quad (6)$$

The last two equations together with (4) give a dynamical system on (k, λ) , with c implicitly as a function of (k, λ) . What are the boundary conditions? As in the discrete time formulation, k_0 is given. The other condition is that the value of capital left over should converge to zero:

$$\lim_{t \rightarrow \infty} \lambda_t k_t = 0$$

In this problem, it is useful to eliminate the costate variable using (5) and express the system as one in (c, k) . Take the derivative with respect to t in both sides of (5) to get:

$$\dot{\lambda}_t = -\delta e^{-\rho t} u'(c_t) + e^{-\rho t} u''(c_t) \dot{c}_t$$

or, dividing both sides by λ ,

$$\frac{\dot{\lambda}_t}{\lambda_t} = -\rho + \frac{u''(c_t) \dot{c}_t}{u'(c_t)}$$

But, using 6 and rearranging, we obtain the *Euler* condition:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma(c_t)} [f'(k_t) - \delta - \rho]$$

where

$$\sigma(c) = -\frac{cu''(c)}{u(c)}$$

is the *coefficient of relative risk aversion*, or *the inverse of the coefficient of intertemporal substitution*.

Note what this tells us: when investment is very productive relative to impatience, consumption grows faster. That is, the agent saves, accumulates more capital, and today's consumption falls relative to tomorrow's. The speed of adjustment depends on σ .

The only solution satisfies the transversality condition:

$$\lim_{t \rightarrow \infty} e^{-\rho t} u'(c_t) k_t = 0$$

Consider the special case in which the CRRA is constant at σ . Then the equations of motion are given by

$$\begin{aligned}\dot{c} &= \frac{1}{\sigma} [f'(k) - \delta - \rho] c \\ \dot{k} &= f(k) - c - \delta k\end{aligned}$$

I have suppressed the t subscripts since they are not needed: this is a system of *autonomous* equations.

A graphical analysis is possible in (c, k) space. One first graphs the \dot{c} locus and the \dot{k} locus. The first one is just the equation:

$$c = f(k) - \delta k$$

The second is the k axis ($c = 0$) and the $f'(k) = \rho + \delta$ parallel to the c axis. A steady state is an intersection of the two locuses.

The global dynamics will be explained in class. In particular, we will find that the motion of the system is given by the *saddlepath*, which has the form $c = g(k)$. This is the same as the policy function in dynamic programming.

Likewise, it should be emphasized that linear methods and undetermined coefficients find a linear approximation to the saddlepath.

(Now you can read Lucas's paper on endogenous growth if you like.)

3. Market Interpretation

We will posit an economy with a representative household and a representative firm.

3.1. Households

The household owns capital and the firms, and has a unit endowment of labor that it supplies inelastically to the market. Its preferences are given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

Its budget constraint:

$$c_t + i_t = w_t + r_t k_t + \pi_t$$

where w_t and r_t denote the real wage rate and the rental rate of capital, respectively, while π_t denote the profits of firms. Capital evolves according to:

$$k_{t+1} = (1 - \delta)k_t + i_t$$

Note that I am allowing for incomplete depreciation of capital here.

You should show that the household's problem implies the Euler equation:

$$u'(c_t) = \beta u'(c_{t+1})(1 + r_{t+1} - \delta)$$

The interpretation is the same as usual, except that the return from saving one unit of goods is $r_{t+1} + (1 - \delta)$.

The household's transversality condition is

$$\lim_{t \rightarrow \infty} \beta^t u'(c_t) k_{t+1} = 0$$

3.2. Firms

The typical firm maximizes profits

$$\pi_t = y_t - r_t k_t - w_t l_t$$

subject to the production function

$$y_t = F(k_t, l_t)$$

As before, an optimal plan involves:

$$\begin{aligned} w_t &= F_2(k_t, l_t) \\ r_t &= F_1(k_t, l_t) \end{aligned}$$

3.3. Equilibrium

A *competitive equilibrium* is an allocation $(c_t, i_t, k_{t+1}, l_t, y_t)$ and price system (w_t, r_t) such that:

- Given the price system, (c_t, i_t, k_{t+1}) is an optimal plan for the household
- Given the price system, (y_t, l_t, k_t) is optimal for the firm (i.e. maximizes profits)
- All markets clear

The market clearing condition implies, in particular, that $l_t = 1$.

Take a competitive equilibrium. Insert the expression for profits, π_t , into the households budget constraint and get:

$$c_t + i_t = y_t$$

that is

$$f(k_t) = F(k_t, 1) = c_t + k_{t+1} - (1 - \delta)k_t$$

Also, from the household's Euler condition,

$$u'(c_t) = \beta u'(c_{t+1})[f'(k_{t+1}) + (1 - \delta)]$$

The last two equations are the same as those in the solution of the optimal growth problem (where we had assumed $\delta = 1$), and the transversality condition is the same. So we have proved that *any competitive equilibrium allocation solves the optimal growth problem*.

The converse is true: the solution of the optimal growth problem can be decentralized as an equilibrium of the market economy. To show this, note that what you need to identify to complete the market equilibrium is the associated price system. But the needed prices are just given by the marginal products of labor and capital.

4. Further Remarks

Let us elaborate on the problem of the household in the preceding section. In particular, we have stated that the budget constraint is:

$$c_t + i_t = w_t + r_t k_t + \pi_t$$

$$k_{t+1} = (1 - \delta)k_t + i_t$$

Consolidating, we have that

$$k_{t+1} = (1 + r_t - \delta)k_t + w_t + \pi_t - c_t$$

Here, k_0 is given.

Let us expand the model a little bit, and allow individual households to borrow and lend from each other. In equilibrium, of course, each household's assets must equal capital, $a_t = k_t$. Also, the rate of interest on borrowing must equal the rate of return on capital accumulation. So, in each period,

$$a_{t+1} = (1 + r_t - \delta)a_t + w_t + \pi_t - c_t \quad (7)$$

This means that, in the first period,

$$\begin{aligned} a_1 &= (1 + r_0 - \delta)a_0 + w_0 + \pi_0 - c_0 \\ &= (1 + r_0 - \delta)a_0 + z_0 \end{aligned}$$

where $z_t = w_t + \pi_t - c_t$. Likewise, in the next period,

$$a_2 = (1 + r_1 - \delta)a_1 + z_1$$

or

$$\frac{1}{R_1}a_2 = a_1 + \frac{1}{R_1}z_1$$

where $R_1 = (1 + r_1 - \delta)$

Same,

$$\begin{aligned} \frac{1}{R_2}a_3 &= \frac{1}{R_1}a_1 + \frac{1}{R_2}a_2 \\ &= a_1 + \frac{1}{R_1}z_1 + \frac{1}{R_2}z_2 \end{aligned}$$

where $R_2 = (1 + r_1 - \delta)(1 + r_2 - \delta)$

We continue and conclude that for any $T \geq 1$,

$$\begin{aligned} \frac{1}{R_T}a_{T+1} &= a_1 + \frac{1}{R_1}z_1 + \frac{1}{R_2}z_2 + \dots + \frac{1}{R_T}z_T \\ &= (1 + r_0 - \delta)a_0 + z_0 + \frac{1}{R_1}z_1 + \frac{1}{R_2}z_2 + \dots + \frac{1}{R_T}z_T \end{aligned}$$

Finally, from the definition of z_t ,

$$\sum_{t=0}^T \frac{c_t}{R_t} + \frac{1}{R_T} a_{T+1} = (1 + r_0 - \delta)k_0 + \sum_{t=0}^T \frac{w_t + \pi_t}{R_t}$$

Now, letting T go to infinity on the RHS, and then on the LHS,

$$\sum_{t=0}^{\infty} \frac{c_t}{R_t} + \lim_{T \rightarrow \infty} \frac{1}{R_T} a_{T+1} \leq (1 + r_0 - \delta)k_0 + \sum_{t=0}^{\infty} \frac{w_t + \pi_t}{R_t}$$

Assuming that

$$\lim_{T \rightarrow \infty} \frac{1}{R_T} a_{T+1} \geq 0 \tag{8}$$

we now have

$$\sum_{t=0}^{\infty} \frac{c_t}{R_t} \leq (1 + r_0 - \delta)k_0 + \sum_{t=0}^{\infty} \frac{w_t + \pi_t}{R_t} \tag{9}$$

This says that the present value of consumption cannot exceed the present value of (nonfinancial income) plus initial wealth.

What is the meaning of (8)? If some household violated it, some other household would have to be accumulating assets with strictly positive value in the limit. This cannot be optimal for the lender. So the assumption is that nobody would agree to lend to a household whose assets do not satisfy 8. This is called a No Ponzi Game condition, for now obvious reasons.

Summarizing, we have shown that the sequence of budget constraints 7 and the No Ponzi Game condition 8 imply the present value budget constraint 9.

Can we show the opposite? Suppose that a consumption plan $\{c_t\}$ satisfies the present value budget constraint 1 with equality. Define assets at the end of any period $T \geq 0$ by:

$$\frac{1}{R_T} a_{T+1} = (1 + r_0 - \delta)k_0 + \sum_{t=0}^T \frac{w_t + \pi_t - c_t}{R_t}$$

Then,

$$\begin{aligned} \frac{1}{R_T} a_{T+1} &= (1 + r_0 - \delta)k_0 + \frac{w_T + \pi_T - c_T}{R_T} + \sum_{t=0}^{T-1} \frac{w_t + \pi_t - c_t}{R_t} \\ &= \frac{1}{R_{T-1}} a_T + \frac{w_T + \pi_T - c_T}{R_T} \end{aligned}$$

which implies, recalling the definition of R_T ,

$$a_{T+1} = (1 + r_T - \delta)a_T + w_T + \pi_T - c_T$$

So the sequence $\{a_t\}_{t=1}^{\infty}$ satisfies the sequence of budget constraints 7. Also, by construction,

$$\lim_{T \rightarrow \infty} \frac{1}{R_T} a_{T+1} = 0$$

In sum: we have shown that the sequence of budget constraints 7 and the No Ponzi condition 8 are equivalent to the present value budget constraint 9. This has very important consequences and applications.

Some remarks are in order:

- One implication is that household's problem can be equivalently formulated as maximizing lifetime utility subject to the present value budget constraint (9). This is, for example, the way the problem is formulated in Romer, ch. 2.
- This view stresses that consumption depends on the *present value* of income, and not on the timing of income. This has important consequences, some of which we will discuss in the course.
- For the sequence of budget constraints to be equivalent to a single present value budget constraint markets must be *complete*. The meaning and significance of market completeness will be investigated later in the course, but suffice it to say here that markets are complete in the case here because there is no uncertainty and we ended up assuming that there is borrowing and lending.

4.1. Application: The Importance (Or Not) of Fiscal Deficits

In the previous setting, suppose that there is a government that needs to spend some exogenous amount g_t in each period. To finance this, the government can either impose lump sum taxes on the representative household, τ_t , or borrow an amount b_{t+1} . The government's constraint is, hence,

$$b_{t+1} = g_t - \tau_t + R_t^* b_t$$

where R_t^* is the gross rate of return (principal plus interest) on government bonds. For simplicity, assume $b_0 = 0$ (i.e. no government debt inherited at time 0).

The representative household's budget constraint is now

$$a_{t+1} = (1 + r_t - \delta)a_t + w_t + \pi_t - c_t - \tau_t$$

where now a_t includes the amount lent to the government. The rest of the model is as before.

A competitive equilibrium is now defined as an allocation, a price system, *and* a government financing policy $\{\tau_t, b_{t+1}\}$ such that:

- given prices, the allocations are optimal for private agents (as before)
- markets clear, *and*
- the government budget constraint is satisfied at all times.

The first thing to note is that $R_t^* = 1 + r_t - \delta$ (why?). Now note the following: per our previous argument, the household's sequence of budget constraints is equivalent to the present value one:

$$\sum_{t=0}^{\infty} \frac{c_t}{R_t} = (1 + r_0 - \delta)k_0 + \sum_{t=0}^{\infty} \frac{w_t + \pi_t}{R_t} - \sum_{t=0}^{\infty} \frac{\tau_t}{R_t}$$

where the only difference is the presence of the discounted value of taxes in the RHS. This says that government policy affects private decisions through the present value of taxes and, importantly, only through that present value.

Assuming, however, that

$$\lim_{T \rightarrow \infty} \frac{b_{T+1}}{R_T} = 0$$

then one can show that

$$\sum_{t=0}^{\infty} \frac{\tau_t}{R_t} = \sum_{t=0}^{\infty} \frac{g_t}{R_t}$$

So the present value of taxes are equal to the present value of expenditure. But the latter is exogenous. This implies that government policy matters *only* because of the present value of expenditure. Whether expenditure is financed via taxes or via debt is immaterial. In other words, fiscal deficits are irrelevant. This is a crucial result known as *Ricardian Equivalence*.

Are we justified in imposing the condition $\lim_{T \rightarrow \infty} \frac{b_{T+1}}{R_T} = 0$? An argument similar to previous ones implies that $\lim_{T \rightarrow \infty} \frac{b_{T+1}}{R_T}$ cannot be strictly greater than zero (it would mean that some household is acting suboptimally). But, can $\lim_{T \rightarrow \infty} \frac{b_{T+1}}{R_T}$ be strictly less than zero? This would imply that the government is taxing households more than it needs to finance expenditure, the difference being lent to the public.