In the market formulation of the optimal growth model, we assumed that the representative household faced a *sequence* of budget constraints:

\[
c_t^h + i_t^h = w_t + r_t k_t^h + \pi_t
\]

\[
k_{t+1}^h = (1 - \delta) k_t^h + i_t^h
\]

for each \( t = 0, 1, \ldots \), with \( k_0^h = k_0 \) given.

One interpretation is that markets open *every period*. 
In general equilibrium theory, it is often assumed (following Arrow-Debreu) that trading only occurs effectively only once, at the beginning of time (i.e., at the start of $t = 0$). Then the economy unfolds over time just executing the trades agreed upon at that initial market.
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In fact, this scenario turns out to be equivalent to the sequential markets formulation under some conditions on the completeness of markets (Arrow).
To see how the equivalence works, let us expand the model a little bit, and allow individual households to *borrow and lend* from each other.

Let:

\[ b_{h}^{t+1} = \text{household } h\text{'s loans in period } t. \]
\[ a_{h}^{t+1} = b_{h}^{t+1} + k_{h}^{t+1} = \text{household } h\text{'s total assets at the end of period } t. \]

Note: the rate of return on loans must equal the rate of return on capital.

Also note that, in equilibrium, \( b_{t}^{h} = b_{t} = 0 \) and \( a_{t}^{h} = k_{t} \). But we need to examine the household’s budget constraint under arbitrary prices.
The *period* $t$ budget constraint is now:

\[ a_{t+1}^h = b_{t+1}^h + k_{t+1}^h \]
\[ = (1 + r_t - \delta)(b_t^h + k_t^h) + w_t + \pi_t - c_t^h \]
\[ = (1 + r_t - \delta)a_t^h + w_t + \pi_t - c_t^h \]
\[ = R_t a_t^h + z_t^h \]

where we have defined, for convenience,

\[ z_t^h = w_t + \pi_t - c_t^h. \]
\[ R_t = (1 + r_t - \delta) \]
\[ a_{t+1}^h = R_t a_t^h + z_t^h \]

Let us look at the implications. For \( t = 0 \),

\[ a_1^h = R_0 a_0^h + z_0^h \]

Likewise, in the next period,

\[ a_2^h = R_1 a_1^h + z_1^h \]

Combining the two,

\[ \frac{a_2^h}{R_1} = \frac{z_1^h}{R_1} + a_1^h \]

\[ = \frac{z_1^h}{R_1} + z_0^h + R_0 a_0^h \]
Repeating, we conclude that for any $T \geq 1$,

$$Q_T a_{T+1}^h = R_0 a_0^h + z_0^h + Q_1 z_1^h + Q_2 z_2^h + \ldots + Q_T z_T^h$$

where we have defined $Q_t$ as the period 0 price of period $t$ consumption:

$$Q_0 = 1$$

$$Q_t = Q_{t-1}.(1/R_t)$$

$$= \Pi_{j=1}^t R_j^{-1}$$

Finally, recalling the definition of $z_t^h$, the preceding gives, for any $T$,

$$\sum_{t=0}^T Q_t c_t^h + Q_T a_{T+1}^h = R_0 a_0^h + \sum_{t=0}^T Q_t (w_t + \pi_t)$$
\[ \sum_{t=0}^{T} Q_t c_t^h + Q_T a_{T+1}^h = R_0 a_0^h + \sum_{t=0}^{T} Q_t (w_t + \pi_t) \]

Now, assuming that
\[ \lim_{T \to \infty} Q_T a_{T+1}^h = 0 \] (1)
we can take limits in both sides and obtain
\[ \sum_{t=0}^{\infty} Q_t c_t^h = R_0 a_0^h + \sum_{t=0}^{\infty} Q_t (w_t + \pi_t) \]

This says that the sequence of budget constraints plus the condition (1) imply a present value budget constraint: at \( t = 0 \), the present value of consumption cannot exceed the present value of (nonfinancial) income plus initial wealth.
What is the meaning of \( \lim_{T \to \infty} Q_T a^h_{T+1} = 0 \)? If some household violated it, some other household would have to be accumulating assets with strictly positive value in the limit. This cannot be optimal for the lender. So the assumption is that nobody would agree to lend to a household whose assets do not satisfy (1). This is called a No Ponzi Game condition, for now obvious reasons.
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Summarizing, we have shown that the sequence of budget constraints and the No Ponzi Game condition (1) imply the *present value* budget constraint.

One can show the converse: the present value budget constraint implies both the sequence of budget constraints and the No Ponzi Game condition (1).
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Summarizing, we have shown that the sequence of budget constraints and the No Ponzi Game condition (1) imply the present value budget constraint.

One can show the converse: the present value budget constraint implies both the sequence of budget constraints and the No Ponzi Game condition.
Take a consumption plan \( \{c_t^h\} \) that satisfies the PV budget constraint, and \textit{define} assets at the end of any period \( T \geq 0 \) by:

\[
Q_T a_{T+1}^h = R_0 a_0^h + \sum_{t=0}^{T} Q_t (w_t + \pi_t - c_t^h)
\]

Then,

\[
Q_T a_{T+1}^h = R_0 a_0^h + \sum_{t=0}^{T-1} Q_t (w_t + \pi_t - c_t^h) + Q_T (w_T + \pi_T - c_T^h)
\]

\[
= Q_{T-1} a_T^h + Q_T (w_T + \pi_T - c_T^h)
\]

Divide both sides by \( Q_T \) gives, recalling the definition of \( Q_T \),

\[
a_{T+1}^h = R_T a_T^h + w_T + \pi_T - c_T^h
\]

So the sequence \( \{c_t^h, a_t^h\}_{t=1}^{\infty} \) satisfies the sequence of budget constraints.

Also, by construction,

\[
\lim_{T \to \infty} Q_T a_{T+1}^h = 0
\]
One implication is that the household’s problem can be equivalently formulated as maximizing lifetime utility subject to the present value budget constraint. (This is, for example, the way the problem is formulated in Romer, ch. 2.)
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This view stresses that consumption depends on the present value of income, and not on the timing of income. This has important consequences, some of which we will discuss later in the course.
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For the sequence of budget constraints to be equivalent to a single present value budget constraint markets must be complete. The meaning and significance of market completeness will be investigated later in the course. Suffice it to say for now that markets are complete in our setting here because there is no uncertainty and we assumed that there is borrowing and lending.
Modifying our setting, suppose that there is a government that needs to spend some exogenous amount $g_t$ in each period.
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To finance this, the government can either impose lump sum taxes on the representative household, $\tau_t$, or borrow an amount $b_{t+1}$. 

$$b_t + 1 = g_t \tau_t + R_t b_t$$

where $R_t$ is the gross rate of return (principal plus interest) on government bonds. For simplicity, assume $b_0 = 0$ (i.e. no government debt inherited at time 0).
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The government’s constraint is, hence,

\[
b_{t+1} = g_t - \tau_t + R_t^* b_t
\]

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For simplicity, assume $b_0 = 0$ (i.e. no government debt inherited at time 0).
The representative household’s budget constraint is now

\[ a_{t+1}^h = (1 + r_t - \delta) a_t^h + w_t + \pi_t - c_t^h - \tau_t \]

where now \( a_{t+1}^h \) includes the amount lent to the government. The rest of the model is as before.
A competitive equilibrium is now defined as an allocation, a price system, and a government financing policy $\{\tau_t, b_{t+1}\}$ such that:

- given prices, the allocations are optimal for private agents (as before)
A competitive equilibrium is now defined as an allocation, a price system, and a government financing policy \( \{\tau_t, b_{t+1}\} \) such that:

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A competitive equilibrium is now defined as an allocation, a price system, and a government financing policy $\{\tau_t, b_{t+1}\}$ such that:

- given prices, the allocations are optimal for private agents (as before)
- markets clear, and
- the government budget constraint is satisfied at all times.
The first thing to note is that $R_t^* = R_t = 1 + r_t - \delta$.

Now note the following: per our previous argument, the household’s sequence of budget constraints is equivalent to the present value one:

$$\sum_{t=0}^{\infty} Q_t c_t^h = R_0 a_0^h + \sum_{t=0}^{\infty} Q_t (w_t + \pi_t) - \sum_{t=0}^{\infty} Q_t \tau_t$$

where the only difference is the presence of the discounted value of taxes in the RHS.

This says that government policy affects private decisions through the present value of taxes and, importantly, only through that present value.
Assuming, however, that

\[ \lim Q_T b_{T+1} = 0 \]

then one can show (using a similar argument as with the household’s budget constraints) that

\[ \sum_{t=0}^{\infty} Q_t \tau_t = \sum_{t=0}^{\infty} Q_t g_t \]

So the present value of taxes are equal to the present value of government expenditure.
Now note:

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- Since the present value of taxes is equal to the present value of government expenditure, only the latter matters for the household’s problem.

This implies that government policy matters only because of the present value of expenditure. Whether expenditure is financed via taxes or via debt is immaterial. In other words, fiscal deficits are irrelevant. This is a crucial result known as Ricardian Equivalence.
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- We showed that the household’s problem depends on fiscal policy only through the present value of taxes.
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- This implies that government policy matters only because of the present value of expenditure. Whether expenditure is financed via taxes or via debt is immaterial. In other words, fiscal deficits are irrelevant. This is a crucial result known as Ricardian Equivalence.
Are we justified in imposing the condition $\lim Q_T b_{T+1} = 0$?

- An argument similar to previous ones implies that $\lim Q_T b_{T+1}$ cannot be strictly greater than zero (it would mean that some household is acting suboptimally).
Are we justified in imposing the condition $\lim Q_T b_{T+1} = 0$?

- An argument similar to previous ones implies that $\lim Q_T b_{T+1}$ cannot be strictly greater than zero (it would mean that some household is acting suboptimally).
- But, can $\lim Q_T b_{T+1}$ be strictly less than zero? This would imply that the government is taxing households more than it needs to finance expenditure, the difference being lent to the public.