Overlapping Generations Models

Econ 504

Roberto Chang

Rutgers

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- But also they have unexpected theoretical properties.
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Samuelson (1953), Diamond (1967)
A Pure Exchange Case

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- In addition, at $t = 1$ there is a generation ("zero") of already *old* agents, which live only for that period.
- Hence, in each $t$, there are two kinds of agents, young and old.
- For simplicity, assume all generations are of equal size (normalized to one).
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Goods and Endowments

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- Generation zero agents are endowed with $e_2$ at $t = 1$
- Hence the *aggregate* endowment is constant
In each period $t$, there is a market for borrowing and lending
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Let $R_t = (1 + r_t)$ denote the interest rate on lending between periods $t$ and $t + 1$
Agent $t$ then solves:

$$\begin{align*}
\text{Max } & u(c_{1t}) + u(c_{2t+1}) \\
\text{s.t. } & c_{1t} + s_t = e_1 \\
& c_{2t+1} = e_2 + R_t s_t \\
& c_{1t}, c_{2t} \geq 0
\end{align*}$$

The solution is given by the present value budget constraint and the Euler equation:

$$\begin{align*}
c_{1t} + \frac{c_{2t+1}}{R_t} &= e_1 + \frac{e_2}{R_t} \\
u'(c_{1t}) &= R_t u'(c_{2t+1})
\end{align*}$$
The solution also gives a savings function

\[ s_t = e_1 - c_{1t} = s(R_t) \]
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At \( t = 1 \), generation zero agents *must* consume the value of their endowments:

\[ c_{21} = e_2 \]
A competitive equilibrium is a sequence of interest rates, \{R_t\}_{t \geq 1} and a consumption allocation \{c_2, (c_1, c_2+1)_{t \geq 1}\} such that:

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- For each $t \geq 1$, $(c_{1t}, c_{2t+1})$ solves agent $t$'s problem given $R_t$
- $c_{21} = e_2$
- For each $t \geq 1$,

$$c_{1t} + c_{2t} = e_1 + e_2$$
Implications of Equilibrium

Claim: In any competitive equilibrium, $c_{1t} = e_1$ and $c_{2t} = e_2$, all $t$.

Proof: We know $c_{21} = e_2$, and hence, from the economy’s resource constraint, $c_{11} = e_1$. 

The equilibrium interest rate sequence must then be:

$$R_t = u_0(c_{1t})u_0(c_{2t+1}) = u_0(e_1)u_0(e_2)$$
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- Hence, $s_1 = e_1 - c_{11} = 0$, and from agent 1’s budget constraint, $c_{22} = e_2$.
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- Therefore \( c_{12} = e_1 \) from resource constraint, etc. (recursive definition)
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- Hence, $s_1 = e_1 - c_{11} = 0$, and from agent 1’s budget constraint, $c_{22} = e_2$
- Therefore $c_{12} = e_1$ from resource constraint, etc. (recursive definition)
- The equilibrium interest rate sequence must then be:

\[
R_t = \frac{u'(c_{1t})}{u'(c_{2t+1})} = \frac{u'(e_1)}{u'(e_2)} \equiv R_a
\]
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So, in the Samuelson case, market equilibria are inefficient!
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This condition has been tested in the past.
Imagine that agents in generation zero are given $M$ units of "money".
The Role of "Fiat Money"

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Is it possible for money to have nonzero value? How?

Agents acquiring money today must believe that they can use it tomorrow to buy goods.
Let $q_t = \text{price of money in terms of goods (the inverse of the price level)}$. 
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Hence

$$R_t = \frac{q_{t+1}}{q_t}$$

And the savings of generation $t$ agents is

$$s_t = s(R_t) = s\left(\frac{q_{t+1}}{q_t}\right)$$
Also, in equilibrium,

\[ s_t = q_t M \]
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Hence an equilibrium with positive valued money exists if there is a positive sequence \( q_t, t \geq 1 \) that solves

\[ s\left(\frac{q_{t+1}}{q_t}\right) = q_t M \]
If there is a monetary steady state, $q_t = q^* > 0$, all $t \geq 1$, and

$$s\left(\frac{q^*}{q^*}\right) = s(1) = q^* M$$
Existence of a Monetary Steady State

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The resulting allocation is efficient (it is in fact a Golden Rule allocation).
Let \( m_t = q_t M \) be the real quantity of money, and rewrite the equilibrium equation as

\[
s\left(\frac{q_{t+1}}{q_t}\right) = s\left(\frac{m_{t+1}}{m_t}\right) = m_t
\]

A monetary equilibrium is given by a positive sequence \( m_t, t \geq 1 \).

The steady state is given by \( m_\infty = q_\infty M \).

There is a continuum of other monetary equilibria in which money loses value in the long run. The monetary steady state may or may not be stable.
Stability and multiplicity of monetary equilibria

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Suppose now that the government consumes a constant amount $g$ in each period.
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Hence

$$g = q_t(M_t - M_{t-1})$$

with $M_0 = M$, in our previous notation.
Everything else is the same, except that:

\[ g = m_t - \frac{q_t}{q_{t-1}} m_{t-1} = m_t - R_t m_{t-1} \]

i.e.

\[ R_t = \frac{m_t - g}{m_{t-1}} \]

and the difference equation is

\[ m_t = s \left( \frac{m_{t+1} - g}{m_t} \right) \]
The Inflation tax and the Laffer curve

- Focus on steady states

\[ g = \gamma R \]

In steady state, we must have:

\[ g = m(R) = \left(1 - \frac{1}{R}\right) s(R) \]

Note that \( \left(1 - \frac{1}{R}\right) \) can be interpreted as a tax on money holdings (the inflation tax).
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If $B_t = B^* > 0$, then focusing in steady states,

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$$g = m_t - R_t m_{t-1} + B^*(1 - R^*)$$
$$g = m(1 - R) + B^*(1 - R^*)$$

This says that, in ss, the higher $B^*$ the more revenue we need from inflation.
**Corollary (unpleasant arithmetic):** A reduction in $M_t - M_{t-1}$ today can lead to higher inflation in the future (in the steady state). This is because, given $g$, a reduction in money growth today can increase $B^*$. 