New Notes on the Solow Growth Model*

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1 The Model

The first ingredient of a dynamic model is the description of the time horizon. In the original Solow model, time is continuous and the horizon is infinite. Without loss of generality assume that time is indexed by t in $[0, \infty)$.

At each point in time, there is only one final good that is produced via the *aggregate production function*:

$$Y = F(K, AL)$$

Here Y, A, K, and L denote output, labor productivity, capital, and labor, and are functions of time (i.e. we should really write $Y(t)$ and so on, but we omit time arguments when not needed). The product AL is called effective labor.

The function F is has neoclassical properties:

- F is continuously differentiable, strictly increasing, strictly concave
- F has constant returns to scale

A key example is the *Cobb Douglas* function:

$$F(K, AL) = K^{\alpha}(AL)^{1-\alpha}, 0 < \alpha < 1$$

Exercise: Show that the Cobb Douglas function enjoys the neoclassical properties listed above.

Once you assume the existence of the aggregate production function, it is clear that the growth of output can be due to growth of A, K, or L.

Solow assumed that A and L grow at exogenous rates:

$$\begin{aligned} \dot{L} &= \frac{dL}{dt} = nL \\ \dot{A} &= \gamma A \end{aligned}$$

*These are revised but still rough notes for Econ 504.

At time $t = 0$, A, and L are given by history. Then one can describe the value of L and A at each point in time:

$$L(t) = L(0)e^{nt}$$

$$A(t) = A(0)e^{\gamma t}$$

So it remains to describe the motion of capital. Assume $K(0)$ is given by history. The accumulation of capital must follow:

$$\dot{K} = I - \delta K$$

where I denotes investment and δ the depreciation rate of capital.

As usual, investment is assumed to equal savings. The key behavioral equation of the Solow model is that savings equal a constant fraction of output, so:

$$I = sY$$

This completes the description of the model.

2 The Solution

A solution is a description of the values of L, A, Y, and K at each point in time. Now, from the capital accumulation equation:

$$\begin{aligned}
\dot{K} &= sY - \delta K \\
&= sF(K, AL) - \delta K \\
&= ALsF(K/AL, 1) - \delta K \\
&= AL[sF(k, 1) - \delta k]
\end{aligned}$$

Here we have defined $k = K/AL$. Define $f(k) = F(k, 1)$, the intensive production function. Now we can rewrite the previous equation as:

$$\left(\dot{K}/K\right) k = sf(k) - \delta k$$

But $k = K/AL$ implies

$$\dot{k}/k = \dot{K}/K - (n + \gamma).$$

Combine the two to obtain:

$$\dot{k} = sf(k) - (n + \gamma + \delta)k$$

This is the *fundamental equation of growth*.

What is the intuition? Capital per effective labor grows faster when there is more investment, hence the role of the savings coefficient s. On the other hand, population growth, productivity growth, and faster capital depreciation tend to reduce capital per unit of effective labor.

Obviously once we have solved for the path of k we can obtain $K = kAL$, and Y. So we can focus on the solution for k.

A *steady state* is a constant value of k, say k^*, such that $k(t) = k^*$ is a solution of the growth equation. Hence k^* must solve:

$$sf(k^*) = (n + \gamma + \delta)k^*$$

As an example, if the production function is CD, $f(k) = k^\alpha$, so:

$$s(k^*)^\alpha = (n + \gamma + \delta)k^*$$

that is

$$k^* = \left[\frac{s}{n + \gamma + \delta}\right]^{1/1-\alpha}$$

More generally, under the Inada conditions $f(0) = 0, f'(0) = \infty, f'(\infty) = 0$ there is a unique nontrivial steady state. This can be shown using the famous Solow diagram. (Do it.)

The same diagram can be employed to show that k must grow when it is between 0 and k^*, and falls if it is greater than k^*. Hence the model implies *global* convergence towards the steady state.

What kind of questions can we answer with this model? In general, we can trace the adjustment path to changes in the parameters of the economy (s, δ, n, γ), or in initial conditions $(K(0), L(0), A(0))$

In the steady state, however, k is constant. Hence the *extensive* variables, such as K and Y, grow at the rate $(n + \gamma)$. Per capita variables, such as Y/L, grow at rate γ. In this sense, the model does *not* explain long run growth. See Romer, chapter 1, for a discussion.

Romer also discusses the effect of a permanent increase in the savings rate s. A most intriguing result is that such a permanent change has only transitory effects on the growth rate (e.g. a *level effect,* but not a *growth effect*).

Another classic question leads to the *Golden Rule*. Suppose that one can choose s. What would an optimal value of s be? In growth theory, it was proposed to choose s so as to maximize steady state consumption. Now, in steady state,

$$c = (1 - s)f(k)$$

so that the optimal rule would require

$$\frac{dc}{ds} = -f(k) + (1 - s)f'(k)\frac{dk}{ds} = 0$$

In steady state, the Solow equation is

$$sf(k) = (n + \gamma + \delta)k$$

so

$$\frac{dk}{ds} = \frac{f(k)}{(n+\gamma+\delta) - sf'(k)}$$

Combining the preceding, we get

$$(1-s)f'(k) = (n+\gamma+\delta) - sf'(k)$$

or

$$f'(k) - \delta = n + \gamma$$

The Golden Rule level of capital is such that the marginal product of capital, net of depreciation, is equal to the growth rate. If k is larger than this level in steady state there is overaccumulation.

2.1 Mathematical aside

The Solow growth equation is an example of a *first order ordinary differential equation.* Such equations are expressed as:

$$\dot{x}(t) = \frac{dx}{dt} = g(x(t)), t \in T$$

for some function g. Here T is the time interval ($[0,\infty)$ in our Solow discussion)

A solution is a function $x(t)$ that satisfies the ODE for each t. A steady state is a constant solution, i.e. x^* such that

$$0 = g(x^*)$$

If x^* is a steady state, it is locally a.s. if $g'(x^*) < 0$, and unstable if the inequality is reversed. (Draw *phase diagram* to see why.)

3 Market interpretation of the Solow Model

The Solow model is an example of a *command* economy, but we usually think of it as giving the outcomes of a related *market* economy. It is useful to think about this connection.

Assume that there is a continuum of identical households of measure one. In each period t, the household can work $L(t)$ hours at wage $W(t)$, in terms of an arbitrary numeraire The household also sells the services of its capital, $K(t)$, at a rental rate $R(t)$. Finally, it receives firm profits, $\Pi(t)$.

$L(t)$ grows at the exogenous rate n. As for capital accumulation, assume that the household saves a fraction s of income in the form of new capital. Hence, if $P(t)$ is the price of output at time t,

$$P\dot{K} = s[\Pi + WL + RK] - P\delta K$$

or

$$\dot{K} = s[\pi + wL + rK] - \delta K$$

The representative firm chooses $y(t), K(t)$, and $L(t)$ to maximize profits:

$$\pi = Y - rK - wL$$

subject to

$$Y = F(K, AL)$$

A *competitive equilibrium* is a collection of paths for prices, $r(t)$ and $w(t)$, output, capital accumulation and profits, such that:

1. Given prices, $Y(t), K(t)$ and $L(t)$ are optimal for the firm at each t
2. The household's budget constraint is satisfied.

In any competitive equilibrium, $K(t)$ is optimal for the firm if:

$$r(t) = F_1(K(t), A(t)L(t))$$

and $L(t)$ if:

$$w(t) = A(t)F_2(K(t), A(t)L(t))$$

Also, substituting π into the household budget constraint, any CE must satisfy:

$$\dot{K} = sY - \delta K = sF(K, AL) - \delta K$$

But the latter is just the Solow equation! It follows that any competitive equilibrium path must satisfy the Solow equation. Likewise, given a solution of the Solow equation, one can construct a CE by defining $r(t)$ and $w(t)$ as above, and setting $\pi = 0$ (prove the last statement).

3.1 Mathematical Notes

$$F_1(K, AL) = F_1(K/AL, 1) = F_1(k, 1) = f'(k)$$

$$F_2(K, AL) = F_2(k, 1) = f(k) - kf'(k)$$

Prove the last (Euler implies: $F(K, AL) = KF_1 + ALF_2$. Divide both sides by AL, you get $f(k) = kF_1 + F_2$.)

Show with Cobb Douglas, and also note there that the income shares are given by α and $(1 - \alpha)$.

4 Discrete Time

Time is indexed by $t = 0, 1, 2, ...$

The equations are:

$$\begin{aligned}
Y_t &= F(K_t, A_t L_t) \\
L_t &= (1+n)L_{t-1} \\
A_t &= (1+\gamma)A_{t-1} \\
K_{t+1} &= (1-\delta)K_t + I_t \\
I_t &= sY_t
\end{aligned}$$

Now the fundamental equation of growth is:

$$(1+\gamma)(1+n)k_{t+1} = (1-\delta)k_t + sf(k_t)$$

A solution is a path for k_t that satisfies this equation for any t. A *steady state* is given by the solution of:

$$(1+\gamma)(1+n)k^* = (1-\delta)k^* + sf(k^*)$$

To study convergence, one often looks at a representation of the growth equation in (k_t, k_{t+1}) space.

The fundamental equation is an example of a difference equation of the form

$$x_{t+1} = g(x_t)$$

If x^* is a steady state, it is locally stable if $|g'(x^*)| < 1$, unstable if the inequality is reversed.

5 Introducing Uncertainty

Follow Farmer and assume that the production function is

$$Y_t = v_t F(K_t, A_t L_t)$$

where v_t is a shock with small, bounded support.

The growth equation now becomes

$$(1+\gamma)(1+n)k_{t+1} = (1-\delta)k_t + v_t s f(k_t)$$

The *nonstochastic steady state* is given by:

$$(1+\gamma)(1+n)\bar{k} = (1-\delta)\bar{k} + \bar{v} s f(\bar{k})$$

where $\bar{v}$ is the mean of v_t.

The growth equation is now nonlinear, but in a neighborhood of the steady state it can be appoximated with linear methods. In particular, defining $\hat{k}_t = k_t - \bar{k}/\bar{k}$, etc. we obtain:

$$\hat{k}_{t+1} = a_1 \hat{k}_t + c_1 \hat{v}_t \tag{1}$$

(The RHS of the Solow equation is:

$$\frac{1-\delta}{(1+\gamma)(1+n)}\hat{k}_t + \left[1 - \frac{1-\delta}{(1+\gamma)(1+n)}\right]\left(\hat{v}_t + \eta_f \hat{k}_t\right)$$

where η_f is the elasticity of f at the ss. This gives a_1 and c_1.)

Usually the linearized law of motion for capital is complemented by some assumption about the shocks. So, for instance, if:

$$\log v_t = (1-\rho)\log \bar{v} + \rho \log v_{t-1} + \varepsilon_t$$

where ε_t is white noise, then

$$\hat{v}_t = \rho \hat{v}_{t-1} + \varepsilon_t \tag{2}$$

5.1 Digression on Stability

How do we know if the dynamic system 1-2 converges to a well behaved stable distribution? The presence of uncertainty complicates matters, but intuitively it is at least necessary that the *deterministic* system:

$$\begin{aligned} E_0\hat{k}_{t+1} &= a_1 E_0\hat{k}_t + c_1 E_0\hat{v}_t \\ E_0\hat{v}_{t+1} &= \rho E_0\hat{v}_t \end{aligned} \quad (3)$$

must converge, for any initial $(\hat{k}_0, \hat{v}_0)$. It turns out that stability of the deterministic system is sufficient as well.

Defining $X_t = (\hat{k}_t, \hat{v}_t)$, the system can be rewritten as

$$X_{t+1} = \Psi X_t \quad (4)$$

where

$$\Psi = \begin{matrix} a_1 & c_1 \\ 0 & \rho \end{matrix}$$

This leads to the question of when a system of the form 4 is stable. A sufficient condition is that the eigenvalues of Ψ be less than one in absolute value. (Prove this!)

In this case, the eigenvalues are ρ and a_1 (why?) So the stability condition is that $|a_1| < 1$. You should show this holds!

5.2 Computing Moments

Consider the system characterized by 1 and 2:

$$\begin{aligned} \hat{k}_{t+1} &= a_1\hat{k}_t + c_1\hat{v}_t \\ &= a_1\hat{k}_t + c_1(\rho\hat{v}_{t-1} + \varepsilon_t) \\ &= a_1\hat{k}_t + c_1\rho\hat{v}_{t-1} + c_1\varepsilon_t \\ \hat{v}_t &= \rho\hat{v}_{t-1} + \varepsilon_t \end{aligned}$$

Define $s_t = (\hat{k}_{t+1}, \hat{v}_t)'$ so that the system is:

$$s_t = M s_{t-1} + N\varepsilon_t$$

where

$$M = \begin{matrix} a_1 & c_1\rho \\ 0 & \rho \end{matrix}$$

$$N = \begin{matrix} c_1 \\ 1 \end{matrix}$$

This representation is convenient to compute variances, etc. For example, the variance of the state vector is the solution Σ_0 of:

$$\Sigma_0 = E s_t s_t' = M\Sigma_0 M' + NN'\sigma_\varepsilon^2$$

We can also compute, say, the autocovariance matrix at first lag:

$$\Sigma_1 = Es_t s'_{t-1} = MEs_{t-1}s'_{t-1} = M\Sigma_0$$

etc.

Exercise: RBCs of ABCs mentions that is $s = 0.2, \delta = 0.1, n = 0.02, \alpha = 0.36, \gamma = 0$, and technology shocks are i.i.d., the variance of capital is 0.0955 times the variance of ε. Check this!

(Hint: You may want to show first that $c_1 = \left[\frac{n+\delta}{(1+n)}\right]$, $a_1 = \alpha c_1 + \frac{1-\delta}{(1+n)}$.).

6 Using the solution to characterize the behavior of other variables

Once we have the motion of k_t we can find the motion of other variables. In particular, from

$$y_t = v_t f(k_t)$$

one gets

$$\hat{y}_t = \hat{v}_t + \eta_f \hat{k}_t$$

or $\eta_f \hat{k}_t = \hat{y}_t - \hat{v}_t$. Inserting in the law of motion for $\hat{k}$,

$$\hat{y}_{t+1} - \hat{v}_{t+1} = a_1(\hat{y}_t - \hat{v}_t) + \eta_f c_1 \hat{v}_t$$

or

$$\hat{y}_{t+1} = a_3 \hat{y}_t + u_{t+1}$$

where

$$u_t = \hat{v}_t + (\eta_f c_1 - a_1)\hat{v}_{t-1}$$

Finally, note that

$$\begin{aligned} \hat{y}_t &= \log y_t - \log \bar{y} \\ &= \log(Y_t/A_t L_t) - \log \bar{y} \\ &= \log Y_t - \log A_t - \log L_t - \log \bar{y} \\ &= \log Y_t - t\log[(1+\gamma)(1+n)] + \text{ constant} \end{aligned}$$

This implies that,in levels,

$$\log Y_{t+1} = b + a_3 \log Y_t + ct + u_{t+1}$$

One is usually interested in the *impulse responses.* To compute them, just set $\hat{v}_0 = E_0\hat{v}_0 = 1$, and $\hat{k}_0 = 0$, compute the expected path of capital from 3, etc.

Likewise, the equation

$$\hat{y}_t = \hat{v}_t + \eta_f \hat{k}_t$$

has the format:

$$z_t = Hs_t$$

where

$$H = (\eta_f, 1)'$$

The variance of z_t is then

$$\Omega_0 = E z_t z_t' = H \Sigma_0 H'$$

7 The Solow Residual

We have postulated a law of motion of v, but in practice we do not observe v directly. How can we learn something about the properties to the productivity shocks?

For simplicity, assume that F is Cobb Douglas, so that the production function is:

$$Y_t = v_t K_t^{\alpha} (A_t L_t)^{1-\alpha}$$

Taking logs and recalling that $A_t = A_0(1+\gamma)^t$

$$\log Y_t = \log v_t + \alpha \log K_t + (1-\alpha)\log L_t + (1-\alpha)[t \log(1+\gamma) + \log A_0]$$

This implies that

$$\varphi_0 + \varphi_1 t + \log v_t = \log Y_t - [\alpha \log K_t + (1-\alpha)\log L_t] \equiv Z_t$$

where $\varphi_0 = (1-\alpha)\log A_0, .\varphi_1 = (1-\alpha)\log(1+\gamma)$. If we could observe Z_t we could regress it against a constant and a time trend, and take the residuals as estimates of log v_t.

Solow reasoned that an estimate of α can be obtained by using the fact that, if factor markets are competitive, $(1-\alpha)$ should be equal to wage income as a fraction of total income. The resulting value of Z_t is called the *Solow residual* and has many applications in macroeconomics.

For example, the difference $\Delta Z_t = Z_t - Z_{t-1}$ is often taken to be a measure of growth not explained by factor accumulation. This is natural since $\Delta Z_t = \Delta \log Y_t - \Delta [\alpha \log K_t + (1-\alpha)\log L_t]$. However, you should be careful with the interpretation. Suppose that there is no uncertainty (so $logv_t = 0$) and that the labor force is constant. Then $\Delta Z_t = (1-\alpha)\log(1+\gamma)$. Hence this procedure would attribute α percent of growth to factor accumulation. But we know that, in this version of the model, the rate of growth is equal to γ and so, in this sense, growth is caused solely by exogenous labor productivity growth.