Cheat Sheet on Linearization

Econ 504

September 18, 2008

A number of people have asked for help with linearization "tricks." This is really a math issue that you should review on your own, but here are some useful hints.

1. Scalar Case

Suppose that you have an equation of the form:

$$Z_t = f(X_t)$$

which you want to linearize around a point $(\bar{Z}, \bar{X})$ such that

$$\bar{Z} = f(\bar{X})$$

The point $(\bar{Z}, \bar{X})$ is often taken to be the nonstochastic steady state.

Assume that Z_t and X_t are scalars, and f is a continuously differentiable function. The starting fact is *Taylor's theorem*, which says that

$$Z_t = f(X_t) = f(\bar{X}) + f'(\bar{X})(X_t - \bar{X})$$

where the last equality is only up to a second order residual, which we ignore in what follows.

If $\bar{Z}$ and $\bar{X}$ are both nonzero, the previous expression can be written (prove this!) as:

$$z_t = \eta_f x_t \tag{1}$$

where lowercase variables denote percentage deviations (i.e. $x_t = (X_t - \bar{X})/\bar{X}$) and

$$\eta_f = \frac{\bar{X} f'(\bar{X})}{f(\bar{X})}$$

is the *elasticity* of f at $\bar{X}$. In the models we have looked at, this means that the percentage deviation from steady state of a function of a variable X_t is equal to the elasticity of that function, evaluated at the steady state, times the percentage deviation of X_t from its own steady state. Note that this is almost the definition of elasticity of f (= the percentage change in the value of the function due to a one percent change in its argument.)

Power trick. An important special case is the power function:

$$Z_t = X_t^{\omega}$$

for which $\eta_f = \omega$, independently of $\bar{X}$ (show this), and hence

$$z_t = \omega x_t$$

You may call this the "power trick." What is important is that the "trick " is just a special case of the more general rule (1).

2. Functions of Many Variables

The argument generalizes to functions of more than one variable: if, for instance,

$$Z_t = f(X_t, Y_t)$$

and we want to approximate its behavior around a point at which

$$\bar{Z} = f(\bar{X}, \bar{Y})$$

you should be able to show that

$$z_t = \eta_f^x x_t + \eta_f^y y_t$$

where

$$\eta_f^x = \frac{\bar{X} f_1(\bar{X}, \bar{Y})}{f(\bar{X}, \bar{Y})}, \eta_f^y = \frac{\bar{Y} f_2(\bar{X}, \bar{Y})}{f(\bar{X}, \bar{Y})}$$

are the *partial* elasticities of f with respect to X and Y respectively.

You should now show various other "tricks " now follow from the above:

Sum trick: if

$$Z_t = X_t + Y_t$$

then

$$z_t = \frac{\bar{X}}{\bar{X}+\bar{Y}} x_t + \frac{\bar{Y}}{\bar{X}+\bar{Y}} y_t$$

In words, the percentage deviation from steady state of a sum is the weighted average of the percentage deviations of the members of the sum.

Product trick: if

$$Z_t = X_t Y_t$$

then

$$z_t = x_t + y_t$$

Ratio trick.

$$Z_t = X_t / Y_t ==> z_t = x_t - y_t$$

(assuming, of course, that $\bar{Y}$ is nonzero).

3. Composite Functions

Finally, suppose that

$$Z_t = f(Y_t), Y_t = g(X_t)$$

so

$$Z_t = f(g(X_t)) \equiv h(X_t)$$

The Chain Rule gives:

$$h'(X_t) = f'(g(X_t))g'(X_t) = f'(Y_t)g'(X_t)$$

so the Taylor approximation around a point $(\bar{Z}, \bar{X}, \bar{Y})$ such that $\bar{Z} = f(\bar{Y}), \bar{Y} = g(\bar{X})$ is

$$Z_t = \bar{Z} + h'(\bar{X})(X_t - X_t) = \bar{Z} + f'(\bar{Y})g'(\bar{X})(X_t - \bar{X})$$

You should show the *composite function trick*:

$$z_t = \eta_f^y \eta_g^x x_t$$

where

$$\eta_f^y = \frac{\bar{Y} f'(\bar{Y})}{f(\bar{Y})}, \eta_g^x = \frac{\bar{X} g'(\bar{X})}{g(\bar{X})}$$

are the elasticities of f and g at the steady state.

Note that this tells that

$$z_t = \eta_f^y \times (\% \text{ deviation of } Y_t \text{ from its steady state}) \tag{2}$$

As an *example*, let us linearize

$$Y_t = (aX_t^\theta + (1-a)Z_t^\theta)^{\gamma/\theta}$$

from Farmer's book, p. 39. Using lowercase letters for percentage deviations,

$$y_t = (\gamma/\theta) \times \text{ the percentage deviation } (p.d.) \text{ of } aX_t^\theta + (1-a)Z_t^\theta$$

using the power trick; but:

$$\begin{aligned} p.d. \text{ of } aX_t^\theta + (1-a)Z_t^\theta &= \frac{a\bar{X}^\theta}{a\bar{X}^\theta + (1-a)\bar{Z}^\theta} \times (p.d.of\ aX_t^\theta) \\ &+ \frac{(1-a)\bar{Z}^\theta}{a\bar{X}^\theta + (1-a)\bar{Z}^\theta} \times (p.d.of\ (1-a)Z_t^\theta) \end{aligned}$$

using the sum trick; finally,

$$p.d.of\ aX_t^\theta = \theta x_t$$

$$p.d.of\ (1-a)Z_t^\theta = \theta z_t$$

using power trick again. Collecting all this:

$$\begin{aligned} y_t &= (\gamma/\theta)[\frac{a\bar{X}^\theta}{a\bar{X}^\theta + (1-a)\bar{Z}^\theta}\theta x_t + \frac{(1-a)\bar{Z}^\theta}{a\bar{X}^\theta + (1-a)\bar{Z}^\theta}\theta z_t] \\ &= \gamma[a\bar{X}^\theta x_t + (1-a)\bar{Z}^\theta z_t]/\bar{Y}^{\theta/\gamma} \end{aligned}$$

The last line is simply a rearrangement. Note also that we have used (2) at each step.

4. Loglinearization vs percentage deviations

Consider the Taylor expansion of

$$Z_t = \log X_t = f(X_t)$$

around $\bar{Z} = \log(\bar{X})$,

$$\log X_t = f(\bar{X}) + f'(\bar{X})(X_t - \bar{X}) = \log \bar{X} + x_t$$

where $x_t = (X_t - \bar{X})/\bar{X}$, as before. So, up to first order,

$$x_t = \log X_t - \log \bar{X} \tag{3}$$

In words, percentage deviations from steady state are the same as log-deviations (differences in logs), up to first order. For most practical purposes in our course, therefore, there is no difference between the two. (Often you will see the term "log linearizing " instead of linear approximation in percentage terms.)

This equivalence (again, up to first order) may help linearizing quickly some expressions that become linear in logs. For example, if the production function is Cobb Douglas,

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$$

taking logs you get

$$\log Y_t = \log A_t + \alpha \log K_t + (1-\alpha) \log L_t$$

The steady state version of this is

$$\log \bar{Y} = \log \bar{A} + \alpha \log \bar{K} + (1-\alpha) \log \bar{L}$$

Substract the two and using (3),

$$y_t = a_t + \alpha k_t + (1-\alpha) l_t$$

5. Caveats

- These derivations become generally invalid if some variable is zero in the nonstochastic steady state, since then you will have to make sure that you have not divided by zero in taking percentage approximations, etc.

- Also, the derivations are only accurate up to first order, as I have emphazised. For some questions (e.g. welfare evaluation) a first order approximation is not good enough, and more accurate methods need to be used. In our course, however, this will most probably not be a concern.