Endogenous Uncertainty, Capital Structure and Financial Amplification

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ABSTRACT

Recent data show that both the stock market volatility (volatility of market return) and the dispersion of firm level stock return (volatility of return on individual firm stock, relative to market), increase during economic recessions. In addition, US firms depend more on equity financing and less on debt financing.

In this paper, I incorporate a costly state verification (CSV) problem into a dynamic model with business cycle feature. Risk neutral borrowers have access to two stochastic technologies, with one being subject to and one not being subject to the costly state verification problem. Optimal contract specifies a repayment consisting of a contingent part and a non-contingent part, which can be replicated using standard debt and equity contract. Both idiosyncratic risk and aggregate risk are shared between borrower and lender. Financial condition of borrower plays a crucial role in determination of repayment rule in optimal contracts. When financial condition is worsened, the repayment specified in optimal contract consists of a larger contingent part and a smaller non-contingent part and in this case lender takes more idiosyncratic and aggregate risks. To implement this optimal contract, the borrower issues more equity and less debt. In addition, the dispersion of equity return and the volatility of equity market return become larger.

Due to financial friction, the wealth distribution across borrowers and lenders matters. In this model, agents with higher net worth and thus suffering less from CSV problem has a higher expected return of investment and then become borrower. Agents with less net worth become lender. A negative aggregate shock not only decrease the total wealth, but also “shifts” wealth from borrowers to lenders, which decreases the wealth concentration and deteriorates the financial market condition further.
1 Introduction

Recent data show that both the stock market volatility (implied volatility of market return, e.g. VXO index) and the dispersion of firm level stock returns (volatility of return on individual firm stock, relative to market) increases during economic recession. In addition, stock market volatility is significantly correlated with different cross-sectional measures of uncertainty. The large bursts of cross-sectional uncertainty in recession are largely interpreted as an outcome of a mean preserving “uncertainty shock”. However, “uncertainty shocks” cannot account for the countercyclical behavior of time series market volatility, since idiosyncratic risks are diversified out when computing the average market return. In addition, it cannot account for the co-movement of the dispersion and market volatility in recessions.

In addition, data also show that US firms issue more debts in economic booms and depend more on equity financing during economic downturns. These cyclical behaviors of firms’ capital structure are documented in Jermann and Quadrini (2012). The findings show that there are underlying substitutability between debt and equity financing.

In this paper, I address these two issues from a different perspective; these two observations reflect the fact that lenders (or holders of financial assets) take more risk (both aggregate and idiosyncratic) during recessions. To analyses this, we need a framework in which borrower raises funds using both contingent and non-contingent financing tools. In classical CSV models, standard debt is the only optimal contract. However, if borrower is allowed to partly eliminate the CSV problem at certain costs, she can thus balance the
tradeoff between bankruptcy costs and costs of eliminating CSV problem. For example, in Boyd and Smith 1999, borrower is allowed to invest in two stochastic technologies with one being subject to and one not being subject to CSV problem. However, the investment technology not being subject to CSV problem has a lower expected return. The difference between expected returns of the two technologies represents the cost of eliminating CSV problem. Repayment specified in optimal contract consists of a contingent and a non-contingent part, which can be replicated using standard debt and equity contract. Financial condition of borrower plays a crucial role in determination of the optimal contract. When the financial condition of the borrower is worsened, financial friction becomes more severe and costs of bankruptcy become relatively higher than costs of eliminating CSV problem. Thus, it becomes optimal for the borrower to use more resources to address the CSV problem, which result in an optimal contract specifying a repayment that has a larger contingent part and a smaller non-contingent part. To decompose this optimal contract into standard debt and equity contract, borrower issues more equity and less debt contracts. In addition, both the volatility of market returns (corresponding to aggregate risks taken by lender) and the dispersion of asset return relative to market (corresponding to idiosyncratic risks taken by lender) are larger.

The crucial variable in this CSV framework is borrower’s financial condition, which determines how the borrower balances the tradeoff between costs of bankruptcy and costs of eliminating CSV problem. Consistent with Bernanke and Gertler (1989), the financial condition of firm is implied by “external finance premium”, which is not observable, but
we can use interest rates spreads as its indicator.

Data shows that there is significant correlation between capital structure and financial condition indicator, which is around -0.4629 during 1986 Q1 to 2010 Q1 and around -0.5481 during 2010 Q1 to 2010 Q1 (period after dot com bubble).

Data also shows that there is a strong positive relationship between stock market volatility and financial condition indicator. The correlation between the two is around 0.7235 during 1986 Q1 to 2010 Q1 and around 0.9570 during 2010 Q1 to 2010 Q1 (period after dot com bubble). More detailed data are presented in the next section.

Finally, in this paper, I study the wealth distribution across borrowers and lenders and the mechanism of financial accelerator. In an economy with financial friction, wealth distribution across borrowers and lenders matters. For example, in financial accelerator framework proposed by Bernanke, Gertler and Gilchrist (2005), when a negative shock hits the economy, the loss of investment is absorbed by the net worth of entrepreneurs, which is similar to a “shift” of wealth from borrower to lender. This decreases their ability to raising funds and thus amplifies the initial shock. In their model, households and entrepreneurs have different risk aversion; household are risk averse while entrepreneurs are risk neutral. Thus, all the aggregate risk is taken by the entrepreneurs. In this model, agents are heterogeneous in wealth. Agents with higher net worth and thus suffering less from CSV problem has a higher expected return of investment and become borrower. Those with less net worth become lender. How the borrower and lender share the aggregate risk is determined by the setup of information friction. A negative aggregate shock
not only decrease the total wealth, but also “shifts” wealth from borrowers to lenders, which decreases the wealth concentration and deteriorates the financial market condition further.
2 Empirical Facts

*Capital Structure over Business Cycle:*

Jermann and Quadrini (2012) uses financial data from “Flow of Funds Accounts” of the Federal Reserve Board and found cyclical movements in equity payout and countercyclical movements for debt repurchase. The equity payout represents “net payments to business owners”, which equals dividend payments plus share repurchase, minus equity issued by nonfinancial corporates, minus net investment in non-corporate businesses. Debt repurchases stand for the decrease in the outstanding debts.

To describe the extent to which firms depend on debt financing over business cycle, I use $DD$ (Dependence on Debt financing) to represent the level of dependence of firms on debt financing, which is defined as

$$DD \equiv \frac{Equity\;Payouts-Debt\;Repurchase}{GDP}$$

![Figure a1: Dependence on Debt Financing and GDP (HP residual, Re-scaled)](image-url)
**Capital Structure and Financial Condition**

According to corporate literatures, a well-known result is that a large proportion of the variation of interest rate spread can be attributed to the financial health of firms. Also, in financial accelerator literature, interest rate spreads are used as an indicator of external finance premium, which is determined by financial conditions.

![Dependence on Debt Financing and Interest Rates Spreads (HP residual, Re-scaled)](image)

Figure a2: Dependence on Debt Financing and Interest Rates Spreads (HP residual, Re-scaled)

Figure a2 shows the level of dependence on debt financing and two financial condition indicators: GZ interest rate spreads and Excess Bond premium which are constructed by Gilchrist and Zakrajšek (2012). GZ-spreads are constructed using individual corporate bond prices in secondary market. Excess Bond Premium is constructed by taking the default risk premium out of GZ-spreads.
Table 1: Correlation between Capital Structure and Other Indicators

<table>
<thead>
<tr>
<th></th>
<th>2001Q1-2010Q1 HP residual</th>
<th>1986Q1-2010Q1 HP residual</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>GDP</td>
<td>GZ Spread</td>
</tr>
<tr>
<td>Dependence on Debt</td>
<td>0.8629</td>
<td>-0.5481</td>
</tr>
</tbody>
</table>

Table 1 shows the correlation between capital structure indicator (DD: Dependence on Debt financing) and the following indicators: business cycle indicator (GDP) and financial condition indicators (GZ interest rate spread and Excess Bond Premium).
Dispersion

Figure a3 (constructed by Gilchrist, Sim and Zakrajsek 2010) shows the correlation between their estimate of time-varying uncertainty using stock returns, and financial condition indicator, interest rate spread. The solid black line, labeled as “aggregate uncertainty”, captures shocks to idiosyncratic volatility, which are common to all firms. It is interpreted as an aggregate “second moment shock” in their paper.
Market Volatility

Figure a4: VXO and GDP (HP residual, Re-scaled)

Figure a4 shows the VXO index, which is an indicator of implied volatility of equity prices from S&P options, during 1986-2010. The market volatility is negatively correlated with GDP.

Figure a5: VXO and Interest Rates Spreads (HP residual, Re-scaled)

Figure a5 show the VXO index and interest rates spreads over 1986-2012. The positively relationship between VXO and interest rates spreads is significant, especially after 2002.
Table 2: Correlation between VXO and Other Indicators

<table>
<thead>
<tr>
<th></th>
<th>2001Q1-2010Q1 HP residual</th>
<th>1986Q1-2010Q1 HP residual</th>
</tr>
</thead>
<tbody>
<tr>
<td>VXO</td>
<td>-0.6267</td>
<td>-0.3996</td>
</tr>
<tr>
<td>GDP</td>
<td></td>
<td></td>
</tr>
<tr>
<td>GZ Spread</td>
<td>0.9570</td>
<td>0.7235</td>
</tr>
<tr>
<td>EBP</td>
<td>0.8320</td>
<td>0.5458</td>
</tr>
</tbody>
</table>

Table 2 shows the correlation between implied equity price volatility index, VXO, and the following indicators: business cycle indicator (GDP) and financial condition indicators (GZ interest rate spread and Excess Bond Premium).
3. A Simple Model

To highlight the mechanism, I start with a tractable two period model with costly state verification (CSV) problem. The full model is to be described in the next section. Some features of the full model which are not crucial are removed here. In this section, I only focus on the borrower-lender relationship.

The main idea is that in an economy where information is asymmetric and state verification is costly, financial condition is a big concern of borrower when she signs funding contract with lender. When financial condition is worsened, the optimal contract provides lender with less insurance against both idiosyncratic and aggregate shocks, which means lender takes more idiosyncratic and aggregate risks. Both the dispersion (measure of idiosyncratic risk) and market volatility (measure of aggregate risk) of return to lender increase.

3.1 Environment

Time is discrete and $t = 1, 2$. Agents are divided into two groups, which are lenders and borrowers. Borrowers (lenders) constitute half of the population. At the beginning of the first period, borrowers are endowed with $\vartheta$ ($\vartheta < 1$), unites of goods and lenders are endowed with $1 - \vartheta$ units of the same goods. Parameter $\vartheta$ represents the wealth distribution across borrowers and lenders. All the agents save and invest in the first period and consume only in the second period. All the agents are risk neutral and care only about the expected value of their consumption in the second period. In addition, all of them have
access to a storage technology that yields \( r \) units of goods tomorrow for each unit goods stored today. Thus, \( r \) is the opportunity cost of lenders.

The problem of multiple monitoring efforts is not taken into account in this model. To avoid this problem, one can think that there is a delegated intermediary, for example, a banker, between the lender and the borrower. Bankers are perfect competitive and earns zero profits by transferring money from lender to borrower. Each banker lends funds to only one borrower. He signs the funding contract with the borrower and is responsible for monitoring if it is needed. The banker collects repayments from the borrower and splits them into two parts: one non-contingent part (if monitor does not happen), which is sold to lender as “debt” and one contingent part, which is sold to lender as “equity”. There is not any friction between banker and lenders. In addition, since each banker only lends to one borrower and does not diversify his portfolio, risks of assets sold by a banker come from uncertainty of investment outcomes of the borrower he lends to.

In the period one, borrowers are endowed with investment opportunities which come in discrete, non-divisible units, called “projects”. To start a project, a borrower needs to invest \( \text{one} \) unit (assumed to be fixed) goods which produce some random quantity of the same goods tomorrow. Technology \( z \) is not subject to CSV problem; outcomes of investment in technology \( z \) can be observed by both lender and borrower. Individual production technology is a linear function. Technology \( z \) produces \( a_z r_k \) units of goods per unit invested. Here \( r_k \) is the aggregate return of investment, which is known in advance, and \( a_z \) is the realized productivity shock. Technology \( u \) is subject to CSV problem. It produc-
es $a_u r_k$ units of goods per unit invested. Here $a_u$ denotes the realized productivity shock, which is only observable by the borrower without any cost. It can be observed by the lender only by bearing an audit cost.

### 3.2 Stochastic Productivities and Bayesian Updating

Let $(z, u)$ denote the state of a borrower. The individual productivity shocks, $a_z$ and $a_u$, are functions of individual state variables, which are given by,

$$a_z \equiv \rho \omega(z) \quad \text{and} \quad a_u \equiv \omega(u) \quad \text{with} \quad \rho < 1$$

where $\omega: R \to R^+$, is a function which has $\omega(\cdot) > 0$ and $\omega'(\cdot) > 0$. Both borrower and lender know the function form of $\omega(\cdot)$. Note that individual productivity shock to technology $z$ (i.e. $a_z$) can be observed by both borrower and lender; however, $a_u$ is observable only to borrower. Since $\omega(\cdot)$ is a monotonically increasing function, both borrower and lender observe $z$ by observing $a_z$. However, only borrower can observe $a_u$ and $u$ without any cost.

Once a project is started, borrower’s state $(z, u)$ are randomly drawn. $z$ (or $u$) have two separate random components: an aggregate component $s$ and an individual component, $\varepsilon_z$ (or $\varepsilon_u$):

$$z = s + \varepsilon_z \quad \text{and} \quad u = s + \varepsilon_u$$

where $s$ denotes the aggregate state and $\varepsilon_z$ and $\varepsilon_u$ are two idiosyncratic shocks, which are i.i.d. across agents and both of them are uniformly distributed on $[0,1]$. For the same borrower, $\varepsilon_z$ and $\varepsilon_u$, are independent from each other. The macro component $s$ has two
possible values: \(s_h\) and \(s_l\). With probability \(\bar{p} \in (0,1)\), \(s = s_h = \bar{s} > 0\); with probability \(1 - \bar{p}\), \(s = s_l = 0\). In addition, aggregate state \(s\) is i.i.d. across time. If \(\bar{p} = 0.75\), then economic recession, i.e. \(s = s_l\), happens every 12 periods on average.\(^1\)

Let \(\underline{z}\) and \(\bar{z}\) (\(\underline{u}\) and \(\bar{u}\)) be the lower and upper bounds of \(z\) (\(u\)). Then, \(\underline{z} = \underline{u} = 0\) and \(\bar{z} = \bar{u} = 1 + \bar{s}\). Let \(\hat{a}_z\) and \(\hat{a}_u\) be the expected value of \(a_z\) and \(a_u\). Since \(\rho < 1\), then \(\hat{a}_z < \hat{a}_u\); otherwise, technology \(z\) is strictly preferable to technology \(u\). Here, \(1 - \rho\) represents the “cost of information transfer”.

Let us take technology \(u\) as an example. The individual productivity shock to technology \(u\) is \(a_u = \omega(u)\), where \(\omega(\cdot)\) is a monotonically increasing function and \(u\) is the borrower’s state, which is a random variable. Then, \(\omega(\cdot)\) is a distribution transformation function that transforms the distribution of \(u\) to the distribution of \(a_u\). I assume that the function \(\omega\) is given as:

\[
a_u = \omega(u) = e^{H^{-1}(u - \bar{s})}
\]

Define \(\bar{a}_u \equiv ln(a_u)\), then \(\bar{a}_u = H^{-1}(u - \bar{s})\). Without loss in generality, choose a function \(H: R \rightarrow R\) and \(H' > 0\). Given that \(s = s_h = \bar{s}\), \(H\) is the cumulative distribution function of \(\bar{a}_u\) on its support \([\epsilon_u, \bar{\epsilon}_u]\), where \(\epsilon_u\) and \(\bar{\epsilon}_u\) are the lower and upper bounds of \(\bar{a}_u\) and \(\epsilon_u = H^{-1}(0), \bar{\epsilon}_u = H^{-1}(1)\). To see this, note that,

\[
F_{\bar{a}}(\kappa|s_h) = P(\bar{a}_u < \kappa|s_h) = P[H^{-1}(u - \bar{s}) < \kappa|s_h] = P[u - \bar{s} < H(\kappa)|s_h]
\]

\[
= P[\epsilon_u < H(\kappa)] = H(\kappa)
\]

The c.d.f. of \(\bar{a}_u\), conditional on \(s = \bar{s}\), can be summarized as:

\(^1\) The economic recession, \(s = s_l\), happens every \(\bar{p}/(1 - \bar{p})^2\) periods on average.
\[ F(\bar{\alpha}|s_h) = \begin{cases} 
0 & \bar{\alpha} < \varepsilon_u \\
H(\bar{\alpha}) & \varepsilon_u \leq \bar{\alpha} < \varepsilon_u \\
1 & \bar{\alpha} \geq \varepsilon_u 
\end{cases} \]

Define \( h(\cdot) \equiv H'(\cdot) \), then the p.d.f. of \( \bar{\alpha}_u \) is \( f(\bar{\alpha}|s_h) = h(\bar{\alpha}) \) on \([\varepsilon_u, \varepsilon_u]\); otherwise, \( f(\bar{\alpha}|s_h) = 0 \).

Given that \( s = s_l = 0 \), it is easy to show that \( H(\bar{\alpha}) + \bar{s} \) is the c.d.f. of \( \bar{\alpha}_u \) on its support \([\varepsilon'_u, \varepsilon'_u]\), where \( \varepsilon'_u \) and \( \varepsilon'_u \) are the lower and upper bounds of \( \bar{\alpha}_u \) and \( \varepsilon'_u = H^{-1}(\bar{s}) \), \( \varepsilon'_u = H^{-1}(1 - \bar{s}) \). To see this, note that, on its support \([\varepsilon'_u, \varepsilon'_u]\):

\[
F_{\bar{\alpha}}(\kappa|s_l) = P(\bar{\alpha}_u < \kappa|s_l) = P[H^{-1}(u - \bar{s}) < \kappa|s_h] = P[u - \bar{s} < H(\kappa)|s_h]
\]

\[
= P[\varepsilon_u < H(\kappa) + \bar{s}] = H(\kappa) + \bar{s}
\]

Given that \( s = s_l = 0 \), the p.d.f. of \( \bar{\alpha} \) is \( f(\bar{\alpha}|s_l) \), with \( f(\bar{\alpha}|s_l) = h(\bar{\alpha}) \) on \([\varepsilon'_u, \varepsilon'_u]\); otherwise, \( f(\bar{\alpha}|s_l) = 0 \). Also, note that \( F_{\bar{\alpha}}(\varepsilon'_u|s_l) = 0 \), \( F_{\bar{\alpha}}(\varepsilon'_u|s_l) = 1 \), and

\[
\int_{\varepsilon'_u}^{\varepsilon'_u} f(\bar{\alpha}|s_h) \, d\bar{\alpha} = H(\varepsilon'_u) - H(\varepsilon'_u) = 1
\]

Thus, \( H(\bar{\alpha}) + \bar{s} \) is the c.d.f function of \( \bar{\alpha}_u \), given that on \( s = s_l = 0 \).

As show by the figure below, a negative aggregate shock, i.e. \( s = s_l \), actually shifts probability density of \( \bar{\alpha} \) left along the function \( h \).
The novel element in this model is the information structure. At the beginning of the second period, borrower draws its states \((z, u)\); Both lender and borrower can observe \(z\); however, \(u\) can only be observed by borrower with zero cost. In addition, neither lender nor borrower can observe the individual components of productivity shock, \(\varepsilon_z\) and \(\varepsilon_u\) or the aggregate state, \(s\). The individual components \((\varepsilon_z, \varepsilon_u)\) hide from borrower the aggregate state \(s\). Therefore, borrower’s state, \(z\) and/or \(u\), can be high for one of two reasons: the aggregate state \(s\) is high, that is, \(s = s_h\), and/or the individual components, \(\varepsilon_z\) and/or \(\varepsilon_u\) is high. Initially, the borrower and lender believe that \(s = s_h\) with probability \(\bar{p}\). Let \(G\) denote cumulative density function of \(z\) (or \(u\)), and \(g\) be their probability density function. Then, conditional on \(s\), \(z\) and \(u\) are uniformly distributed. After observing \(z\), borrower and lender update their beliefs by Bayesian rule to:

\[
p(s_h|z) = \frac{g(z|s_h)}{g(z|s_h)p + g(z|s_l)p(s_l)}
\]

Figure 1 shows the graph of posterior probability \(p(s_h|z)\). In this case, borrower gets little information by observing \(z\). Only when \(z\) is extremely high (low), borrower realizes that \(s_h (s_l)\) happens; otherwise, \(p(s_h|z) = p\).
3.3 From Idiosyncratic to Aggregate

Borrowers are indexed by \( i \) and uniformly distributed on \([0, 1]\). Given interest rates \( r \) and \( r_k \), agent \( i \) starts a project and invests \( k^i_z \) and \( k^i_u \) in technology \( z \) and \( u \). The aggregate return \( r_k \) is given by:

\[
r_k = \bar{A} K_t^{\theta - 1} \quad \text{with} \quad K_t \equiv \int_0^1 \left( \rho k^i_z + k^i_u \right) di
\]

where \( K_t \) is the aggregate capital stock. Let \( \omega_z \equiv \omega(z) \), and \( \bar{\omega}_{z|s} \) be the average value of \( \omega_z \) given that the aggregate state is \( s \). Then, \( \bar{\omega}_{z|s} \) is only contingent on \( s \). The aggregate production function is then given as:

\[
Y_t = r_k \int_0^1 \left( \rho \omega^i_z k^i_z + \omega^i_u k^i_u \right) di = r_k \left( \bar{\omega}_{z|s} + \bar{\omega}_{u|s} \right) K_t = A_t K_t^\theta
\]

where \( A_t \equiv \left( \bar{\omega}_{z|s} + \bar{\omega}_{u|s} \right) \bar{A} \) is the aggregate productivity shock.

3.4 Financial Contract

Borrower has his own wealth, that is, net worth, and also raises funds by borrowing from lenders. Let \( b \) be the amount borrowed from outside and \( n \) be his own net worth, the borrower’s balance sheet is given as: \( n + b = 1 \). Let \( k_u \) and \( k_z \) be the amounts invested in technology \( u \) and \( z \) respectively. Define \( \alpha \equiv k_u \) to be the fraction of total funds invested in technology \( z \). Thus, \( 1 - \alpha \) is the fraction invested in technology \( u \).

The financial contract signed between borrower and lender specifies: (1) Investment Plan: how the funds to be allocated between technology \( u \) and \( z \); (2) Repayment and Auditing Rules. Repayment rules are functions of states, that is, \( u \) and \( z \). Since \( z \) is observa-
ble, the repayment and audit rules can always be contingent on $z$. Let $A(z)$ be the audit region while $A^c(z) \equiv [u, \bar{u}] - A(z)$ be the non-audit region, where $[u, \bar{u}]$ is the support of random variable $u$. For any $u \in A^c(z)$, promised repayments should not be contingent on $u$. Let $r(z, u)$ denote the promised repayment made to the lender in audit region and $x(z)$ be the promised repayment in non-audit region.

The funding contract must be feasible. Under limited liability, the borrower’s return should not be negative. Also, repayments to lender cannot be negative either. Thus,

$$0 \leq x(z) \leq [\alpha a_z + (1 - \alpha) a_u] r_k \quad (1)$$

The funding contract should be incentive compatible, so that the borrower never has any incentive to cheat. This requires that repayment in the audit region is less than that in the non-audit region. Thus,

$$x(z) > r(z, u) \quad \text{if } u \in A(z) \quad (2)$$

Let $r$ be the opportunity cost of lender. Note that $x(z)$ denotes the promised payment in the non-auditing region, while $r(z, u)$ is the payment in auditing region. The difference between them two is so called “outstanding claims”, unpaid portion of a firm’s debt and obligations when it goes to bankruptcy. I assume that the auditing cost is a fraction $\mu$ of “outstanding debt”$^2$. The net repayments to lender should satisfy the expected return constraint:

---

$^2$ This interpretation of auditing cost is mentioned in Townsend 1979.
Let \( \pi \) be profit of borrower, thus his expected profit can be written as:

\[
E(\pi) = [\alpha \hat{a}_z + (1 - \alpha) \hat{a}_u] r_k - br
\]

\[
- \int_\mathbb{Z} g(z) \left[ \int_{u \in A(z)} \mu [x(z) - r(z, u)] g(u | z) du \right] dz
\]

(4)

Given interest rates, \( r \) and \( r_k \), and borrower’s net worth, the optimal contract maximizes (4) subject to (1) – (3). The borrower chooses repayment and audit rules and investment plan to maximize her expected profits, that is, borrower chooses \( (x(\cdot), r(\cdot, \cdot), A(\cdot), \alpha) \) to maximize \( E(\pi) \). Let us define \( \Omega \) to be the set of all feasible contracts:

\[
\Omega \equiv \{(x(\cdot), r(\cdot, \cdot), A(\cdot), \alpha)|(1) - (3) holds\}
\]

If the borrower’s net worth is too low, constraint (3) can never be satisfied no matter what repayment and audit rules are, thus, the set \( \Omega \) is empty, i.e. \( \Omega = \emptyset \).

It will be helpful to define:

\[
\bar{u}_z = \omega^{-1} \left[ \frac{x(z) - \alpha a_z r_k}{(1 - \alpha)r_k} \right]
\]

(5)

In the following part, it will be shown that \( \bar{u}_z \) is the critical value of \( u \), under which auditing will happen.
**Lemma 1**: the solution to the borrower’s optimization problem has:

1. \( A^c(z) = [u, \bar{u}_z] \)
2. If \( u \in A(z) \), then \( r(u, z) = [\alpha a_z + (1 - \alpha)a_u]r_k \).

The proof of lemma 1 appears in Appendix A\(^3\). Lemma 1 assets that audit never happens if it is feasible to fully pay \( x(z) \). In addition, when the payment \( x(z) \) is infeasible, auditing occurs and lender takes away all the residual. This result is standard in CSV models.

By re-arranging (5), promised repayment \( x(z) \) in non-auditing region and the “outstanding claims” in the auditing region can be written as:

\[
x(z) = [\alpha_2 a_z + (1 - \alpha)\omega(\bar{u}_z)]r_k
\]

\[
x(z) - r(z, u) = (1 - \alpha)[\omega(\bar{u}_z) - \omega(u)]r_k
\]

Then, the expected payment to lender can be written as:

\[
a\hat{a}_z r_k + (1 - \alpha)r_k \int \bar{z} g(z) \left\{ \omega(\bar{u}_z)[1 - G(\bar{u}_z | z)] + \int \omega(u)g(u|z)du \right\} dz
\]

- \[
\int \bar{u}_z \mu[\omega(\bar{u}_z) - \omega(u)]g(u|z)du \] dz

It will useful to define \( \Gamma(\bar{u}_z | z) \) to be “lender’s share” and \( \Psi(\bar{u}_z | z) \) to be “dead weight loss of auditing”, which are given by,

\[
\Gamma(\bar{u}_z | z) \equiv \omega(\bar{u}_z)[1 - G(\bar{u}_z | z)] + \int \omega(u)g(u|z)du
\]

\[
\Psi(\bar{u}_z | z) \equiv \int \mu[\omega(\bar{u}_z) - \omega(u)]g(u|z)du
\]

\(^3\) This result is also proved in Boyd and Smith 1989.
Then, the optimization problem of borrower can be re-written as:

$$\max_{\{\tilde{u}_z, \alpha\}} \left[ \alpha \tilde{a}_z + (1 - \alpha) \tilde{a}_u \right] r_k - bR - (1 - \alpha) r_k \int_{\tilde{z}} g(z) \psi(\tilde{u}_z | z) dz$$  \hspace{1cm} (6)$$

subject to

$$\alpha \tilde{a}_z r_k + (1 - \alpha) r_k \int_{\tilde{z}} g(z) [\Gamma(\tilde{u}_z | z) - \psi(\tilde{u}_z | z)] dz = br$$ \hspace{1cm} (7a)$$

$$0 \leq \alpha \leq 1$$ \hspace{1cm} (7b)$$

The borrower chooses $\tilde{u}_z$ and $\alpha$ to maximize his expected profits. The first order conditions for $\tilde{u}_z$ is given by:

$$\tilde{u}_z: \quad 1 + \lambda = \frac{\Gamma'(\tilde{u}_z | z)}{\Gamma'(\tilde{u}_z | z) - \psi'(\tilde{u}_z | z)}$$ \hspace{1cm} (8)$$

Here $1 + \lambda$ denotes the marginal rate of transformation (MRT): to increase the expected return of lender by one unit, it requires more than one unit decrease in borrower’s expected profit if there is CSV problem. When MRT is larger than 1, i.e. $\lambda > 0$, the positive transformation cost, which leads to a wedge between marginal return of borrower and the opportunity cost of lender, the so called “external finance premium”. Borrower chooses $\tilde{u}_z$ to smooth the transformation cost. Note that,

$$\Gamma'(\tilde{u}_z | z) = \alpha'(\tilde{u}_z) [1 - G(\tilde{u}_z | z)]$$

$$\psi'(\tilde{u}_z | z) = \mu \alpha'(\tilde{u}_z) G(\tilde{u}_z | z)$$

Then, the first order condition for $\tilde{u}_z$ can be written as:

$$\lambda = \frac{1}{[1/G(\tilde{u}_z | z) - 1] \mu^{-1} - 1}$$

Note that $\lambda$ is constant and the left hand side is strictly increasing in $G(\tilde{u}_z | z)$. Thus, $\tilde{u}_z$
should be chosen such that $G(\bar{u}_z|z)$ is constant across states, which implies $G(\bar{u}_z|z) = \bar{G}$ for any $z$, where $\bar{G}$ is a constant number. Also, note that the c.d.f at $u = \bar{u}_z$, conditional on $z$ is given as:

$$G(\bar{u}_z|z) = p(s_h|z)G(\bar{u}_z|s_h) + p(s_l|z)G(\bar{u}_z|s_l)$$

$$= p(s_h|z)(\bar{u}_z - s_h) + p(s_l|z)(\bar{u}_z - s_l) = \bar{u}_z - \hat{s}_z = \bar{G}$$

Thus, $\bar{u}_z = \bar{G} + \hat{s}_z$, with $\hat{s}_z \in [s_l, s_h]$. Define $u_0 \equiv \bar{G}$, then $\bar{u}_z$ can be expressed as,

$$\bar{u}_z = u_0 + \hat{s}_z$$ (9)

which consists of two parts: a constant part $u_0$ and a contingent (and non-negative) part $\hat{s}_z$. Define $\omega_0 \equiv \omega(u_0)$, then

$$\omega(\bar{u}_z) = \omega_0 + \xi_z$$

where $\xi_z \equiv \omega(u_0 + \hat{s}_z) - \omega(u_0)$.

Now, let us turn to the optimal choice on $\alpha$. Since we are looking for the interior solution and assume that the second constraint (6b) never binds, thus the Lagrangian multipliers associated with that constraint are set to zero. The first order condition is then given as,

$$(1 + \lambda) = \frac{\hat{a}_u - \bar{F}}{\bar{a}_z - (\bar{F} - \bar{\Phi})}$$ (10)

where $\bar{F} \equiv \int_z^\infty g(z)\Gamma(\bar{u}_z|z)dz$ and $\bar{\Phi} \equiv \int_z^\infty g(z)\Psi(\bar{u}_z|z)dz$. Then, the expected return to lender is, equation (7a) can be written as:

$$\alpha \bar{a}_z r_k + (1 - \alpha)(\bar{F} - \bar{\Phi})r_k = br$$ (11)

By combing (11) and (6), the borrower’s expected profit can be written as:
\[(1 - \alpha)[\hat{a}_u - \hat{f}]r_k \]  

(12)

The dead weight loss because of auditing cost is then \((1 - \alpha)\hat{p}r_k\).

According to (11) and (12), an increase in \(\alpha\) increases the expected return of lender by \([\hat{a}_z - (\hat{f} - \hat{\Phi})]r_k\) units, while decreasing borrower’s expected return by \([\hat{a}_u - \hat{f}]r_k\) units. Note that \(1 + \lambda\) is the marginal rate of transformation (MRT). In other words, per unit increase in \(\alpha\) transfer \([\hat{a}_z - (\hat{f} - \hat{\Phi})]r_k\) units of goods from the borrower to the lender and the transformation rate is \(1 + \lambda\).

Note that the optimal contract is defined by first order condition (8) (10) and constraint (7a). In addition, the first order conditions does not contain any of \(\{b, r, r_k\}\). The optimal \((\bar{u}_z^*, \lambda^*)\) is determined by solving (8), (10) and \(\alpha^*\) is determined by the expected return constraint (7a).

In equilibrium where the expected return constraint binds, that is, \(\lambda > 0\), note that first order condition (10) also implies \(\hat{a}_z - (\hat{f} - \hat{\Phi}) > 0\). Thus, an increase in \(\alpha\) loosens the expected return constraint (7a) and a decrease in \(\alpha\) tightens it. When the net worth is low (i.e. \(b = 1 - n\) is larger), the borrower increases \(\alpha^*\) while keeping \((\bar{u}_z^*, \lambda^*)\) constant.

**Proposition 1**: An decrease in borrowers net worth \(\vartheta\) leads to an increase in \(\alpha^*\). In addition, the optimal \(\bar{u}_z^*\), which is determined by (8) and (10) is kept constant.
3.5 Capital Structure

The promised payment in the non-auditing region, \( x(z) \) is given by,

\[
x(z) = [\alpha a_z + (1 - \alpha)(\omega_0 + \xi_z)]r_k \quad (12)
\]

which consists of two parts: one \((1 - \alpha)\omega_0 r_k\), which does not depend on any state and one \([\alpha a_z + (1 - \alpha)\xi(z)]r_k\), which depends on the observable performance of the borrower, which is observable. The optimal contract can be replicated using standard debt and equity contracts. The first part is repayment to debt holders and the second is the repayment to equity holders in the non-auditing region. Let \( e(z) \) denotes the payment to equity holder and \( d \) denotes the payment to the debt holder. In the non-auditing region,

\[
e(z) = [\alpha a_z + (1 - \alpha)\xi_z]r_k \quad (13a)
\]

\[
d = (1 - \alpha)\omega_0 r_k \quad (13b)
\]

If auditing happens, debt holder get paid first and the equity holder gets the residual. It should be noted that there are actually many ways to split the repayments between equity holders and debt holders. However, this is not likely to change the result that given the way to split repayment between equity and debt holders, borrower issues more equity and less debt to replicate the optimal contract.

3.6 Dispersion and Volatility

In statistics, coefficient of variation (CV henceforth) is often used as a measure of dispersion of a distribution,\(^4\) which is the ratio of standard deviation over mean. In most finan-

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\(^4\) Coefficient of variation is defined as \( \sigma / \mu \), where \( \sigma \) is the standard deviation and \( \mu \) is mean.
cial friction literature, the dispersion of assets return is defined as the volatility of individual stock return relative to market, which is actually square of CV.

Given the aggregate state $s$, let $r_s$ be the return of a lender and $\hat{r}_s$ be the market return (or average return). Then, dispersion of lender’s return, conditional on the aggregate state $s$ is defined as:

$$\sigma_{x|s}^2 = Var\left(\frac{r_s}{\hat{r}_s}\right)$$

The expected dispersion is then given by:

$$\bar{\sigma}_{x}^2 = \bar{p}\sigma_{x|s_{h}}^2 + (1 - \bar{p})\sigma_{x|s_{l}}^2$$

Since the expected return constraint holds in equilibrium, the expected return of lender is always equal to their opportunity cost, $r$. Note that $\hat{r}_s$ is the market return given the aggregate state $s$, then market volatility is defined as:

$$\sigma_{x,m}^2 = \bar{p}\left(\frac{\hat{r}_{s_{h}} - r}{r}\right)^2 + (1 - \bar{p})\left(\frac{\hat{r}_{s_{l}} - r}{r}\right)^2$$

Note that, the payment to lender is $x(z)$, if bankruptcy does not happen, and $r(z, u)$, if bankruptcy happens. To keep tractability and get some intuition, let us focus only on the part of assets which are fully repaid. In addition, in the event of a default, it is hardly to observer the recovery rate, that is, fraction of the obligations recovered through bankruptcy proceedings. Data collected from financial market actually only show the dispersion and volatility of returns of assets issued by those firms which are not in bankruptcy.

Define $x_s$ to be the random repayment to lender (in non-auditing region) given the

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5. The calibrated bankruptcy rate is very small, which is around 3% (BGG). Thus, taking those assets which are not fully repaid is not likely to affect the results.
aggregate state \( s \) and \( \hat{s} \) to be the average repayment to lender conditional on \( s \). Define \( e_s \) to be the random repayment to equity holders given the aggregate state \( s \) and thus \( \hat{e}_s \) is the average payment to equity holders. According to (12), \( x_s = e_s + d \). Thus, the return to lender (in non-auditing region) is \( r_s = x_s/b \) and the dispersion of return to lender is then,

\[
\sigma_{x|s}^2 = \text{Var} \left( \frac{x_s}{\hat{x}_s} \right) = \text{Var} \left( \frac{x_s}{b} \right) = \frac{\text{Var} \left( x_s \right)}{\hat{x}_s^2} = \eta_s^2 \cdot \sigma_{e|s}^2
\]

where \( \eta_s \equiv \hat{e}_s/\hat{x}_s \) and \( \sigma_{e|s}^2 \equiv \text{Var} \left( \frac{e_s}{\hat{e}_s} \right) \). Note that \( \sigma_{e|s}^2 \) can be interpreted as the dispersion of equity return given the aggregate state \( s \).

When the financial condition is worsened, that is, the net worth of borrower, \( \vartheta \), is less, borrower increases \( \alpha^* \) fraction invested in technology \( z \) while keeping \( \bar{u}_z \) the same (Proposition 1). This will increase the dispersion of return of lender in two ways: (1) it increases \( \eta_s \), which is ratio of contingent part over total repayment and (2) it increases volatility of contingent part of repayment, that is, \( \sigma_{e|s}^2 \).

Note that \( e_s = [\alpha a_z|s + (1 - \alpha)\xi_z|s] r_k \), according to (13). The first component of \( e_s \), \( a_z|s \), is much volatile than \( \xi_z|s \). Note that \( a_z \equiv \omega(z) \) and \( \xi_z \equiv \omega(u_0 + \hat{s}_z) - \omega(u_0) \). Also note that \( \hat{s}_z = \hat{s} \) whenever \( z \in [\bar{s}, 1] \) and \( \hat{s}_z \) is nearly constant since agent gets information on the aggregate state \( s \) only when \( z \) turns out to be extremely small (or large). When \( \alpha^* \) increases, the volatility of contingent part of payment is also increased. Since \( \sigma_{e|s}^2 \) can be interpreted as the dispersion of equity return given the aggregate state \( s \). Thus, it also means that the dispersion of stock return increases when the financial condition is worsened.
The market volatility of return to lender is then,

\[ v_x = \text{Var}\left(\frac{\hat{x}_s}{\hat{x}}\right) = \eta^2 \cdot v_e \]

where \( \eta \equiv \hat{\epsilon}/\hat{x} \) and \( v_e \equiv \text{Var}\left(\frac{\hat{\epsilon}_s}{\hat{\epsilon}}\right) \). Note that \( v_e \) can be interpreted as the market volatility of stock return, which is given by

\[ v_e = \bar{p} \left(\frac{\hat{\epsilon}_{sh} - \hat{\epsilon}}{\hat{\epsilon}}\right)^2 + (1 - \bar{p}) \left(\frac{\hat{\epsilon}_{sl} - \hat{\epsilon}}{\hat{\epsilon}}\right)^2 \]

When the financial condition of borrower is worsened, she increases \( \alpha^* \) and thus the fraction of contingent part, \( \eta \), increases. In addition, an increase in \( \alpha^* \) also increases the volatility of the contingent part of repayment, that is, \( v_e \).
4. The Full Model

4.1 Setup

Consider an OLG model with infinite periods, \( t = 1, 2 \ldots \). All the agents live three periods. For each period, economy consists of young, middle age and old generations. All the agents works in the young, save and invest in their middle age, and consume only when they are old. The utility function of agent \( i \) born in \( t \) is given as:

\[
 u_{i,t} = c_{i,t+2}^{\gamma} q_{i,t+2}^{1-\gamma}
\]

with

\[
 n_{i,t+1} = w_{i,t} + q_{i,t+1} \quad \text{and} \quad \pi_{i,t+2} = c_{i,t+2} + q_{i,t+2}
\]

where \( c_{i,t+2} \) is the agent’s consumption when old, \( q_{i,t+2} \) is the bequest form the old to the middle age. The initial net worth of the agent \( i \) is \( n_{i,t+1} \), which consists of two parts: the wage she earns when she is young, i.e. \( w_{i,t} \) and bequest from her parents \( q_{i,t+1} \). In the middle age, she can choose to lend money to others and then becomes lender or to start her own project and thus become borrower. When she is old, she gets \( \pi_{i,t+2} \) units of goods from saving or operating a project. She consumes \( c_{i,t+2} \) of them and leave the rest, \( q_{i,t+2} \), to her children as banquets. The function form implies that agent always consumes a fraction \( \gamma \) of his wealth when old and only cares about the expected value of her wealth when old, i.e. \( E_t(\pi_{i,t+1}) \).
4.2 Equilibrium of Funding Market

Note that the expected return to borrower is given as:

\[ \hat{\pi} = (1 - \alpha^*)[\hat{\alpha}_u - \hat{\beta}]r_k \]

Also, note that according to proposition 1, the optimal fraction invested in technology \( z \), that is, \( \alpha^* \) is larger when the net worth of the borrower is lower. Thus, given interest rates, \( r_k \) and \( r \), the more net worth the borrower has, the more expected return she can have by starting a project is. For those with lower net wealth, the expected returns of starting a project are below the opportunity cost of funds, \( r \), then they will choose to lend money to others or invest in the storage technology. The marginal borrower should be indifferent between being a borrower or a lender. Thus, the expected return of starting a project for the marginal borrower is equal to the opportunity cost of funds, which is \( r \).
**Proposition 2** Given \( r_k \) and \( r \), there exists \( n_1 \) and \( n_2 \) such that \( 0 < n_1 < n_2 < \bar{n} \), and the saving/investment decision made by the agents with wealth \( n \) will be one of the following:

1. If \( 0 \leq n < n_1 \), she becomes lender and lend money to others;
2. If \( n_1 \leq n < n_2 \), she chooses to start her project and becomes borrower. She will invest on both technology \( u \) and \( z \). The optimal fraction investment on technology \( z \), \( \alpha^* \), is linearly decreasing in her net worth.
3. If \( n_2 \leq n < \bar{n} \), she chooses to start her project and becomes borrower. She will invest on only technology \( u \).

**4.3 Wealth Distribution and Financial Accelerator**

One of the most important results of financial accelerator literatures is that the wealth distribution across lenders and borrowers matters. For example, in the BGG model, when a negative shock hits the economy, the loss of investment is absorbed by the net worth of borrowers. This decreases their ability to raising funds and thus amplifies the initial shock.
In their framework, how borrower and lender share the aggregate risk is important. By assuming that borrower is risk neutral while lender is risk averse, aggregate risk is taken by the borrower.

In this model, the way borrower and lender share the aggregate risk is determined by the information structure, rather than their risk aversion. Since the optimal contract provides the lenders with insurance against aggregate risk, when a negative aggregate shock hits the economy, borrowers lose relatively more and the wealth concentration decreases; however, when positive shock hits the economy, borrowers earn relatively more and the wealth concentration increases. Wealth distribution across agents (borrowers and lenders) is important in an environment with financial friction. When the wealth concentration increases, the information friction is mitigated, which amplifies positive shock; however, when the wealth concentration decreases, the information friction becomes more severe, which amplifies the negative shock.
Figure 3: I reorder the indexes of agents, such that \( n^{1'} \geq n^{2'} \) if \( i_1' < i_2' \). Assuming that the total amount of wealth is given, the solid line shows wealth of agents when the wealth concentration is high and the dashed line shows wealth of agents when the wealth concentration is low. The financial friction is more severe in the economy with low wealth concentration. The net worth of marginal borrower is lower in the economy with low wealth concentration.
Reference