Problem Set 4: Proposed solutions
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1 IS equation with external habit formation

1.1 Log-linear Approx:

The IS equation is given by:

$$\beta E_t \left\{ \frac{Y_t^{\sigma-(1-\sigma)\lambda} Y_{t-1}^{(1-\sigma)\lambda}}{Y_{t+1}^{\sigma}} \frac{P_t}{P_{t+1}} \right\} = \frac{1}{1+i_t}$$

First take logs of everything:

$$\begin{aligned} \ln\beta + \sigma - \underbrace{(1-\sigma)\lambda}_{\equiv\eta} \ln Y_t + \underbrace{(1-\sigma)\lambda}_{\equiv\eta} \ln Y_{t-1} - \sigma E_t \ln Y_{t+1} + \underbrace{E_t \ln\left(\frac{P_t}{P_{t+1}}\right)}_{\approx E_t\pi_{t+1}} &= -\underbrace{\ln(1+i_t)}_{\approx i_t} \\ \ln\beta + (\sigma-\eta)\, y_t + \eta y_{t-1} - \sigma E_t y_{t+1} - E_t\pi_{t+1} &= -i_t \end{aligned}$$

next, to obtain the log-differences we need to substract the log of the steady state values. Notice that the coefficients on the y_i terms sum to zero so if we add $(-(\sigma-\eta)\,\bar{y} - \eta\bar{y} + \sigma\bar{y}) = 0$ we will not modify the above equation. Thus we get:

$$\ln\beta + (\sigma-\eta)\,\hat{y}_t + \eta\hat{y}_{t-1} - \sigma E_t\hat{y}_{t+1} - E_t\pi_{t+1} = -i_t$$

Finally, notice that in SS $\beta = 1/(1+\bar{\imath})$ so that substracting $\ln\beta$ from both sides:

$$\begin{aligned} (\sigma-\eta)\,\hat{y}_t + \eta\hat{y}_{t-1} - \sigma E_t\hat{y}_{t+1} - E_t\pi_{t+1} &= -i_t - \ln\beta \\ &= -i_t - \ln\left(\frac{1}{1+\bar{\imath}}\right) \\ &= -i_t + \bar{\imath} \qquad (\because \ln(1+\bar{\imath}) \approx \bar{\imath}) \end{aligned}$$

so we have:

$$(\sigma-\eta)\,\hat{y}_t + \eta\hat{y}_{t-1} - \sigma E_t\hat{y}_{t+1} - E_t\pi_{t+1} = -\hat{\imath}_t$$

rearranging terms:

$$\hat{y}_t = \frac{\sigma}{(\sigma-\eta)}\hat{y}_{t+1} - \frac{\eta}{(\sigma-\eta)}\hat{y}_{t-1} - \frac{1}{(\sigma-\eta)}\left\{\hat{\imath}_t - E_t\pi_{t+1}\right\}$$

2 Determinacy of Equilibrium

Simply use the MP rule to replace in the AD equation and you will get the system:

$$E_t \begin{bmatrix} x_{t+1} \\ \pi_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 + \frac{\phi_x}{\sigma} + \frac{\kappa}{\beta\sigma} & \frac{\phi_\pi}{\sigma} - \frac{1}{\beta\sigma} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}}_{M} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \hat{r}_t^n$$

as in your lecture notes, set up the characteristic polynomial:

$$\begin{aligned} f(\lambda) &= \lambda^2 + tr(M)\lambda + \det(M) \\ &= \lambda^2 - \left(1 + \frac{\phi_x}{\sigma} + \frac{\kappa}{\beta\sigma} + \frac{1}{\beta}\right)\lambda + \left[\left(1 + \frac{\phi_x}{\sigma} + \frac{\kappa}{\beta\sigma}\right)\frac{1}{\beta} + \frac{\kappa}{\beta}\left(\frac{\phi_\pi}{\sigma} - \frac{1}{\beta\sigma}\right)\right] \\ &= \lambda^2 - \left(1 + \frac{\phi_x}{\sigma} + \frac{\kappa}{\beta\sigma} + \frac{1}{\beta}\right)\lambda + \frac{1}{\beta} + \frac{\phi_x}{\beta\sigma} + \frac{\kappa\phi_\pi}{\beta\sigma} \end{aligned}$$

Next, notice that $\sigma > 0$; otheriwse $u(\cdot)$ is not concave. Also, $\beta > 0$, otherwise positive period utility yields negative NPV utility in odd periods. We are told that $\phi_x > 0$ and finally recall that:

$$\begin{aligned} \kappa &= \frac{(1-\alpha)(1-\alpha\beta)}{\alpha}\gamma \\ &= \frac{(1-\alpha)(1-\alpha\beta)}{\alpha}\frac{(\sigma+\varphi)}{1+\theta\varphi} \\ &> 0 \end{aligned}$$

so we have that $\sigma, \beta, \phi_x, \kappa > 0 \Rightarrow f(-1) > f(0) > f(1)$. Thus, if $f(1) > 0$ all three are and we need:

$$\begin{aligned} f(1) &= -\frac{\phi_x}{\sigma} - \frac{\kappa}{\beta\sigma} + \frac{\phi_x}{\beta\sigma} + \frac{\kappa\phi_\pi}{\beta\sigma} > 0 \\ &\Rightarrow \frac{\phi_x(1-\beta)}{\kappa} + \phi_\pi > 1 \end{aligned}$$

3 Time varying government subsidy

3.1 The firm's problem:

Rememeber the maximization problem is:

$$\begin{aligned} \max_{\{P(i),Y(i),N(i)\}} \Pi(i) &= P_t(i)Y_t(i) - (1-\tau)W_t N_t(i) \\ s.t. &: Y_t(i) = A_t N_t(i) \\ Y_t(i) &= Y_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\theta} Y_t \end{aligned}$$

so the unconstrained maximization problem is:

$$\max_{P(i)} P_t(i)\left(\frac{P_t(i)}{P_t}\right)^{-\theta} Y_t - (1-\tau_t)W_t\left(\frac{P_t(i)}{P_t}\right)^{-\theta}\frac{Y_t}{A_t}$$

with FOC:

$$(\theta-1)\frac{(P(i))^{-\theta}}{P^{-\theta}}Y_t=(1-\tau_t)\,\theta W\left(\frac{P(i)}{P}\right)^{-\theta-1}\frac{Y_t}{P_tA_t}$$

or:

$$\begin{aligned}(\theta-1)\frac{P_t(i)}{P_t} &= (1-\tau_t)\,\theta\frac{W_t}{P_t}\frac{1}{A_t}\\ \frac{P(i)}{P} &= \left(\frac{\theta}{\theta-1}\right)(1-\tau_t)\,MC_t\end{aligned}$$

3.2 The NSS for the subsidy:

Rewrite the above condition:

$$\frac{P_t(i)}{P_t}=\left(\frac{\theta}{\theta-1}\right)(1-\tau_t)\frac{W_t}{A_tP_t}$$

now in steady state $N(i)=N$ and using the equilibrium conditions from the HH $\frac{W}{P}=Y^{\sigma}N^{\varphi}$ so:

$$1=\left(\frac{\theta}{\theta-1}\right)(1-\tau)\frac{Y^{\sigma}N^{\varphi}}{A}$$

and replacing with the technological constraint $N=Y/A$ we have:

$$Y=\left[\left(\frac{\theta}{\theta-1}(1-\tau)\right)^{-1}\right]A^{\frac{1+\varphi}{\varphi+\sigma}}$$

so we need:

$$\left(\frac{\theta}{\theta-1}(1-\tau)\right)^{-1}=1\Rightarrow\tau=\frac{1}{\theta}$$