Trends and Balanced Growth in a simple RBC

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Some of you were worried about the formalities and nitty-gritty details of the whole detrending exercise so here's an attempt to clarify some things. Recall the nonlinear system of stochastic difference equations found in pages 23-24 of Dave and Dejong (2011, henceforth DD:2011). For simplicity, I will assume that $\delta = 1$ so that:

$$Y_t = Z_t K_t^{\alpha} N_t^{1-\alpha}, \qquad K_{t+1} = I_t = Y_t - C_t \tag{1}$$

$$Z_t = Z_0 \left(1+g\right)^t \exp\left(\rho w_{t-1} + \epsilon_t\right) \tag{2}$$

$$\left(\frac{C_t}{1-N_t}\right) = \left(\frac{\varphi}{1-\varphi}\right)\left(1-\alpha\right) Z_t \left(\frac{K_t}{N_t}\right)^{\alpha} \tag{3}$$

$$C_t^{\kappa}\left(1-N_t\right)^{\kappa} = \beta \mathbb{E}_t \left\{ C_{t+1}^{\kappa}\left(1-N_{t+1}\right)^{\kappa} \alpha Z_{t+1} \left(\frac{N_{t+1}}{K_{t+1}}\right)^{1-\alpha} \right\} \tag{4}$$

The first question to answer is which variables, if any, will be trended (i.e. they will grow even in the absence of shocks). The key idea here is the realization that in a first-order accurate solution to the model above, the unconditional expectation of a random variable (RV) is equal to its non-stochastic steady state (NSSS). The strategy for answering this first question is then as follows:

1. We can show that a variable will trend by showing that its unconditional expectation (i.e. its NSSS) does not exist.
2. We can show that the unconditional expectation of a RV does not exist by showing that it is non-stationary
3. We can show that a RV is nonstationary by proving the contrapositive of the following elementary proposition:

Definition 1 *A Markov process is time homogeneous if its conditional distribution does not depend explicitely on time.*

Proposition 2 *A stationary Markov process is time homogeneous.*

That is, we will show that, because our RV of interest is not time-homogeneous, then it cannot be stationary ($\neg Q \Rightarrow \neg P$). So, take logs of the first four equations in the system (1)-(4):

$$\begin{aligned}
\ln Y_t &= \ln Z_t + \alpha \ln K_t + (1-\alpha)\ln N_t \\
\ln Z_t &= \ln Z_0 + tg + \rho w_{t-1} + \epsilon_t \\
\ln C_t - \ln\left(1-N_t\right) &= \ln\left(\frac{\varphi}{1-\varphi}\right) + \ln\left(1-\alpha\right) + \ln Z_t + \alpha \ln K_t - \alpha \ln N_t
\end{aligned}$$

where I have used $\ln(1+g) \approx g$ for g "small", and using the factorization $\ln(a+b) = \ln(a(1+b/a)) = \ln(a) + \ln(1+b/a)$ for the investment equation:

$$\begin{aligned}
\ln I_t &= \ln Y_t + \ln\left(1 - C_t/Y_t\right) \\
&= \ln Y_t + \ln\left(1 - \Psi_t\right)
\end{aligned}$$

where $\Psi_t = \frac{(1-N_t)(1-\alpha)\varphi}{N_t(1-\varphi)}$. Now take conditional expectations of all RVs:

$$\mathbb{E}_{t-1}\ln Y_t = \ln Z_0 + tg + \mathbb{E}_{t-1}w_t + \alpha \ln K_t + (1-\alpha)\,\mathbb{E}_{t-1}\ln N_t \tag{5}$$

$$\mathbb{E}_{t-1}\ln C_t = \mathbb{E}_{t-1}\ln\left(\frac{1-N_t}{N_t^{\alpha}}\right) + \ln\left(\frac{\varphi}{1-\varphi}\right) + \ln(1-\alpha)\,Z_0 + tg + \mathbb{E}_{t-1}w_t + \alpha\ln K_t \tag{6}$$

$$\mathbb{E}_{t-1}\ln I_t = \ln Z_0 + tg + \mathbb{E}_{t-1}w_t + \alpha \ln K_t + (1-\alpha)\,\mathbb{E}_{t-1}\ln N_t + \mathbb{E}_{t-1}\ln(1-\Psi_t) \tag{7}$$

First we notice that given the assumption that $1-L_t = N_t$ it is easy to see that N_t must have a NSSS (i.e. it will not trend), for, otherwise leisure would diverge to $\pm\infty$. Moreover, w_t is a stationary Markov process and thus time homogeneous so $\mathbb{E}_{t-1}w_t$ cannot depend explicitely on time. Given these two findings, it is easy to see that, unless $\alpha\ln K_t = \Phi - tg$, for some constant Φ, the conditional expectation of the RVs Y_t, C_t, I_t (and therefore K_t) will depend explicitely on time (i.e. these RVs will not be time-homogeneous). We use the equation $K_{t+1} = I_t$ to check if this is the case. Suppose that $\alpha\ln K_t = \Phi - tg$ (and find a contradiction). Then:

$$\begin{aligned}
\alpha\mathbb{E}_{t-1}\ln K_{t+1} &= \alpha\mathbb{E}_{t-1}\ln I_t \\
&= \alpha\left[\ln Z_0 + tg + \mathbb{E}_{t-1}w_t + \alpha\ln K_t + (1-\alpha)\,\mathbb{E}_{t-1}\ln N_t + \ln(1-\Psi_t)\right] \\
&= \alpha\left[\ln Z_0 + tg + \mathbb{E}_t w_t + (1-\alpha)\,\mathbb{E}_{t-1}\ln N_t + \ln(1-\Psi_t)\right] + \alpha\left[\Phi - tg\right] \\
&= \alpha\left[\ln Z_0 + \mathbb{E}_{t-1}w_t + (1-\alpha)\,\mathbb{E}_{t-1}\ln N_t + \mathbb{E}_{t-1}\ln(1-\Psi_t)\right] + \alpha\Phi
\end{aligned}$$

but $\alpha\ln K_t = \Phi - tg \Rightarrow \alpha\ln K_{t+1} = \Phi - (t+1)\,g$, a contradiction since we know that $w_t,\ N_t$ are time-homogeneous. This shows that the "t" in the expressions for conditional expectations in (5)-(7) will not dissapear and therefore, Y_t, C_t, I_t (and therefore K_t) will not be time-homogeneous (i.e. they will trend).[1]

Now we know that the behavior of our endogenous variables will have a nonstationary component (since they are trended) and a stationary component (given that stationary shocks perturb the system). Thus, the second question to answer is, how can we derive explicitely their trends? Recall that *in continuous* time the growth rate of x_t is given by $\ln(x_t/x_{t-1})$ so in this case:[2]

$$\begin{aligned}
\Delta\ln Y_t &= \Delta\ln Z_t + \alpha\Delta\ln K_t + (1-\alpha)\,\Delta\ln N_t \\
&= \Delta\ln Z_t + \alpha\Delta\ln Y_t + \alpha\Delta\ln(1-\Psi_t) + (1-\alpha)\,\Delta\ln N_t \\
\Delta\ln Y_t &= \frac{g}{1-\alpha} + \Delta w_t + \frac{\alpha}{1-\alpha}\Delta\ln(1-\Psi_t) + \Delta\ln N_t
\end{aligned}$$

Clearly, the first term is the nonstationary component of the behavior of Y_t. To see this, notice that absent any shocks ($\epsilon_t = 0\ \forall\ t$), we obtain $\Delta\ln Y_t = g/(1-\alpha)$, or, $Y_t = Y_0 e^{tg/(1-\alpha)}$ for some initial value Y_0. The discrete time analogue of this is of course $Y_t = Y_0\left[1 + \frac{g}{1-\alpha}\right]^t$, again when $\epsilon_t = 0\ \forall\ t$ (if you want, try all this with $1 - Y_{t-1}/Y_t$ instead of $\ln Y_t/Y_{t-1}$). Next, do the same for I_t :

$$\begin{aligned}
\Delta\ln I_t &= \Delta\ln Y_t + \Delta\ln(1-\Psi_t) \\
&= \frac{g}{1-\alpha} + \Delta w_t + \frac{1}{1-\alpha}\Delta\ln(1-\Psi_t) + \Delta\ln N_t
\end{aligned}$$

and again, the last three terms will be the stationary component of I_t and, absent any shocks, $\Delta\ln I_t = g/(1-\alpha)$. By my assumption that $\delta = 1$, this implies that K_t will also grow at rate $g/(1-\alpha)$ but it is trivial to show that this is also the case even for $\delta < 1$. Finally, using the equation for consumption:

$$\ln C_t = \ln\left(\frac{\Psi_t}{N_t^{\alpha-1}}\right) + \ln Z_t + \alpha\ln K_t$$

[1] If the presence of logs in the expressions for conditional expectations bothers you, simply recall that $\mathbb{E}_{t-1}Y_t \geq \log\mathbb{E}_{t-1}Y_t \geq \mathbb{E}_{t-1}\log Y_t$, where the second result follows from Jensen's inequality.

[2] Of course, our model is in discrete time, but you get the idea.

so:

$$
\begin{aligned}
\Delta \ln C_t &= \Delta \ln \left(\frac{\Psi_t}{N_t^{\alpha-1}} \right) + g + \Delta w_t + \alpha \Delta \ln K_t \\
&= \Delta \ln \left(\frac{\Psi_t}{N_t^{\alpha-1}} \right) + g + \Delta w_t + \alpha \Delta \ln I_t \\
&= \frac{g}{1-\alpha} + (1+\alpha)\Delta w_t + \frac{\alpha}{1-\alpha} \Delta \ln (1-\Psi_t)\, \Psi_t + \Delta \ln N_t
\end{aligned}
$$

so we conclude, again, that, absent any shocks, $\Delta \ln C_t = g/(1-\alpha)$. Thus, we've shown that Y_t, C_t, I_t, K_t will be trended and that in fact their secular trend will be identical (i.e. the model delivers balanced growth). This, by the way, is not an accident and it's not true of every model. The functional forms in this model have been carfully chosen so as to be consistent with the data, which is suggestive of balanced growth.

We're now ready to de-trend the variables. Given our findings so far, we can write, say, C_t as a combination of trend and cycle:

$$
\begin{aligned}
C_t &= C_{t-1} [1+\gamma] (1+\xi_t) \\
&= C_0 [1+\gamma]^t (1+\xi_t)
\end{aligned}
$$

where $\gamma = g/(1-\alpha)$ and ξ_t is a stationary stochastic process (which of course, will be a function of the shocks and parameters of the model). It is clear now that to make C_t stationary, we need to divide by $[1+\gamma]^t$. This is exactly what DD:2011 say you should do in order to remove the (common) trend (third paragraph, page 25 of the 2nd edition). Define $\tilde{C}_t = C_t/[1+\gamma]^t$ and suppose now that all shocks are zero so that $\xi_t = \xi^* \ \forall\ t$ (naturally, a special case of this is $\xi^* = 0$). In that case, clearly $\tilde{C}_t = \tilde{C}_{t-1} = C_0 (1+\xi^*) = \tilde{C}^*$ which shows that the de-trended variables indeed have a NSSS.

Finally, some of you were puzzled by how to obtain equation (3.26) in DD:2011. The confusion, I think has to do with the slightly abuse of notation in the book; they use the same symbol, Z_t, to denote two different processes (one stationary and the nonstationary). First, let's express our Euler equation (4) in terms of the detrended variables $\tilde{Y}_t, \tilde{C}_t, \tilde{I}_t, \tilde{K}_t$:

$$
\begin{aligned}
C_t^\kappa (1-N_t)^\kappa &= \beta \mathbb{E}_t \left\{ C_{t+1}^\kappa (1-N_{t+1})^\kappa \left[\alpha Z_{t+1} \left(\frac{N_{t+1}}{K_{t+1}} \right)^{1-\alpha} \right] \right\} \\
\left(\frac{\tilde{C}_t^t (1+\gamma)^t}{\mathbb{E}_t \tilde{C}_t (1+\gamma)^{t+1}} \right)^\kappa \left(\frac{1-N_t}{1-\mathbb{E}_t N_{t+1}} \right)^\kappa &= \beta \mathbb{E}_t \left\{ \alpha Z_{t+1} \left(\frac{N_{t+1}}{\tilde{K}_{t+1} (1+\gamma)^{t+1}} \right)^{1-\alpha} \right\}
\end{aligned}
$$

next, recall that $Z_{t+1} = Y_{t+1}/K_{t+1}^\alpha N_{t+1}^{1-\alpha}$ so that:

$$
\left(\frac{\tilde{C}_t^t (1+\gamma)^t}{\mathbb{E}_t \tilde{C}_{t+1} (1+\gamma)^{t+1}} \right)^\kappa \left(\frac{1-N_t}{1-\mathbb{E}_t N_{t+1}} \right)^\kappa = \beta \mathbb{E}_t \left\{ \alpha \frac{\tilde{Y}_{t+1}}{\tilde{K}_{t+1}^\alpha N_{t+1}^{1-\alpha}} \left(\frac{N_{t+1}}{\tilde{K}_{t+1}} \right)^{1-\alpha} \right\}
$$

now, define $V_t = \tilde{Y}_t/\tilde{K}_t^\alpha N_t^{1-\alpha}$ then the Euler equation is given by:

$$
\left[\frac{\tilde{C}_t (1-N_t)}{1+\gamma} \right]^\kappa = \beta \mathbb{E}_t \left\{ \tilde{C}_{t+1}^\kappa (1-N_{t+1})^\kappa \alpha V_{t+1} \left(\frac{N_{t+1}}{\tilde{K}_{t+1}} \right)^{1-\alpha} \right\}
$$

where V_t is approximated by a stationary AR(1) process such as $\ln V_t = (1-\rho) \ln \bar{V} + \rho \ln V_{t-1} + \varsigma_t$, and $\varsigma_t \sim \mathcal{N}(0, \sigma^2)$. To obtain estimates of $\bar{V}, \rho, \sigma^2$ you can get data on $\tilde{Y}_t, \tilde{K}_t$, (i.e., data on Y_t, K_t detrended), fit the OLS regression $\ln \tilde{Y}_t = (1-\alpha) \ln N_t + \alpha \ln \tilde{K}_t + \eta_t$, let $\hat{\eta}_t = \ln V_t$ and then fit an AR(1) to $\hat{\eta}_t$.