Problem Set 4: Proposed solutions
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1 Lucas' Tree

(i) & (iii) I will not solve exactly the exercise in the problem set (so you can actually think about it) but instead I will solve a very similar problem. This is a simplified version of Lucas' (1978) tree model. Suppose that there is no production. Agents can hold assets whcih yield exogenous **stochastic** dividends y_t. In each period, the rep. agent's choice variables are, consumption, c_t and share holdings, θ_t, (share of the tree). In turn, the state of this economy at t is composed of shares holdings from previous period, θ_{t-1}, and the divident shock, y_t. Since there is one good and one asset, we introduce p_t, the relative price of shares (in terms of consumption goods). We should also allow for capital gains from selling shares carried from the previous period $p_t(\theta_t - \theta_{t-1})$. The planner's problem is therefore:

$$\max_{\{\theta_t, C_t\}} \mathbb{E} \sum_{t=0}^{\infty} \beta^t U(c_t)$$
$$s.t.$$
$$c_t + p_t(\theta_t - \theta_{t-1}) \leq y_t \theta_{t-1}$$

A Pareto optimal allocation is comprised of sequences $\{c_t, \theta_t\}_{t=0}^{\infty}$ that, given a sequence of shocks, $\{y_t\}_{t=0}^{\infty}$, and a sequence of prices $\{p_t\}_{t=0}^{\infty}$, solve the rep. agent's problem, i.e.:

$$p_t = \beta \mathbb{E}_t \left\{ \frac{U'(c_{t+1})}{U'(c_t)} (y_{t+1} + p_{t+1}) \right\}$$
$$c_t + p_t(\theta_t - \theta_{t-1}) = y_t \theta_{t-1}$$

along with the usual TVC for θ_t.

(ii) & (iv) Analogous to (ii) and (iv) in the problem set, consider a multi-asset environment. There are k different risky assets and a riskless asset, B. The planners problem now becomes:

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t)$$
$$s.t.$$
$$c_t + \sum_{j=1}^{k} p_{jt}(\theta_{jt} - \theta_{jt-1}) + B_t \leq \sum_{j=1}^{k} y_{jt}\theta_{jt-1} + (1 + r_{t-1}) B_{t-1} + \omega_t$$

notice that obviously $\sum_j \theta_{jt} = 1 \; \forall \; t$ (think about $\sum_j \pi_{ij} = 1$ in the problem set). Now the FOC for this problem are:

$$\begin{aligned} [C_t] &: \quad \beta^t U'(c_t) = \lambda_t \\ [B_t] &: \quad \lambda_t = \mathbb{E}_t \lambda_{t+1} (1 + r_t) \end{aligned}$$

and k (one for each of the j assets) FOCs of the form:

$$\lambda_t p_{jt} = \mathbb{E}_t \left[\lambda_{t+1} (p_{jt+1} - y_{jt+1}) \right]$$

hence the $k + 1$ Euler equations are:

$$\begin{aligned} U'(c_t) &= \beta \mathbb{E}_t \left[U'(c_{t+1}) \right] (1 + r_t) \\ p_{jt} &= \beta \mathbb{E}_t \left[\frac{U'(c_{t+1})(p_{jt+1} - y_{jt+1})}{U'(c_t)} \right] \quad \text{for } j = 1, ...k \end{aligned}$$

2 Competitive equilibrium

The solution to the simple RBC model can in fact be descentralized as the outcome of a competitive equilbrium. To see this, state the problem of the RH and the firm separately.

Households

The representative household maximizes lifetime discounted utility subject to its resource constraint. Households own the factors of production k, l and own the firms. At each period, the RH receives income from renting all of its available capital at rate r_t, working a fraction of its endowed labor at wage w_t, and earning profits from the firms. Since there is only one final good, we normalize its price to one ($p_t c_t = c_t$). With this income, the RH and decides how much to consume and how much to invest (save):

$$\max_{c_t, l_t} \mathbb{E} \sum_{t=0}^{\infty} \beta^t \left[\log c_t - \chi \frac{l_t^2}{2} \right]$$
$$s.t.$$
$$c_t + k_{t+1}^h \leq (1 + r_t - \delta) k_t^h + w_t l_t^h + \pi_t = y_t^h$$

Firms

Firms produce a single good by renting production factors from the RH and maximize profits subject to their production technology:

$$
\begin{aligned}
\max\sum_{t=0}^{\infty}\pi_t &= \max\sum_{t=0}^{\infty}\left(y_t^f - w_t l_t^f - r_t k_t^f\right) \\
& \quad s.t. \\
y_t^f &\leq F(k_t^f, l_t^f) \equiv A_t k_t^{\alpha} l_t^{1-\alpha}
\end{aligned}
$$

Since firms don't discount the future, lifetime profits are maximized $\Leftrightarrow$ profits are maximized at every period t.[1]

Equilibrium

A competitive equilibrium consists of a set of prices $\{p_t = 1, w_t, r_t\}_{t=0}^{\infty}$ and allocations $\{k_t^*, l_t^*, y_t^*, c_t^*\}_{t=0}^{\infty}$ such that $\forall\ t$:

1. The firm maximizes profits. To do so, note that since $F(\cdot)$ is strictly increasing, the technology constraint will hold with equality $\left(y_t^f = F(k_t^f, l_t^f)\right)$. Thus, the F.O.C.s of the firm are:

$$
\begin{aligned}
\frac{\partial \pi(k_t^f, l_t^f)}{\partial l_t^f} &= 0 \Longrightarrow w_t = F_l(k_t^f, l_t^f) = (1-\alpha)\, A_t \left(k_t^f\right)^{\alpha}\left(l_t^f\right)^{-\alpha} \\
\frac{\partial \pi(k_t^f, l_t^f)}{\partial k_t^f} &= 0 \Longrightarrow r_t = F_k(k_t^f, l_t^f) = \alpha A_t \left(k_t^f\right)^{\alpha-1}\left(l_t^f\right)^{1-\alpha}
\end{aligned}
$$

2. The RH maximizes utility. The F.O.C.s for the RH are usual:

$$
\begin{aligned}
\left(c_t^h\right)^{-1} &= \beta\mathbb{E}_t\left(c_{t+1}^h\right)^{-1} r_{t+1} \\
\left(c_t^h\right)^{-1} w_t &= \chi l_t^h \\
c_t^h + k_{t+1}^h &= (1 + r_t - \delta)\, k_t^h + w_t l_t^h + \pi_t
\end{aligned}
$$

3. Markets clear in all periods ($t = 1, 2...$):

$$
\begin{gathered}
c_t^* = c_t^h = y_t^f = y_t^* \\
l_t^h = l_t^f = l_t^* \\
k_t^h = k_t^f = k_t^*
\end{gathered}
$$

[1] It is straightforward to extend this model to the case where firms discount future profits. A natural candidate for discounting would be $\frac{1}{R}$ where R is the gross interest rate (in this economy all assets would earn R).

Next, replace the F.O.C.s for the firm in the profit function at t:

$$\pi_t = F(k_t^*, l_t^*) - F_l(k_t^*, l_t^*) - F_k(k_t^*, l_t^*)k_t^*$$

and because $F(\cdot)$ is homogeneous of degree one, Euler's theorem ($\mathbf{x} \cdot \boldsymbol{\nabla} f(\mathbf{x}) = f(\mathbf{x})$) implies that $\pi_t = 0$ so that $\sum_{t=0}^{\infty} \pi_t = 0$. Replacing in the F.O.C.s for the RH yields the same optimality conditions derived under the centralized approach. Hence we have found a vector of prices that delivers the (planned) Pareto optimal allocation. That is, the optimal allocation has been 'descentralized' as a competitive equilibrium of the economy This is an ilustration of the **second fundamental theorem of welfare economics**. [2]

3 Complete markets (I)

(i) Let $\mathbb{P}$ be the transition matrix which is row stochastic. Finding the probability of a particular history in this case is trivial: $\pi\left(s^t\right) = (1,0,1,0)$ given $s(0) = 0$ is simply $(\mathbb{P}_{12})^4 = 0.2^4 = 0.001\,6$. A more interesting question is how to derive the unconditional distribution π_t (i.e., a vector of unconditional probabilities given a matrix of conditional probabilities) and its relationship with stationary distributions. The unconditional probability of a Markov process are determined by:

$$\pi_t = \Pr(x_t) = \pi_0' \mathbb{P}^t \Rightarrow \pi_{t+1} = \pi_t' \mathbb{P}$$

since $\pi_t'\mathbb{P} = \left(\pi_0'\mathbb{P}^t\right)\mathbb{P} = \pi_0'\mathbb{P}^{t+1}$. An unconditional distribution is said to be time-invariant or stationary if

$$\begin{aligned} \pi &= \pi'\mathbb{P} \\ \pi'(I - \mathbb{P}) &= 0 \\ \left(I - \mathbb{P}'\right)\pi &= 0 \end{aligned}$$

that is, the stationary distribution π can be found as the eigenvector (normalized to satisfy $\sum_{j=1}^{S} P_{ij} = 1$) associated with the unit eigenvalue of $\mathbb{P}'$. Notice that $\mathbb{P}$ stochastic $\Rightarrow \exists$ at least one unit eigenvalue. Furthermore, the stationary distribution may not be unique because $\mathbb{P}$ may have a repeated unit eigenvalue. When do unconditional distributions π_t approach a stationary distribution? That is, does the following condition hold:

$$\lim_{t \to \infty} \pi_t = \pi_\infty$$

where $(I - \mathbb{P}')\pi_\infty = 0$? And if it does hold, does this depend upon the initial distribution π_0?. If the condition holds regardless of the initial distribution then the process is *asymptotically stationary with a unique invariant distribution.* Markov chains whose matrix $\mathbb{P}$ has all nonzero elements satisfy this condition (Theorem 1 LS, pp33)

[2] Recall that the first welfare theorem states that whenever households are non-satiated, a competitive equilibrium allocation is Pareto optimal.

(ii) The Pareto optimal allocation must solve:

$$L = \sum_{t=0} \sum_{s^t} \left\{ \omega \beta^t \log \left[c_t^1 \left(s^t \right) \right] \pi \left(s^t \right) + (1 - \omega) \beta^t \log \left[c_t^2 \left(s^t \right) \right] \pi \left(s^t \right) + \theta \left(s^t \right) \left[1 + s_t - c_t^1 \left(s^t \right) - c_t^2 \left(s^t \right) \right] \right\}$$

the FOC are:

$$\begin{aligned} \frac{c_t^2 \left(s^t \right)}{c_t^1 \left(s^t \right)} &= \frac{(1-\omega)}{\omega} \\ 1 + s_t &= c_t^1 \left(s^t \right) + c_t^2 \left(s^t \right) \end{aligned}$$

therefore, $c_t^2 \left(s^t \right) = \frac{(1-\omega)}{\omega} \left[1 + s_t - c_t^2 \left(s^t \right) \right]$ so that:

$$\begin{aligned} c_t^2 \left(s^t \right) &= (1 - \omega) \left[1 + s_t \right] \\ c_t^1 \left(s^t \right) &= \omega \left[1 + s_t \right] \end{aligned}$$

(iii) A competitive equilibrium is composed of feasible allocations $\left\{ c_t^1 \left(s^t \right), c_t^2 \left(s^t \right) \right\}$ and price sequences $\left\{ q_t^0 \left(s^t \right) \right\}$ $\forall\, t$ and $\forall\, s^t$ such that for $i = 1, 2$, the consumption allocation $c_t^i \left(s^t \right)$ solves the i-th household problem given prices and shocks. Now we need to solve for the competitive equilibrium. Let μ_i be the multiplier on the resource constraint for each HH. Household 2 solves:

$$\begin{aligned} &\max_{c_t^2(s^t)} \sum_{t=0} \sum_{s^t} \left\{ \beta^t \log \left[c_t^2 \left(s^t \right) \right] \pi \left(s^t \right) \right\} \\ s.t. \quad : \quad &\sum_{t=0} \sum_{s^t} q_t^0 \left(s^t \right) c_t^2 \left(s^t \right) \leq \sum_{t=0} \sum_{s^t} q_t^0 \left(s^t \right) \end{aligned}$$

with FOC:

$$\beta^t \pi \left(s^t \right) = \mu_2 q_t^0 \left(s^t \right) c_t^2 \left(s^t \right)$$

Next, household 1 problem is:

$$\begin{aligned} &\max_{c_t^1(s^t)} \sum_{t=0} \sum_{s^t} \left\{ \omega \beta^t \log \left[c_t^1 \left(s^t \right) \right] \pi \left(s^t \right) \right\} \\ s.t. \quad : \quad &\sum_{t=0} \sum_{s^t} q_t^0 \left(s^t \right) c_t^1 \left(s^t \right) \leq \sum_{t=0} \sum_{s^t} q_t^0 \left(s^t \right) s^t \end{aligned}$$

with FOC:

$$\beta^t \pi \left(s^t \right) = \mu_1 q_t^0 \left(s^t \right) c_t^1 \left(s^t \right)$$

let $\lambda_i = \mu_i^{-1}$. Now, market clearing requires:

$$Y_t \left(s_t \right) = c_t^1 \left(s^t \right) + c_t^2 \left(s^t \right)$$

so using the FOCs:

$$Y_t \left(s_t \right) = \frac{\lambda_1 \beta^t \pi \left(s^t \right)}{q_t^0 \left(s^t \right)} + \frac{\lambda_2 \beta^t \pi \left(s^t \right)}{q_t^0 \left(s^t \right)}$$

so that the prices that support the competitive equilibrium are given by:

$$q_t^0\left(s^t\right) = \frac{\left(\lambda_1 + \lambda_2\right)\beta^t \pi\left(s^t\right)}{Y_t\left(s_t\right)}$$

(iv) For an appropriately chosen set of Pareto weights, the two allocations coincide. In particular, $\omega = \lambda_1$ and $(1 - \omega) = \lambda_2$. In that case, $q_t^0\left(s^t\right) = \beta^t \pi\left(s^t\right) / Y_t\left(s_t\right) = \theta\left(s^t\right)$. See LS pp. 202.