

Problem Set 3: Proposed solutions
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1 Balanced growth in the OGM

Instead of going over the FOCs, log-linearization, etc again, I will explore one interesting feature of the OGM, namely, the BGP result. Simplify problem 1 by assuming $\delta = 1$, $\alpha = 1$, $A_t = 1 \forall t$ (i.e., the non-stochastic so-called AK model under CRRA utility and full depreciation). The social planner's problem is:

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \\ \text{s.t.} \quad & : y_t \geq c_t + i_t \\ & : y_t = f(k_t) = Ak_t \\ & : i_t = k_{t+1} \end{aligned}$$

the BFE for this problem is:

$$V(k_t) = \max_{k_{t+1}} \{u(Ak_t - k_{t+1}) + \beta V(k_{t+1})\}$$

with F.O.C.:

$$-u'(c_t) + \beta V'(k_{t+1}) = 0$$

and envelope condition:

$$V'(k_{t+1}) = u'(c_{t+1}) A$$

so the Euler equation becomes:

$$u'(c_t) = \beta u'(c_{t+1}) A$$

or, using the functional form for $u(c_t)$:

$$\frac{c_{t+1}^\sigma}{c_t^\sigma} = \beta A$$

Since $c_t = Ak_t - k_{t+1}$:

$$(Ak_{t+1} - k_{t+2})^\sigma = \beta A (Ak_t - k_{t+1})^\sigma$$

A second order difference equation that, without a (on k) that, without a terminal condition has multiple solutions for any given k_0 . The interest is in that which satisfies the TVC (??) which in this case becomes (using the EC):

$$\lim_{t \rightarrow \infty} \beta^{t+1} V'(k_{t+t}) k_{t+1} = \lim_{t \rightarrow \infty} \beta^{t+1} u'(c_{t+1}) A k_{t+1}$$

Now to find conditions that ensure holding of the TVC note that in a BGP capital and consumption grow at the same constant rate so $k_{t+1} = \gamma k_t$ and $c_{t+1} = \gamma c_t$ for some γ . Thus:

$$\frac{(\gamma c_t)^\sigma}{c_t^\sigma} = \beta A \Rightarrow \gamma = (\beta A)^{1/\sigma}$$

which brings positive growth iif $\beta > 1/A$. Next, rewrite $u'(c_{t+1}) = c_t^{-\sigma}/\beta A$ and $k_{t+1} = \gamma^{t+1} k_0$ in the TVC:

$$\begin{aligned} \lim_{t \rightarrow \infty} \beta^{t+1} u'(c_{t+1}) A k_{t+1} &= \lim_{t \rightarrow \infty} \beta^{t+1} \frac{c_t^{-\sigma}}{\beta A} A \gamma^{t+1} k_0 \\ &= \lim_{t \rightarrow \infty} \beta^{t+1} \frac{c_t^{-\sigma}}{\beta} \gamma^{t+1} k_0 \\ &= \lim_{t \rightarrow \infty} (\beta \gamma)^t c_t^{-\sigma} \gamma k_0 \end{aligned}$$

then the TVC holds iif $\beta \gamma < 1 \Rightarrow \beta (\beta A)^{1/\sigma} < 1$. Hence by assuming that $\beta < A^{-\frac{1}{1+\sigma}}$ one can ensure that there is balanced growth and the TVC holds.

2 Tobin's q

Using the results from the appendix of this piece (see below), I will solve a special case of problem 2, namely, that in which $T(\cdot) = \Phi/2 [I(t)/k(t)]$. Once you see how this is solved, it is trivial to solve question 2 of the PS. The problem can be stated as:

$$\begin{aligned} \max_{\{c_t, i_t\}_{t=0}^{\infty}} & \int_0^{\infty} e^{-\rho t} u(c(t)) dt & (1) \\ \text{s.t.} & \dot{k}(t) = I(t) \\ & c(t) + I(t) = f(k(t)) - \frac{\Phi}{2} \left(\frac{I^2(t)}{k(t)} \right) \end{aligned}$$

where $I(t)$ is investment and $\Phi > 0$ is a constant.

Note that in this problem, the control variables are $c(t)$ and $I(t)$ (or k_{t+1} in the OGM), while the state variable is $k(t)$. On what follows, the present-value optimization problem is solved and then it is expressed in current value terms since the latter form lends itself to intuitive interpretation. First, we get rid of $c(t)$ by using the second constraint. Notice that we can do this only because the constraint is specified with equality. Notice also that

after eliminating this constraint, the Lagrangian and the Hamiltonian are obviously the same so that in the language of Appendix, $G(\cdot)$ disappears and $\mathcal{L}_I = H_I$. Thus, set up the present-value Hamiltonian:

$$H^{pv} = e^{-\rho t} u \left[f(k(t)) - I(t) - \frac{\Phi}{2} \left(\frac{I^2(t)}{k(t)} \right) \right] + \pi(t) I(t)$$

The F.O.C. w.r.t. the control is simply obtained:

$$\pi(t) = e^{-\rho t} u'(c(t)) \left[1 + \Phi \left(\frac{I(t)}{k(t)} \right) \right] \quad (2)$$

Next, using the Pontryagin conditions corresponding to the present-value problem (15)-(16):

$$\dot{\pi}(t) = -e^{-\rho t} u'(c(t)) \left[f'(k(t)) + \frac{\Phi}{2} \left(\frac{I(t)}{k(t)} \right)^2 \right] \quad (3)$$

$$\dot{k}(t) = I(t) \quad (4)$$

Or, using the F.O.C. to solve for $I(t)$ we can rewrite (4) as:

$$\dot{k}(t) = \frac{k(t)}{\Phi} \left[\frac{e^{\rho t} \pi(t)}{u'(c^*(t))} - 1 \right]$$

And just as in the discrete time problem, the TVC is given by: $\lim_{t \rightarrow \infty} \pi(t)k(t)$. Naturally, with an explicit functional form for u , we could solve for $c^*(t)$, $I^*(t)$ using the F.O.C. (2) and the constraint, replace this in (3)-(4) to obtain a pair of differential equations on $\pi(t)$ and $k(t)$ that would characterize the solution to the problem.

An interesting avenue to take in this problem is to express the equilibrium conditions in current-value terms. To do so, multiply (3) by $e^{\rho t}$ on both sides (the other conditions do not involve $\pi(t)$) and define $q(t) = e^{\rho t} \pi(t)$. Then, since $\dot{q}(t) = \dot{\pi}(t)e^{\rho t} + \rho\pi(t)e^{\rho t}$, one has that $\dot{\pi}(t)e^{\rho t} = \dot{q}(t) - \rho q(t)$ and therefore the Pontryagin conditions can be written:

$$\dot{q}(t) - \rho q(t) = -u'(c(t)) \left[f'(k(t)) + \frac{\Phi}{2} \left(\frac{I(t)}{k(t)} \right)^2 \right] \quad (5)$$

$$\dot{k}(t) = \frac{k(t)}{\Phi} \left[\frac{q(t)}{u'(c^*(t))} - 1 \right] \quad (6)$$

Finally, using (5) we can arrive at:

$$q(t) = \int_t^\infty \underbrace{e^{-\rho(s-t)} u'(c(s))}_{\text{disc. marg. util of output}} \left[\underbrace{f'(k(s)) + \frac{\Phi}{2} \left(\frac{I(s)}{k(s)} \right)^2}_{\text{marg. prod. of k - marg. adj cost}} \right] ds \quad (7)$$

that is, *Tobin's q* summarizes the information of the discounted social benefit of installing an additional unit of capital.

A Appendix: The Maximum Principle

The Maximum Principle gives an approach to dynamic optimization that is alternative to the Dynamic Programming approach. It also exploits the concepts of states, controls, state-transition functions and the Envelope Theorem. This section follows closely Dixit (1990).

A.1 Discrete time

Let:

- z_t be the control variable and y_t the state variable.
- The objective function be defined by:

$$F(y_t, z_t)$$

- The transition function be defined by:

$$Q(y_t, z_t) = y_{t+1} - y_t$$

- The additional constraints:

$$G(y_t, z_t) \geq 0$$

The dynamic optimization problem is therefore:

$$\begin{aligned} \max \sum_{t=0}^T F(y_t, z_t) \\ \text{s.t.} \\ Q(y_t, z_t) = y_{t+1} - y_t \\ G(y_t, z_t) \geq 0 \end{aligned}$$

with $y_0 \geq 0$ given and a terminal condition on y_{T+1} . The Lagrangian is:

$$\mathcal{L} = \sum_{t=0}^T \{F(y_t, z_t) + \pi_{t+1} [Q(y_t, z_t) - y_{t+1} + y_t] + \lambda_t G(y_t, z_t)\}$$

where π_{t+1} and λ_t are the multipliers associated with each constraint. The F.O.C. for the control variable is easy to obtain:

$$\frac{\partial \mathcal{L}}{\partial z_t} = 0 \implies F_z(y_t, z_t) + \pi_{t+1} Q_z(y_t, z_t) + \lambda_t G_z(y_t, z_t) \quad (8)$$

but the condition for the state variable is not so straightforward since each y_t appears in two terms of the infinite sum. To circumvent this issue, re-write the relevant part of the Lagrangean as:

$$\begin{aligned}
\sum_{t=0}^T \pi_{t+1} [y_t - y_{t+1}] &= \pi_1 [y_0 - y_1] + \pi_2 [y_1 - y_2] + \dots + \pi_{T+1} [y_T - y_{T+1}] \\
&= \pi_1 y_0 - \pi_1 y_1 + \pi_2 y_1 - \pi_2 y_2 \dots \pi_{T+1} y_T - \pi_{T+1} y_{T+1} \\
&= \sum_{t=1}^T y_t (\pi_{t+1} - \pi_t) + y_0 \pi_1 - y_{T+1} \pi_{T+1}
\end{aligned}$$

so that the problem becomes:

$$\begin{aligned}
\mathcal{L} &= \sum_{t=1}^T [F(y_t, z_t) + \pi_{t+1} Q(y_t, z_t) + y_t (\pi_{t+1} - \pi_t) + \lambda_t G(y_t, z_t)] \\
&\quad + \underbrace{F(y_0, z_0) + \pi_1 Q(y_0, z_0) + y_0 \pi_1 - y_{T+1} \pi_{T+1}}
\end{aligned}$$

and note that the terms in braces pertain to $t = 0, T + 1$ whose values are given by initial and terminal conditions so no need to worry about them. From this formulation, it is clear why π_{t+1} is given the name "co-state". Now, the F.O.C. for the state variable can be derived more easily:

$$\frac{\partial \mathcal{L}}{\partial y_t} = 0 \implies \pi_{t+1} - \pi_t + [F_y(y_t, z_t) + \pi_{t+1} Q_y(y_t, z_t) + \lambda_t G_y(y_t, z_t)] = 0 \quad \forall t \neq 0, T+1 \quad (9)$$

This optimality condition states that, at the optimum, the overall marginal return from increasing y_t is zero; that is, the shadow prices prevent pure or excess return from holding y_t . Now, rearranging:

$$\pi_{t+1} - \pi_t = - [F_y(y_t, z_t) + \pi_{t+1} Q_y(y_t, z_t) + \lambda_t G_y(y_t, z_t)] \quad (10)$$

next define the Hamiltonian:

$$H(y_t, z_t, \pi_t) = F(y_t, z_t) + \pi_{t+1} Q(y_t, z_t) \quad (11)$$

and note that the optimization problem does not consist simply in maximizing the instantaneous reward function $F(\cdot)$ since future reward depends upon future values of the state variable, which in turn is related to its current value and the choice variable via the state-transition function $Q(\cdot)$. Next, define the Lagrangian, \mathfrak{L} , for the single-period problem:

$$\mathfrak{L} = H(y_t, z_t, \pi_{t+1}) + \lambda_t G(y_t, z_t) \quad (12)$$

and here, $H(y_t, z_t, \pi_t)$ is the objective function of this single-period problem. Following F.O.C. (8), it is clear that z_t is chosen so as to maximize (11), so let $H(y_t, z_t^*, \pi_{t+1}) = H^*(y_t, \pi_{t+1})$. Next, notice that:

$$\frac{\partial \mathfrak{L}}{\partial y_t} = F_y(y_t, z_t) + \pi_{t+1} Q_y(y_t, z_t) + \lambda_t G_y(y_t, z_t)$$

so we can replace this in (10):

$$\pi_{t+1} - \pi_t = -\frac{\partial \mathcal{L}}{\partial y_t} = -\mathcal{L}_y$$

But notice, in the static problem (12) the Envelope Theorem applies and thus $\mathcal{L}_y = H_y^*$ so:

$$\pi_{t+1} - \pi_t = -H_y^*(y_t, \pi_t) \quad (13)$$

and a similar envelope condition for the co-state variable gives $\mathcal{L}_\pi = H_\pi^*(y_t, \pi_t) = Q(y_t, z_t)$ which replaced in the definition of the state-transition equation yields:

$$y_{t+1} - y_t = H_\pi^*(y_t, \pi_t) \quad (14)$$

so the Maximum Principle states that first order necessary and sufficient conditions for the optimization problem above are:

1. For each t , z_t maximizes the Hamiltonian $H(y_t, z_t, \pi_t)$ subject to the single period constraint(s) $G(y_t, z_t)$.
2. The changes in y_t, π_t over time are governed by the pair of difference equations (13)-(14).

A.2 Continuous time

State the problem above in continuous time:

$$\begin{aligned} \max \int_0^T F(y(t), z(t)) dt \\ \text{s.t.} \\ Q(y(t), z(t)) = \dot{y}(t) \\ G(y(t), z(t)) \geq 0 \end{aligned}$$

so that the (rearranged) Lagrangean is:

$$\begin{aligned} \mathcal{L} = \int_0^T [F(y(t), z(t)) + \pi(t)Q(y(t), z(t)) + y(t)(\dot{\pi}(t)) + \lambda(t)G(y(t), z(t))] \\ + F(y(0), z(0)) + \pi_1 Q(y(0), z(0)) + y(0)\pi(0) - y(T)\pi(T) \end{aligned}$$

The condition for z_t to maximize the Hamiltonian is (assuming it is legitimate to differentiate under the integral sign):

$$\frac{\partial \mathcal{L}}{\partial z(t)} = 0 \implies F_z(y(t), z(t)) + \pi(t)Q_z(y(t), z(t)) + \lambda(t)G_z(y(t), z(t)) = 0$$

while the Hamiltonian itself is defined as:

$$H(y(t), z(t), \pi(t)) = F(y(t), z(t)) + \pi(t)Q(y(t), z(t))$$

and the pair of differential equations (Pontryagin conditions) governing the behavior of the state and co-state variables:

$$\dot{y}(t) = H_{\pi}^*(y(t), \pi(t)) \quad (15)$$

$$\dot{\pi}(t) = -H_y^*(y(t), \pi(t)) \quad (16)$$

A.3 Current value vs. present value Hamiltonian

Using the notation above, suppose that:

$$F(y(t), z(t)) = e^{-\rho t} f(y(t), z(t))$$

so that the underlying objective function is the present-discounted value of the stream of instantaneous utility functions $f(y(t), z(t))$. Then the present value hamiltonian above can be written as:

$$H^{pv}(y, z, \pi) = e^{-\rho t} f(y(t), z(t)) + \pi(t)Q(y(t), z(t))$$

Now suppose that it is desirable to state the problem in current value terms; the Hamiltonian would be:

$$H^{cv}(y, z, \pi) = f(y(t), z(t)) + q(t)Q(y(t), z(t))$$

where:

$$q(t) = \pi(t)e^{\rho t} \quad (17)$$

is the current-value shadow multiplier. Now revisit the Pontryagin conditions for the *present value* problem:

$$\begin{aligned} \frac{\partial H^{pv}}{\partial z(t)} &= 0 \implies e^{-\rho t} f_z(y(t), z(t)) + \pi(t)Q_z(y(t), z(t)) = 0 \\ \dot{y}(t) &= H_{\pi}^{pv}(y, z, \pi) = Q(y(t), z(t)) \\ \dot{\pi}(t) &= -H_y^{pv}(y, z, \pi) = e^{-\rho t} f_y(y(t), z(t)) + \pi(t)Q_y(y(t), z(t)) \end{aligned}$$

Only the first and last of these conditions involve discounting so, rewrite the first in *current-value* terms:

$$\begin{aligned} f_z(y(t), z(t)) + \pi(t)e^{\rho t}Q_z(y(t), z(t)) &= 0 \\ f_z(y(t), z(t)) + q(t)Q_z(y(t), z(t)) &= 0 \end{aligned} \quad (18)$$

and rewrite the last condition, still in *present-value* terms as:

$$\dot{\pi}(t)e^{\rho t} = f_y(y(t), z(t)) + \pi(t)e^{\rho t}Q_y(y(t), z(t)) \quad (19)$$

now, since from (17):

$$\begin{aligned}\dot{q}(t) &= \dot{\pi}(t)e^{\rho t} + \rho\pi(t)e^{\rho t} \\ &= \dot{\pi}(t)e^{\rho t} + \rho q(t) \\ \dot{\pi}(t)e^{\rho t} &= \dot{q}(t) - \rho q(t)\end{aligned}$$

one can replace in (19) and:

$$\begin{aligned}\dot{q}(t) - \rho q(t) &= f_y(y(t), z(t)) + \pi(t)e^{\rho t} Q_y(y(t), z(t)) \\ \dot{q}(t) - \rho q(t) &= f_y(y(t), z(t)) + q(t) Q_y(y(t), z(t))\end{aligned}\tag{20}$$

is the Pontryagin condition for the costate variable corresponding to the *current-value* optimization problem.