Perfect and limit admissible perfect equilibria in discontinuous games

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ABSTRACT

We compare the properties of several notions of trembling-hand perfection within classes of compact, metric, and possibly discontinuous games, and show that in the presence of payoff discontinuities, standard notions of trembling-hand perfection fail a weakening of admissibility termed limit admissibility. We also provide conditions ensuring the existence of a limit admissible perfect equilibrium.

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1. Introduction

The notion of trembling-hand perfect equilibrium was introduced by Selten (1975) as a refinement of the Nash equilibrium concept. Nash equilibria survive trembling-hand perfection if they are “good” predictors of equilibrium behavior in some perturbed games in which the players make slight mistakes in the execution of their strategies.

It is well-known that for finite games, trembling-hand perfect equilibria exist in the set of Nash equilibria and put no mass on weakly dominated strategies. In other words, trembling-hand perfect equilibria satisfy existence, (E), and admissibility, (Ad). Moreover, a trembling-hand perfect equilibrium is the limit of a sequence of exact equilibria in perturbed games in which the players are constrained to play completely mixed strategies, (P).

Ideally, one would like an infinite-game extension of the standard solution concept that satisfies (E), (Ad), (P), and reduces to the set of trembling-hand perfect equilibria in finite games, (R). For infinite games, however, properties (E) and (Ad) are not generally compatible. In fact, there are (compact, continuous) games that have a unique Nash equilibrium in weakly dominated strategies (e.g., Simon and Stinchcombe, 1995, Example 2.1). However, condition (E) is compatible with a weaker property, termed limit admissibility (LA) in Simon and Stinchcombe (1995). This property requires that equilibria put mass only on the limits of weakly un-dominated strategies. For compact, continuous games, Simon and Stinchcombe (1995) present three infinite-game generalizations of Selten’s (1975) trembling-hand perfect equilibrium: strong perfection (satisfying the four criteria, (E), (LA), (P), and (R)), weak perfection (satisfying all but (P)), and limit-of-finite perfection (which fails (LA) but satisfies the remaining three properties). Of the three approaches, it is argued in Simon and Stinchcombe (1995) that strong perfection most closely respects the strategic aspects of infinite games.

Our treatment allows for potential discontinuities in the payoffs of a game. In the presence of discontinuities, condition (E) is not guaranteed by standard arguments. This, in turn, influences (P) (through the potential nonexistence of exact equilibria in perturbations of the original game). Moreover, discontinuities may affect (LA) via their effect on the topological properties of the set of weakly dominated strategies.

Carbonell-Nicolau (2011b) considers an infinite-game extension of the standard notion of perfection and identifies a rich class of possibly discontinuous games for which the set of trembling-hand perfect equilibria is nonempty. The extension considered in Carbonell-Nicolau (2011b) is termed trembling-hand perfection. Building on the existence results of Carbonell-Nicolau (2011b), in this paper we compare several notions of trembling-hand perfection in terms of their properties. We first present the equilibrium concepts of Simon and Stinchcombe (1995) and

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state an analogue of the standard three-way characterization of tremble-hand perfection (e.g. van Damme, 2002, p. 28) for discontinuous games (Theorem 2). We then study a class of possibly discontinuous games (subsuming the set of all compact, metric, continuous games) for which the behavior of the existing concepts of perfection is standard (i.e., as in Simon and Stinchcombe, 1995). The properties of the various solution concepts within this class are summarized in Theorem 4. This result considers two notions of weak domination. Weak dominance confines attention to domination by pure strategies, while weak domination allows for domination by both pure and mixed strategies. Limit admissibility (LA) is based on weak* domination, while limit admissibility (LA*) is consistent with weak domination.

In Section 4, we show, by means of an example, that trembling-hand perfection and strong (or weak) perfection fail limit admissibility* (and hence limit admissibility) in rather well-behaved discontinuous games. Thus, as the universe of games is expanded to include larger classes of discontinuous games, the strong approach, which is equivalent to trembling-hand perfection, loses some of its appeal: it ceases to satisfy (LA*) and (LA). It is therefore natural to ask whether there are alternative approaches to trembling-hand perfection (or stronger refinements) that do not suffer from this drawback.

We argue that for the classes of discontinuous games considered in this paper, natural strengthenings of trembling-hand perfection for infinite games are likely to pose existence problems, in the sense that, for the said classes, conditions leading to existence (e.g., better-reply security in perturbed games) fail. By contrast, the set of trembling-hand perfect equilibria minus the interior of the set of weakly dominated (resp. weakly* dominated) strategy profiles, a solution concept that we call limit admissible perfection (resp. limit admissible perfection*), need not be subject to this trade-off: there are rich collections of games for which the set of limit admissible perfect (resp. limit admissible perfect*) equilibria is nonempty. For these collections, limit admissible perfection (resp. limit admissible perfection*) selects a subset of the set of trembling-hand perfect equilibria and, unlike the existing formulations of perfection, meets (LA) (resp. (LA*)) and (E), as well as the other desiderata. These properties, along with those of the other solution concepts, are summarized in Theorems 6 and 7.

2. Preliminaries

A metric game is a collection
\[ G = (X_i, u_i)_{i=1}^N, \]
where \( N \) is a finite number of players, each \( X_i \) is a nonempty metric space, and each \( u_i : X \to \mathbb{R} \) is bounded and Borel measurable, with domain \( X = \times_{i=1}^N X_i \). A metric game \( G = (X, u)_{i=1}^N \) such that each \( X_i \) is compact is called a compact metric game.

In the sequel, by \( X_{-i} \) we mean the set \( \times_{j \neq i} X_j \), and, given \( i, x_i \in X_i \), and
\[ X_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \in X_{-i}, \]
we slightly abuse notation and denote the point \((x_1, \ldots, x_N)\) by \((x_i, x_{-i})\).

The mixed extension of \( G \) is the game
\[ \mathcal{G} = (M_i, U_i)_{i=1}^N, \]
where each \( M_i \) represents the set of Borel probability measures on \( X_i \), endowed with the weak* topology, and \( U_i : M \to \mathbb{R} \) is defined by
\[ U_i(\mu) := \int_X u_i d\mu, \]
where \( M := \times_{i=1}^N M_i \).

Henceforth, the set \( \times_{i=1}^N M_i \) is denoted by \( M_{-i} \), and given \( i, \mu_i \in M_i \), and
\[ \mu_{-i} := (\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_N) \in M_{-i}, \]
we sometimes represent the point \((\mu_1, \ldots, \mu_N)\) as \((\mu, \mu_{-i})\).

Given \( x_i \in X_i \), let \( \delta_i \) be the Dirac measure on \( X_i \) with support \( \{x_i\} \). We sometimes write, by a slight abuse of notation, \( x_i \) in place of \( \delta_i \). For \( \delta \in [0, 1] \) and \((\mu, v_i) \in M^2 \),
\[ (1 - \delta)v_i + \delta \mu, \]
denotes the member \( \sigma_i \) of \( M_i \) for which \( \sigma_i(B) = (1 - \delta)v_i(B) + \delta \mu_i(B) \) for every Borel set \( B \subseteq X_i \). When \( v_i = \delta_i \), for some \( x_i \in X_i \), we sometimes write \((1 - \delta)x_i + \delta \mu_i \) for \((1 - \delta)v_i + \delta \mu_i \). Similarly, given \((v, \mu) \in M^2 \),
\[ (1 - \delta)v + \delta \mu \]
denotes the point \((1 - \delta)v_i + \delta \mu_1, \ldots, (1 - \delta)v_N + \delta \mu_N \), where \( v = (v_1, \ldots, v_N) \) and \( \mu = (\mu_1, \ldots, \mu_N) \).

The following notation shall be used in Section 3 to formally define various notions of trembling-hand perfection. A Borel probability measure \( \mu_i \) on \( X_i \) is said to be strictly positive if \( \mu_i(O) > 0 \) for every nonempty open set \( O \subseteq X_i \).

For each \( i \), let \( M_i \) represent the set of all strictly positive members of \( M_i \). Set \( \tilde{M} := \times_{i=1}^N \tilde{M}_i \). For \( v_i \in \tilde{M}_i \) and \( \delta = (\delta_1, \ldots, \delta_N) \in [0, 1]^N \), define
\[ M_i(\delta v_i) := \{\mu_i \in M_i : \mu_i \geq \delta v_i\} \]
and \( M(\delta v) := \times_{i=1}^N M_i(\delta v_i) \). Given \( \delta = (\delta_1, \ldots, \delta_N) \in [0, 1]^N \) and \( v = (v_1, \ldots, v_N) \in \tilde{M} \), the game
\[ \mathcal{G}_\delta = (M_i(\delta v_i), U_i|_{M_i(\delta v_i)})_{i=1}^N \]
is called a Selten perturbation of \( G \). We often work with perturbations \( \mathcal{G}_\delta \), satisfying \( \delta_1 = \cdots = \delta_N \). When referring to these objects, we simply write \( \mathcal{G}_\delta \), with \( \delta = \delta_1 = \cdots = \delta_N \).

3. Comparison of perfect equilibria

In this section, we introduce several formulations of perfection and compare their properties in families of possibly discontinuous games. Let \( \mathcal{G}_\delta \) designate the set of compact, metric, and continuous games. We show that the strengths of the various solution concepts within \( \mathcal{G}_\delta \), as highlighted by Simon and Stinchcombe (1995), need not extend to larger classes of games. In this section, we consider a strict superset of \( \mathcal{G}_\delta \) for which the existing solution concepts behave as in \( \mathcal{G}_\delta \). In a subsequent section (Section 4), we point to the notion: in the presence of discontinuities, the existing concepts fail limit admissibility. We then study refinements of trembling-hand perfection that do not suffer from this limitation.

**Definition 1.** A strategy profile \( x = (x_1, \ldots, x_N) \in X \) is a Nash equilibrium of \( G \) if for each \( i \), \( u_i(x) \geq u_i(y_i, x_{-i}) \) for every \( y_i \in X_i \).

A Nash equilibrium of the mixed extension \( \mathcal{G} \) is called a mixed-strategy Nash equilibrium of \( G \). By a slight abuse of terminology, we sometimes refer to a mixed-strategy Nash equilibrium of \( G \) simply as a Nash equilibrium of \( G \).

We first define trembling-hand perfection in terms of Selten perturbations.

**Definition 2.** A strategy profile \( \mu \in M \) is a trembling-hand perfect (thp) equilibrium of \( G \) if there are sequences \((\delta^0), (\nu^0)\), and \((\mu^0)\) such that \((0, 1)^N \ni \delta^0 \to 0, \nu^0 \in M, \mu^0 \to \mu \), and each \( \mu^0 \) is a Nash equilibrium of the perturbed game \( \mathcal{G}_{\nu^0} \).

In words, \( \mu \) is a thp equilibrium of \( G \) if it is the limit of some sequence of equilibria of neighboring Selten perturbations of \( G \). See Carbonell-Nicolau (2011b) for an intuitive interpretation of Definition 2.
Remark 1. Note that, in Definition 2, we do not require that $\mu$ be a Nash equilibrium of $G$. It is well-known that, for continuous games, the fact that a strategy profile $\mu$ is the limit of some sequence of equilibria of Selten perturbations of $G$ guarantees that $\mu$ is a Nash equilibrium of $G$. While we do not impose continuity of payoff functions, our conditions also ensure that the limit point is an equilibrium (cf. Carbonell-Nicolau, 2011b).

The following definitions adapt the perfection concepts in Simon and Stinchcombe (1995) to potential discontinuities in the payoff functions of a game.

For $\mu \in M$, let $Br_i(\mu)$ denote player $i$’s set of best responses in $M_i$ to the vector of strategies $\mu$:

$$Br_i(\mu) := \{\sigma_i \in M_i : U_i(\sigma_i, \mu_{-i}) \geq \sup_{\sigma_{-i} \in M_{-i}} U_i(\sigma, \mu_{-i})\}.$$  

Consider the following distance functions between members of $M_i$:

$$\rho_i^\mu(\mu, \nu) := \sup_{B} |\mu_i(B) - \nu_i(B)|,$$

and

$$\rho_i^\nu(\mu, \nu) := \inf_{\epsilon > 0} : \forall B, \mu_i(B) \leq \nu_i(N_i(B)) + \epsilon$$

and

$$\nu_i(\nu, \mu) := \inf_{\epsilon > 0} : \forall B, \mu_i(B) \leq \nu_i(N_i(B)) + \epsilon,$$

where $N_i(B)$ denotes the $\epsilon$-neighborhood of the (measurable) set $B$ (i.e., $N_i(B) := \bigcup_{x \in B} N_i(x)$).

Definition 3 (Simon and Stinchcombe, 1995). Given $\epsilon > 0$, a strong $\epsilon$-perfect equilibrium of $G$ is a vector $\mu^\epsilon \in M$ such that for each $i$,

$$\rho_i^\epsilon(\mu^\epsilon_i, Br_i(\mu^\epsilon)) < \epsilon,$$

and a weak $\epsilon$-perfect equilibrium of $G$ satisfies

$$\rho_i^\epsilon(\mu^\epsilon_i, Br_i(\mu^\epsilon)) < \epsilon.$$

A strategy profile $\mu$ in $G$ is a strong (resp. weak) perfect equilibrium of $G$ if it is the weak* limit as $\epsilon^* \to 0$ of strong (resp. weak) $\epsilon^*$-perfect equilibria.

Strong closedness implies weak closedness, and hence all strong perfect equilibria are weak perfect.\footnote{The converse is false (see Simon and Stinchcombe, 1995, Example 2.3).}

It is well-known that for finite games, $thp$ equilibria exist in the set of Nash equilibria and put no mass on weakly dominated strategies. In other words, $thp$ equilibria satisfy existence, (E), and admissibility, (Ad). Moreover, a $thp$ equilibrium is the limit of a sequence of exact equilibria in perturbed games in which players are constrained to play completely mixed strategies, (P).

As demonstrated by Simon and Stinchcombe (1995, Example 2.1), (E) and (Ad) are not generally compatible, for there are infinite games with a unique Nash equilibrium in weakly dominated strategies. Simon and Stinchcombe propose a weakening of (Ad), termed limit admissibility (LA), which states that a Nash equilibrium is limit admissible if it puts mass only on the limits of dominated strategies, or, equivalently, if no player assigns positive mass to the interior of the set of weakly dominated strategies.

For compact, continuous games, the set of $thp$ equilibria (according to Definition 2) satisfies (E), (LA), (P), and reduces to the set of trembling-hand perfect equilibria for finite games, (R) (Theorem 3). Our purpose is to compare the properties of the solution concepts in Definitions 2 and 3 in the presence of payoff discontinuities.\footnote{Simon and Stinchcombe (1995) introduced a third equilibrium refinement, termed limit-of-finite (lof) perfection. Roughly speaking, $\mu$ is a limit-of-finite perfect equilibrium in $G$ if it is the limit of $\epsilon$-perfect equilibria for successively larger finite approximations of $G$. As pointed out in Simon and Stinchcombe (1995), the lof approach is ill-suited as a general solution concept even in continuous games. We shall not reiterate the limitations of the lof approach. These limitations extend to families of possibly discontinuous games. See Carbonell-Nicolau and McLean (2011) for a limit-of-finite formulation that does not suffer from the drawbacks of Simon and Stinchcombe’s (1995) approach.}

We first state the existence result of Carbonell-Nicolau (2011b).

Condition (A). There exists $(\mu_1, \ldots, \mu_n) \in M$ such that for each $i$ and every $\epsilon > 0$ there is a Borel measurable map $f : X_i \to X_i$ such that the following is satisfied:

(a) For each $x_i \in X_i$ and every $y_{-i} \in X_{-i}$, there is a neighborhood $O_{x_i} \subset X_i$ such that $u_i(f(x_i), y_{-i}) > u_i(x_i, y_{-i}) - \epsilon$ for all $z_{-i} \in O_{y_{-i}}$.

(b) For each $y_{-i} \in X_{-i}$, there is a subset $Y_i$ with $\mu(Y_i) = 1$ such that for every $x_i \in Y_i$, there is a neighborhood $V_{x_{-i}} \subset X_{-i}$ such that $u_i(f(x_i), z_{-i}) < u_i(x_i, z_{-i}) + \epsilon$ for all $z_{-i} \in V_{y_{-i}}$.\footnote{Condition (A) can be slightly weakened. See footnote 11 in Carbonell-Nicolau (2011b).}

Theorem 1 (Carbonell-Nicolau, 2011b, Theorem 2). Suppose that a compact, metric game $G$ satisfies Condition (A). Suppose further that $\sum_{i=1}^{N} u_i$ is upper semicontinuous. Then $G$ has a trembling-hand perfect equilibrium, and all trembling-hand perfect equilibria of $G$ are Nash.

The following result establishes the relationship between trembling-hand perfection and strong perfection. The equivalence of (1)–(3) is analogous to the standard characterization of trembling-hand perfect equilibria in finite games (e.g. van Damme, 2002, p. 28).

Theorem 2. For a metric game $G = (X_i, u_i)_{i=1}^{N}$, the following three conditions are equivalent:

(1) $\mu$ is a trembling-hand perfect equilibrium of $G$.

(2) $\mu$ is a strong perfect equilibrium of $G$.

(3) $\mu$ is the limit of a sequence ($\mu^n$) in $M$ with the property that for each $i$ and every $\epsilon > 0$, $\mu^n_i(\{x_i \in X : U_i(x_i, \mu^n_{-i}) \geq \sup_{y \in N_i} U_i(y, \mu^n_{-i})\}) \geq 1 - \epsilon$, for any sufficiently large $n$.

Proof. Let $G = (X_i, u_i)_{i=1}^{N}$ be a metric game. Suppose that $\mu$ is a $thp$ equilibrium of $G$. Then there are sequences $(\delta^n)$, $(\nu^n)$, and $(\mu^n)$ such that (0, 1) $\delta^n \to 0$, $\nu^n \in M$, $\mu^n \to \mu$, and each $\mu^n$ is a Nash equilibrium of the perturbed game $G_{\delta^n, \nu^n}$. Since $\mu^n \in M(\delta^n, \nu^n)$ for each $n$, it is easily seen that for each $i$ and $n$ there exists $\rho_i^n \in M_i$ such that

$$\mu^n_i = (1 - \delta^n_i)\mu_i^n + \delta^n_i\nu^n_i.$$  

Because each $\mu^n$ is a Nash equilibrium of $G_{\delta^n, \nu^n}$, $\rho_i^n \in Br_i(\mu^n_i)$ for each $i$ and $n$. In fact, $\rho_i^n \not\in Br_i(\mu^n_i)$ implies that

$$U_i(\rho_i^n, \mu^n_i) > U_i(\mu^n_i, \mu^n_i),$$

for some $\rho_i^n \in M_i$, so

$$U_i((1 - \delta^n_i)\rho_i^n + \delta^n_i\nu^n_i, \mu^n_i) > U_i((1 - \delta^n_i)\rho_i^n + \delta^n_i\nu^n_i, \mu^n_i) = U_i(\mu^n_i, \mu^n_i),$$

thereby contradicting that $\mu^n_i$ is a Nash equilibrium of $G_{\delta^n, \nu^n}$. Consequently, because $\mu^n_i = (1 - \delta^n_i)\rho_i^n + \delta^n_i\nu^n_i$ and $\rho_i^n \in Br_i(\mu^n_i)$ for each $i$ and $n$, and since $\delta^n_i \to 0$ for each $i$ and $n$, we have

$$\rho_i^n(\mu^n_i, Br_i(\mu^n)) \to 0.$$
for each i, so that $\mu$ is a strong perfect equilibrium. Hence, (1) implies (2).

If $\mu$ is a strong perfect equilibrium of $G$, then there is a sequence $(\mu^n_i)$ in M such that for each i and every $\epsilon > 0$, $\rho^i(\mu^n_i, Br(\mu^n)) < \epsilon$ for any sufficiently large n, and this implies

$$\mu^n_i \left( \{ x_i \in X_i : U_i(x_i, \mu^n_i) \geq \sup_{y_i \in X_i} U_i(y_i, \mu^n_i) \} \right) \geq 1 - \epsilon$$

for any sufficiently large n. Hence, (2) implies (3).

Now assume (3). Let

$$X^n_i := \{ x_i \in X_i : U_i(x_i, \mu^n_i) < \sup_{y_i \in X_i} U_i(y_i, \mu^n_i) \}.$$

Let $\sigma^n_i \in M_i$ be defined by

$$\sigma^n_i(B) := \frac{\mu^n_i(B \cap X^n_i)}{\mu^n_i(X^n_i)} + \frac{1}{\mu^n_i(B \cap (X_i \setminus X^n_i))} \mu^n_i(B \cap (X_i \setminus X^n_i)).$$

For large enough n, $\mu^n_i(X^n_i) + \frac{1}{\mu^n_i(X^n_i)} < 1$, and $\mu^n_i$ is an equilibrium of $G_{\sigma^n_i, \rho}$, where

$$\delta^n_i = (\delta^n_{i,1}, \ldots, \delta^n_{i,n_i}) := \left( \mu^n_i(X^n_i) + \frac{1}{\mu^n_i(X^n_i)} \right) \mu^n_i(X_i \setminus X^n_i) + \frac{1}{\mu^n_i(X_i \setminus X^n_i)} \mu^n_i(X_i \setminus X^n_i).$$

In fact, for large enough n,

$$\mu^n_i(B) \geq \mu^n_i(B \cap X^n_i) + \frac{1}{\mu^n_i(B \cap (X_i \setminus X^n_i))} \mu^n_i(B \cap (X_i \setminus X^n_i)) = \delta^n_i \sigma^n_i(B)$$

for every B, and so $\mu^n_i \in M_i(\delta^n_i, \sigma^n_i)$ for each i. Moreover, for each i and every $p_i \in M_i(\delta^n_i, \sigma^n_i)$, if $p^n_i(X^n_i) \geq \delta^n_i \sigma^n_i(X^n_i) = \mu^n_i(X^n_i)$, it follows that $U_i(p_i, \mu^n_i) \leq U_i(\mu^n_i)$. Consequently, $\mu^n_i$ is a best response to $\mu^n_i$.

Because $\mu^n_i$ is an equilibrium of $G_{\sigma^n_i, \rho}$ (for large enough n) and we have $\delta^n_i \to 0$ and $\mu^n_i \to \mu$, $\mu$ is a thp equilibrium of G. Hence, (3) implies (1). □

**Theorems 1 and 2 give the following:**

**Corollary 1.** Suppose that a compact, metric game $G$ satisfies Condition (A). Suppose further that $\sum u_i$ is upper semicontinuous. Then $G$ possesses a trembling-hand perfect equilibrium, which is also strong perfect and weak perfect, and all trembling-hand perfect equilibria of $G$ are Nash.

In this section we consider the family of games satisfying the conditions of Theorem 1, and impose an additional constraint on the dominance relations between strategies (to be formally defined below). We drop the extra requirement in Section 4.

Recall that $\bar{G}$ denotes the set of compact, metric, and continuous games $G = (X, u^n_i)_{i=1}^N$ satisfying Condition (A) and upper semicontinuity of the sum of payoffs $\sum_{i=1}^N u_i$.

**Remark 2.** Not all games satisfy Condition (A) (or some of its weakenings (such as Reny’s (1999) payoff security) for that matter) or upper semicontinuity of $\sum_{i=1}^N u_i$. For example, the game $G = ([0, 1], \{0, 1\}, u_1, u_2)$, where $u_1(0, 0) = 1$, $u_1(0, 0) = 1$ everywhere, and $u_2$ is identically zero fails Condition (A) (even payoff security). In fact, we have $u_1(0, 0) = 1$ and for each $x_1 \in [0, 1]$ and every neighborhood $[0, \epsilon)$ of 0, $u_1(x_1, 1) = 0 < u_1(0, 0) = 1$. Nevertheless, $\bar{G}$ is a rich class in that it contains several economic games of interest (cf. Carbonell-Nicolau (2011b, Section 3) and Carbonell-Nicolau (2011c, Section 4)).

On the other hand, for fixed action spaces $X_1, \ldots, X_N$ (with $X := X_1 \times \ldots \times X_N$), it can be shown that the class $\mathcal{G}(X)$ of games $G = (X, u^n_i)_{i=1}^N$ in $\bar{G}$ is closed when viewed as a metric subspace of the metric space $(B(X)^N, \rho_X)$, where $B(X)$ represents the set of bounded maps $f : X \to \mathbb{R}$ and $\rho_X : B(X)^N \times B(X)^N \to \mathbb{R}$ is defined by

$$\rho_X((f_1, \ldots, f_N), (g_1, \ldots, g_N)) := \sum_{x \in X} |f(x) - g(x)|.$$

Consequently, this class is complete and metric (since $B(X)$ is complete and metric) and hence (by the Baire category theorem) a Baire space.

To see that $\mathcal{G}(X)$ is closed in $B(X)^N$, suppose that $(u^n) = (u^n_1, \ldots, u^n_N)$ is a sequence of games in $\mathcal{G}(X)$ with $u^n_i \to u = (u_1, \ldots, u_N)$. We need to show that u lies in $\mathcal{G}(X)$. The proof that $\sum_{i=1}^N u_i$ is upper semicontinuous is left to the reader. We show that $(X_i, u^n_i)_{i=1}^N$ satisfies Condition (A).

For each n there exists $(\mu^n_1, \ldots, \mu^n_N) \in \hat{M}$ such that for each i and every $\alpha > 0$ there is a Borel measurable map $f^n_i(x) : X_i \to X_i$ such that the following is satisfied:

(a) For each $x_i \in X_i$ and every $y_i \in X_i$, there is a neighborhood $O_{\alpha, x_i, y_i}$ of $y_i$, for which $u^n_i(f^n_i(x_i), y_i) > u^n_i(x_i, y_i) - \alpha$ for all $z_i \in O_{\alpha, x_i, y_i}$.

(b) For each $y_i \in X_i$, there is a subset $Y_i$ of $X_i$ with $\mu^n_i(Y_i) = 1$ such that for every $x_i \in Y_i$, there is a neighborhood $O_{\alpha, x_i, y_i}$ of $y_i$, such that $u^n_i(f^n_i(x_i), z_i) < u^n_i(x_i, z_i) + \alpha$ for all $z_i \in O_{\alpha, x_i, y_i}$.

Fix i and $\epsilon > 0$. In light of (a), and since $u^n_i \to u$, if n is large enough and $\alpha > 0$ is small enough, then, for every $(x_i, y_i) \in X_i$, $u^n_i(x_i, y_i) - \alpha$ is close enough to $u_i(x_i, y_i)$ to ensure that

$$u^n_i(f^n_i(x_i), z_i) > u_i(x_i, y_i) - \frac{\alpha}{2}$$

for all $z_i \in O_{\alpha, x_i, y_i}$.

and, since $u^n_i \to u$, we see that for any large enough n, $u_i(f^n_i(x_i), z_i) > u_i(x_i, y_i) - \epsilon$ for all $z_i \in O_{\alpha, x_i, y_i}$.

We conclude that given i and $\epsilon > 0$, and for any sufficiently large n and any sufficiently small $\alpha$, the map $f^n_i$ satisfies the following: for each $x_i \in X_i$ and every $y_i \in X_i$, there is a neighborhood $O_{\alpha, x_i, y_i}$ of $y_i$ such that $u_i(f^n_i(x_i), z_i) > u_i(x_i, y_i) + \epsilon$ for all $z_i \in O_{\alpha, x_i, y_i}$.

Hence, to show that $(X_i, u^n_i)_{i=1}^N$ satisfies Condition (A), it suffices to prove that for any large enough n and any small enough $\alpha$, the map $f^n_i$ satisfies the following: For every $y_i \in X_i$, there is a subset $Y_i$ of $X_i$ with $\mu^n_i(Y_i) = 1$ such that for every $x_i \in Y_i$, there is a neighborhood $O_{\alpha, x_i, y_i}$ of $y_i$ such that $u^n_i(f^n_i(x_i), z_i) < u_i(x_i, z_i) + \epsilon$ for all $z_i \in O_{\alpha, x_i, y_i}$. But this follows from the following observations. In light of (b), because $u^n_i \to u$, for large enough n and small enough $\alpha$, and given any $y_i \in X_i$, given a subset $Y_i$ satisfying condition (b), an $x_i \in Y_i$, and a $z_i \in O_{\alpha, x_i, y_i}$ (where $O_{\alpha, x_i, y_i}$ is given by condition (b)), $u^n_i(x_i, z_i)$ is close enough to $u_i(x_i, z_i)$ to ensure that $u^n_i(f^n_i(x_i), z_i) < u_i(x_i, z_i) + \epsilon$. Consequently, since $u^n_i \to u$, for any large enough n and any small enough $\alpha$ we have $u_i(f^n_i(x_i), z_i) < u_i(x_i, z_i) + \epsilon$.

Two notions of weak domination have been considered in the literature.

**Definition 4.** A strategy $x_i \in X_i$ is weakly domination for if there exists a strategy $y_i \in X_i$ such that $u(y_i, x_i) \geq u(x_i, x_i)$ for all $x_i \in X_i$, with strict inequality for some $x_i$.

**Definition 5.** A strategy $x_i \in X_i$ is weakly dominated for if there exists a strategy $mu_i \in M_i$ such that $u_i(x_i, x_i) \geq u_i(x_i, x_i)$ for all $x_i \in X_i$, with strict inequality for some $x_i$.

The first definition is less restrictive and has been considered in Salonen (1996), while the second definition is used in Simon and Stinchcombe (1995) to formulate limit admissibility.

Recall that $\mathcal{G}$ denotes the set of compact, metric games $G = (X_i, u^n_i)_{i=1}^N$ satisfying Condition (A) and upper semicontinuity of
the sum of payoffs $\sum_{i=1}^N u_i$. Define the set $g_d$ of all members $G = \{X_i, u_i\}_{i=1}^N$ of $\bar{g}_d$ with the following property: if $x_i$ is weakly dominated in $G$ for player $i$, then for some $\mu_i$ that weakly dominates $x_i$, there exists $y_{-i} \in X_{-i}$ with $U_i(\mu_i, z_{-i}) > u_i(x_i, z_{-i})$ for all $z_{-i} \in O_{-i}$ and for some neighborhood $O_{-i}$ of $y_{-i}$.

In words, $g_d$ is the set of members of $\bar{g}_d$ for which any strategy $x_i$ that is weakly dominated for $i$ has the property that there is some $\mu_i$ that weakly dominates $x_i$ and a better response than $x_i$ to some neighborhood of action profiles of the other players. This condition is clearly satisfied by all continuous games, but need not be met in the set $\bar{g}_d$ (Example 1, Section 4). We have $g_c \subseteq \bar{g}_d \subseteq g_d$.

The next results are concerned with the properties of the various notions of perfection in the families of games $g_c, g_d$, and $g_{\bar{d}}$. For the axioms that are not affected by potential payoff discontinuities, the characterization of the solution concepts follows from the analysis in Simon and Stinchcombe (1995). We expand on the cases when Simon and Stinchcombe (1995) uses continuity to verify a property.

**Definition 6.** A strategy profile $\mu \in M$ is admissible if if $\mu_i(D^*_i) = 0$ for all $i$, where $D^*_i$ denotes the set of strategies weakly dominated for $i$.

**Definition 7.** A strategy profile $\mu \in M$ is limit admissible if $\mu_i(O^*_i) = 0$ for all $i$, where $O^*_i$ denotes the interior of the set of strategies weakly dominated for player $i$.

**Definition 8.** A strategy profile $\mu \in M$ is admissible if $\mu_i(D_i) = 0$ for all $i$, where $D_i$ denotes the set of strategies weakly dominated for $i$.

**Definition 9.** A strategy profile $\mu \in M$ is limit admissible if $\mu_i(O_i) = 0$ for all $i$, where $O_i$ denotes the interior of the set of strategies weakly dominated for player $i$.

We say that a refinement specification satisfies property (N) (resp. (W), (S)) if it selects a set of strategy profiles that is closed in the set of Nash equilibria (resp. the set of weak perfect equilibria and the set of strong perfect equilibria); property (E) if it chooses a nonempty set of strategy profiles; and property (LA*) (resp. (LA)) if it selects a set of limit admissible (resp. limit admissible) strategy profiles.

A notion of perfection satisfies property (R) if it reduces to the standard set of trembling-hand perfect equilibria in finite games; and property (P) if it chooses strategy profiles that can be viewed as limits of equilibria in perturbed games, so that the agents can be thought of as optimizing among the play of others, subject to the constraint that they and the others make mistakes.

We follow Simon and Stinchcombe (1995), where the interpretation of condition (P) varies across the various equilibrium refinements: in each case, the conception of a perturbed game arises “naturally” from the “spirit” of the corresponding approach to refinement.

Formally, a perturbation for the strong approach is a game $(M^*_\epsilon(u), U)$, where $M^*_\epsilon(u)$ denotes a convex, weak compact subset of $M_u$ within Hausdorff distance $\epsilon$ of $M_u$, where the strong distance, $\rho^*_i$, is the associated metric. The definition of a perturbation for the weak approach is analogous, with the weak distance, $\rho^w_i$, replacing $\rho^*_i$.

When applied to strong perfection, condition (P) requires that any strong perfect equilibrium be the limit of a sequence $(\mu^n)$, where each $\mu^n$ is a Nash equilibrium of a perturbation $(M^*_\epsilon(u^n), U)$, for some $\epsilon^n \to 0$.\footnote{Even when the choice of a perturbation for $G$ has been made, one can furnish (at least) two definitions of (P). The alternative to the above definition (for strong perfection) imposes the equivalence of the set of strong perfect equilibria and the set of limits of sequences $(\mu^n)$ with the said properties. We adopt the weak criterion.}

Simon and Stinchcombe (1995) study the properties of the strong and weak approaches within the class $g_c$.

**Theorem 3 (Simon and Stinchcombe, 1995).** For the family of games $g_c$, we have the following:

- Strong perfection and trembling-hand perfection satisfy (E), (LA) (hence (LA*)), (P), (R), and (W).
- Weak perfection satisfies (E), (LA) (hence (LA*)), (R), and (N), and fails (P).

**Theorem 4.** For the family of games $g_d$, we have the following:

- Strong perfection and trembling-hand perfection satisfy (E), (LA) (hence (LA*)), (P), (R), and (W).
- Weak perfection satisfies (E), (LA) (hence (LA*)), (R), and (N), and fails (P).

**Proof.** Theorems 1 and 2 give (E) in $g_d$ for all the equilibrium concepts.

We now show that weak perfect equilibrium profiles must be limit admissible. Suppose that $\mu$ is a weak perfect equilibrium of $G \in g_d$. Let $O_i$ denote the interior of the set of strategies weakly dominated for $i$. Because $\mu$ is a weak perfect equilibrium, there are sequences $(\mu^n)$ and $(\epsilon^n)$ with $M \ni \mu^n \to \mu$ and $\epsilon^n \to 0$ such that for each $i$,

$$
\rho^n_i(\mu^n - Br_i(\epsilon^n)) < \epsilon^n, \text{ for all } n.
$$

Fix $i$. Suppose that $\nu_i(O_i) > 0$ and $v_i \in Br_i(\epsilon^n)$. Then there exists $z_i \in Br_i(\mu^n) \cap O_i$. But this is a contradiction, since there is a strategy $\sigma_i \in M_i$ such that $U_i(\sigma_i, x_{-i}) \geq U_i(z_i, x_{-i})$ for all $x_{-i} \in X_{-i}$ and, for some $y_{-i}$ and some neighborhood $O_{-i}$ of $y_{-i}$, $U_i(\sigma_i, z_{-i}) > U_i(z_i, z_{-i})$ for all $z_{-i} \in O_{-i}$. (recall that $G \in g_d$, so $(\mu^n, \epsilon^n) \in M$ one cannot have $z_i \in Br_i(\mu^n)$). Consequently, $\nu_i(O_i) = 0$ for every $v_i \in Br_i(\epsilon^n)$, and this implies that for every sequence $(v_i^n)$ in $M_i$ such that $v_i^n \in Br_i(\epsilon^n)$ for each $n$, we have

$$
\lim_{n \to \infty} \inf v_i^n(O_i) = 0.
$$

Now, given (1), one can choose a sequence $(v_i^n)$ in $M_i$ such that $v_i^n \in Br_i(\epsilon^n)$ for each $n$ and

$$
\rho^n_i(\mu_i, v^n_i) \to 0.
$$

Consequently, because $\mu^n \to \mu$, we have $v^n_i \to \mu_i$. This yields

$$
\lim_{n \to \infty} \inf v^n_i(O_i) \geq \mu_i(O_i).
$$

Since $O_i$ is open. Therefore, using (2), we see that $\mu_i(O_i) = 0$.

Thus, $\mu_i(O_i) = 0$ for each $i$, and $\mu$ is limit admissible. Because weak perfection satisfies (LA) (and the equilibria are weak perfect), so do strong perfection and trembling-hand perfection (Theorem 2).

Now consider condition (P). It is clear that trembling-hand perfection satisfies (P), and hence so does strong perfection (Theorem 2). Simon and Stinchcombe (1995) show, in their Example 2.6, that there are games in which the limit of some sequence $(\mu^n)$, where each $\mu^n$ is a Nash equilibrium of $(M^*_\epsilon(u^n), U)$, and $\epsilon^n \to 0$, fails to be a weak perfect equilibrium. Moreover, according to Simon and Stinchcombe (1995) (footnote 23) one can construct a (five-agent) game with a weak perfect equilibrium that is not the limit of any sequence of weakly perturbed games. Thus, weak perfection fails (P).

Because the metrics $\rho^w$ and $\rho^*$ are equivalent for mixtures on finite sets, for finite games weak and strong perfection coincide, and are identical to Selten’s (1975) definition.
By Theorem 1, trembling-hand perfection selects Nash equilibria. Furthermore, the set of thp equilibria is closed. In fact, for \( \alpha \in (0, 1) \), let \( P(\alpha) \) be the set of profiles \( \mu \) such that \( \mu \) is a Nash equilibrium of \( G_\alpha \), for some \( \delta \in (0, \alpha) \) and some \( v \in M \). Since the set of thp equilibria can be expressed as \( \cap_{\alpha} \text{cl}(P(\alpha)) \), trembling-hand perfection and strong perfection satisfy (W).

To see that weak perfect equilibria are Nash, let \( \mu \) be a weak perfect equilibrium of \( G \). Then there is a sequence \( (\varepsilon^n) \) with \( \varepsilon^n \to 0 \) such that \( \mu^n \to \mu \), where each \( \mu^n \) is a weak \( \varepsilon^n \)-perfect equilibrium. Hence, for each \( i \) and every \( n \),

\[
\mu^n_i(\pi_0) = \frac{\mu^n_i(\pi_0 + \varepsilon^n_i)}{1 + \varepsilon^n_i}
\]

where \( \pi_0 \) is the profile of the initial state.

and, since \( \mu^n \to \mu, \mu_i \) puts mass 1 on \( L_{\text{thp}}(\mu^n_i) \), the limes superior (in the sense of Kuratowski) of \( L_{\text{thp}}(\mu^n_i) \). Thus it suffices to show that \( L_{\text{thp}}(\mu^n_i) \subseteq L(\mu) \). To establish this containment, let \( x^*_n \to x_i \), where \( x^*_n \in L(\mu^n_i) \), for some subsequence \( (\mu^n_i) \) of \( (\mu^n_i) \), and \( x \not\in L(\mu) \). Because \( x^*_n \to x_i, \mu^k \to \mu \), and each \( u_i \) is bounded, we have (passing to a subsequence if necessary)

\[
(x^*_n, \mu^k), (U(x^*_n, \mu^k), \ldots, U_N(x^*_n, \mu^k)) \to ((x_i, \mu_{-i}), (\alpha_1, \ldots, \alpha_N))
\]

(3)

and, since \( \mu^n \to \mu, \mu_i \) puts mass 1 on \( L_{\text{thp}}(\mu^n_i) \), the limes superior (in the sense of Kuratowski) of \( L_{\text{thp}}(\mu^n_i) \). Thus it suffices to show that \( L_{\text{thp}}(\mu^n_i) \subseteq L(\mu) \). To establish this containment, let \( x^*_n \to x_i \), where \( x^*_n \in L(\mu^n_i) \), for some subsequence \( (\mu^n_i) \) of \( (\mu^n_i) \), and \( x \not\in L(\mu) \). Because \( x^*_n \to x_i, \mu^k \to \mu \), and each \( u_i \) is bounded, we have (passing to a subsequence if necessary)

\[
(x^*_n, \mu^k), (U(x^*_n, \mu^k), \ldots, U_N(x^*_n, \mu^k)) \to ((x_i, \mu_{-i}), (\alpha_1, \ldots, \alpha_N))
\]

(3)

for some \( \alpha := (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N \). Therefore, \( (x_i, \mu_{-i}, \alpha) \) belongs to the closure of the set

\[
\{(\rho, \sigma) \in M \times \mathbb{R}^N : U_i(\rho) = \sigma_i, \text{ for all } i\}.
\]

Consequently, since \( G \in G_{\text{dt}} \), and Condition (A), together with upper semicontinuity of \( \sum_{i=1}^n u_i \), implies Reny’s (1999) better-reply security of \( G \), the fact that \( x_i \not\in L(\mu) \) (so that \( \delta_0 \not\in L(\mu) \)) gives i and \( \sigma_i \in M_i \) such that, for some neighborhood \( O_{\mu_{-i}} \) of \( \mu_{-i} \) and some \( \gamma > 0 \),

\[
U_i(\sigma_i, \mu_{-i}) \geq \sigma_i + \gamma, \quad \text{for all } \sigma_i \in O_{\mu_{-i}}.
\]

We therefore have, in view of (3),

\[
U_i(\sigma_i, \mu_{-i}) > U_i(x^*_n, \mu^k)
\]

for any sufficiently large \( k \), thereby contradicting that \( x^*_n \) is a best response to \( \mu^k \) (i.e., that \( x^*_n \in L(\mu^k) \)). Therefore, \( x_i \in L_{\text{thp}}(\mu^n_i) \) implies \( x_i \in L(\mu) \), as desired. We omit the proof that the set of weak perfect equilibrium profiles is closed. □

4. Limit admissibility in discontinuous games

While trembling-hand perfection and strong perfection appear to better respect the structure of infinite games relative to weak perfection (cf. Simon and Stinchcombe, 1995), expanding the domain of games beyond \( G_{\text{dt}} \) or \( G_{\text{dt}} \) affects the properties of these solution concepts. In fact, there are members of \( G_{\text{dt}} \) in which carefully chosen “trembles” may render a weakly dominated strategy no less appealing than a corresponding dominant strategy: if the opponent’s “tremble” gives measure zero to the strategies against which the dominant strategy is superior, associated Selten perturbations may exhibit equilibria assigning large mass to the interior of the set of weakly” dominated strategies. As a result, trembling-hand perfection and strong perfection fail (LA* ) and (LA). This is illustrated in the following example.

Example 1. Define \( f : [0, 1] \to \mathbb{R} \) by

\[
f(x) :=
\begin{cases}
0 & \text{if } x > 0, \\
1 & \text{if } x = 0.
\end{cases}
\]

Let \( f^n \) be a sequence of continuous, strictly decreasing functions \( f^n : [0, 1] \to \mathbb{R} \) with the following properties:

(a) Each \( f^n \) intersects with \( f \) only once at \( x^n_2 \in (0, \frac{1}{2}) \), where \( x^n_2 \to 0, \) and \( f^n(1) = -\frac{1}{2} < 0 = f(1) \) for each \( n \).

(b) \( f^n \) converges to \( f \) pointwise.

Let \( (x^n_i) \) be a sequence from \( (\frac{1}{2}, 1) \) with \( x^n_2 \to \frac{1}{2} \). Consider the two-player game \( G = ([0, 1], [0, 1], u_1, u_2) \), where

\[
u_1(x_1, x_2) :=
\begin{cases}
0 & \text{if } (x_1, x_2) \in \left[ 0, \frac{1}{2} \right] \times [0, 1],
\alpha f^n(x_1) + (1 - \alpha)f^{n+1}(x_2) & \text{if } x_1 = 0, x_2 = x^n_1,
\end{cases}
\]

and \( u_2(x_1, x_2) := x_2 \) (Fig. 1).

For this game, \( \sum_{i=1}^n u_i \) is upper semicontinuous (in fact, each \( u_i \) is upper semicontinuous), and Condition (A) is fulfilled.6

The action profile \( (0, 1) \) is clearly a Nash equilibrium in weakly dominated strategies. In fact, 0, and any sufficiently small neighborhood around 0, is weakly dominated for player 1 by \( \frac{1}{2} \).

We claim that \( (0, 1) \) is a thp equilibrium of \( G \). In fact, consider a tremble sequence \( (\nu^n_2) \in M_2 \) for player 2 such that \( \nu^n_2 \) is the Lebesgue measure on \( [0, 1] \). Note that for large enough \( n \), choosing the action 0 (in fact, choosing any action in \( [0, \frac{1}{2}] \)) is a best response for player 1 to the perturbed strategy \( (1 - \frac{1}{n}) \cdot \frac{1}{2} + (\frac{1}{n}) \cdot \nu^n_2 \).

To see this, note that for every \( x_1 \in [0, \frac{1}{2}] \) we have

\[
u_1(x_1, (1 - \frac{1}{n}) \cdot \frac{1}{2} + \frac{1}{n} \cdot \nu^n_2) = 0,
\]

and for every \( x_1 \in (\frac{1}{2}, 1] \) we have

\[
u_1(x_1, (1 - \frac{1}{n}) \cdot \frac{1}{2} + \frac{1}{n} \cdot \nu^n_2) \leq (1 - \frac{1}{n}) \cdot u_1(x_1, 1) + \frac{1}{n} \cdot u_1(x_1, \nu^n_2)
\]

\[
\leq (1 - \frac{1}{n}) \cdot \frac{1}{2} + \frac{1}{n} \cdot \frac{1}{2},
\]

6 The game \( G \) satisfies even Carbonell-Nicolau's (2011b) generic entire payoff security and generic local equi-upper semicontinuity.
so, for large enough \( n \),

\[
U_i \left( \begin{pmatrix} 0, \left(1 - \frac{1}{n}\right) 1 + \frac{1}{n} v_i^3 \end{pmatrix} \right) \geq U_i \left( \begin{pmatrix} x_1, \left(1 - \frac{1}{n}\right) 1 + \frac{1}{n} v_i^3 \end{pmatrix} \right)
\]

for every \( x_1 \in [0, 1] \). Consequently, since 0 is a best response for player 1 to the perturbed strategy \( \left(1 - \frac{1}{n}\right) 1 + \frac{1}{n} v_i^3 \), and since the action 1 is strictly dominant for player 2, for large enough \( n \),

\[
\left( \begin{pmatrix} 1 - \frac{1}{n} \end{pmatrix} 0 + \frac{1}{n} v_i^3, \left(1 - \frac{1}{n}\right) 1 + \frac{1}{n} v_i^3 \right)
\]

is an equilibrium of the Selten perturbation \( G_{i,n-1}^{n} \) for any \( v_i \in \mathcal{M}_i \). Hence, \( (0, 1) \) is trembling-hand perfect, and therefore strongly (hence weakly) perfect (Theorem 2).

Since the action 0 lies in the interior of the set of strategies weakly dominated for 1, it follows that trembling-hand perfection and strong perfection fail limit admissibility* and limit admissibility.

The following result summarizes the properties of strong perfection, trembling-hand perfection, and weak perfection in \( g_{ad} \). The statements that are given with no direct proof can be established as in the proof of Theorem 4.

**Theorem 5.** For the family of games \( g_{ad} \), we have the following:

- Strong perfection and trembling-hand perfection satisfy (E), (P), (R), and (W), and fail (LA*) and (LA).
- Weak perfection satisfies (E), (R), and (N), and fails (P), (LA*), and (LA).

### 5. Limit admissible perfect equilibria

Because (limit) admissibility is regarded as a basic requirement for stability of equilibria (e.g. Kohlberg and Mertens, 1986, and references listed therein), the fact that the existing notions of trembling-hand perfection for infinite games fail (LA*) and (LA) raises questions on the existence of alternative refinement specifications that do not suffer from this drawback. We argue that, even if there were refinements of trembling-hand perfection, along the lines of Kohlberg and Mertens’ (1986) stability, satisfying (LA*) or (LA), these refinements would pose existence problems for the classes of games considered in this paper. By contrast, there are rich families of games for which the concept of limit admissible perfect (lap) equilibrium (defined below) is immune to this problem, and meets other criteria.

We first define a lap equilibrium and compare the properties of the various solution concepts.

We consider two concepts for limit admissible perfection. The first one uses the less restrictive notions of weak domination or (limit) admissibility in Definitions 4 and 7. The second one is consistent with Definitions 5 and 9.

**Definition 10.** A strategy profile \( \mu \) in \( G \) is a limit admissible perfect* (lap*) equilibrium if it is a limit admissible*, trembling-hand perfect equilibrium of \( G \).

**Definition 11.** A strategy profile \( \mu \) in \( G \) is a limit admissible perfect (lap) equilibrium if it is a limit admissible, trembling-hand perfect equilibrium of \( G \).

**Remark 3.** A related solution concept eliminates the interior of the set of weakly dominated strategies and applies trembling-hand perfection to the resulting reduced form. Formally, given a game \( G = (X, u_i)^N \), and letting \( G_{r} = (Y, u_i)^N \) stand for the ‘subgame’ of \( G \) that results from eliminating, for each \( i \), the interior of the set of all weakly dominated strategies in \( X_i \), this variant picks the set of trembling-hand perfect equilibria in \( G_r \).

Like limit admissible perfection, this specification satisfies (LA). In finite games, it chooses thep equilibria, but there are finite games with the property that adding dominated strategies enlarges the set of trembling-hand perfect equilibria (cf. Myerson, 1978). In infinite games, however, it need not select thep equilibria or even Nash equilibria, unlike limit admissible perfection. Thus, for infinite games, adding weakly dominated strategies may alter the set of thep equilibria in such a way that some strategy profiles cease to be thep or even Nash.

This is illustrated in the following example. Define the maps \( f : [0, 1] \rightarrow \mathbb{R} \) and \( g : [0, 1] \rightarrow \mathbb{R} \) by

\[
f(x) := \begin{cases} -1 & \text{if } x = 0, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad g(x) := \begin{cases} 1 & \text{if } x \in (0, 1), \\ 0 & \text{otherwise.} \end{cases}
\]

Let \( g^n \) be a sequence of continuous maps \( g^n : [0, 1] \rightarrow \mathbb{R} \) with the following properties:

- \( g^n \) converges pointwise to \( g \); and
- \( g^n(x) < 1 \) for all \( x \).

Let \( G = ([0, 1], [0, 1], u_1, u_2) \) be a two-player game with

\[
u_1(x_1, x_2) := \begin{cases} f(x_2) & \text{if } x_1 = 1, \\ 0 & \text{if } (x_1, x_2) = \left(0, \frac{1}{2}\right), \\ x_1 - 1 & \text{otherwise,} \end{cases}
\]

and

\[
u_2(x_1, x_2) := \begin{cases} g(x_1) & \text{if } x_2 = 0, \\ 1 & \text{if } (x_1, x_2) = \left(1, \frac{1}{2}\right), \\ g^n(x_1) & \text{if } x_2 = \frac{1}{2} + \frac{1}{n} \text{ and } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}
\]

The sum \( u_1 + u_2 \) is upper semicontinuous, and \( G \) satisfies Condition (A), so \( G \) is a member of \( g_{ad} \). Moreover, it is easily seen that the action 0 is weakly dominated by 1 for player 1, and that any action \( x_1 \in (0, 1) \) is dominated by any action in \( (x_1, 1) \) for player 1. In addition, any action in \( \left(0, \frac{1}{2}\right) \cup \left(\left(\left(\frac{1}{2}, 1\right) \setminus \left(\bigcup \frac{1}{n} + \frac{1}{2} \right)\right)\right) \) is weakly dominated by 0 for player 2. Consequently, deleting the interior of the set of weakly dominated strategies yields the reduced form

\[
G_r = \left(\{1\}, [0, 1] \cup \left(\bigcup \left(\frac{1}{n} + \frac{1}{2}\right)\right)\right), u_1, u_2).
\]

which has two Nash equilibria, \( (1, \frac{1}{2}) \) and \( (1, 0) \), and it is easy to see that \( (1, 0) \) is thep in \( G_r \) (note that \( \frac{1}{2} \) is weakly dominated by 0 for player 2). However, the strategy profile \( (1, 0) \) is not even a Nash equilibrium in \( G \), for

\[
u_1 \left(\frac{1}{2}, 0\right) = -\frac{1}{2} > -1 = u_1(1, 0).
\]

Proving or disproving the existence of lap equilibria (resp. lap*) equilibria in \( g_{ad} \) seems highly nontrivial. We can furnish results for other rich collections of games. We first define these collections and then derive corresponding analogues of Theorems 3–5 for the refinement specifications considered in this section and in Section 4.

Given \( i \), recall that \( O_i^r \) denotes the interior of the set of strategies weakly* dominated for player \( i \). Given \( G = (X_i, u_i)^N \), define, for each \( i \), \( X^*_i := X_i \setminus O_i^r \).

**Condition (A).** There exists \( \mu_1, \ldots, \mu_N \in \mathcal{M} \) such that for each \( i \) and every \( \varepsilon > 0 \) there is a Borel measurable map \( f : X_i \rightarrow X_i \) such that the following is satisfied:
(a) For each \( x_i \in X_i^* \) and every \( y_i \in X_i \), there is a neighborhood \( O_{y_i} \) of \( y_i \) such that \( u_i(f(x_i), y_i) > u_i(x_i, y_i - \varepsilon) \) for all \( x_i \in O_{y_i} \).

(b) For each \( y_i \in X_i \), there is a subset \( Y_i \) of \( X_i \) such that for every \( x_i \in Y_i \), there is a neighborhood \( V_{y_i} \) of \( y_i \) such that \( u_i(f(x_i), y_i) > u_i(x_i, y_i - \varepsilon) \) for all \( x_i \in V_{y_i} \).

Given \( i \), recall that \( O_i \) denotes the interior of the set of strategies weakly dominated for player \( i \). Let \( \bar{O}_i \) be the set of \( x_i \in X_i \) such that for every \( \sigma_i \) that weakly dominates \( x_i \), if \( U_i(\sigma_i, x_i) > u_i(x_i, y_i) \), there is no neighborhood \( V_{x_i} \) of \( x_i \) such that \( U_i(\sigma_i, V_{x_i}) > u_i(x_i, y_i) \).

Define \( \bar{M} \) as the set of \( \mu \in \bar{M} \) such that there exist \( Q_1, \ldots, Q_n \), where each \( Q_i \) is a countable dense subset of \( X_i \), satisfying the following: for each \( i \) and every \( x_i \in Q_i \cap \bar{O}_i \), \( \mu \in \{ \phi_i(x_i) \} \) for some \( \phi_i : Q_i \cap \bar{O}_i \to X_i \) is defined as follows: \( \phi_i(x_i) := y_i \), where \( y_i \) is such that, for some \( \rho_i \in M \), \( \mu(y_i) \geq u_i(x_i, y_i) \) for all \( y_i \in X_i \) and \( U_i(\rho_i, y_i) > u_i(x_i, y_i) \).

Note that when each \( X_i \) is compact and metric (and hence separable) the set \( \bar{M} \) is nonempty. In fact, if each \( X_i \) is separable there exists, for each \( i \), a countable dense subset \( Q_i \) of \( X_i \). Letting \( p_i = (\rho_i(1), \ldots, \rho_i(n)) \) be a member of \( M \) such that \( \rho_i(i)(\phi_i(x_i)) > 0 \) for every \( x_i \in Q_i \cap \bar{O}_i \) and each \( i \), and defining \( \mu := \alpha + (1 - \alpha) \sum_i \rho_i(i) \) for any \( \alpha \in (0, 1) \) and \( \sigma \in \bar{M} \), we see that \( \mu \in \bar{M} \).

Remark 4. If \( \bar{O}_i = \emptyset \) for each \( i \), then \( \bar{M} = \hat{M} \).

Let \( G = (X_i, u_i^{N_i}) \) be a Nash equilibrium of \( G \). Fix \( (\delta, \mu) \in [0, 1) \times M \). Define \( u_i^{(\delta, \mu)}(x, y) := u_i((1 - \delta)x + \delta y) \) for all \( x, y \in X_i \).

Definition 12. The game \( G = (X_i, u_i^{N_i}) \) is payoff secure if for each \( \varepsilon > 0 \), \( x \in X \), and \( i \), there exists \( y_i \in X_i \) such that \( u_i(y_i, y_i) > u_i(x, y_i - \varepsilon) \) for all \( y_i \in X_i \).

The proofs of the following lemmas are similar to the proof of Lemma 1 in Carbonell-Nicolau (2011b). We omit the details and refer the reader to Carbonell-Nicolau (2011b).

Lemma 1. Suppose that a compact, metric game \( G = (X_i, u_i^{N_i}) \) with \( X_i^* \neq \emptyset \) for each \( i \) satisfies Condition (A). Then there exists \( \mu \in \bar{M} \) such that \( C_{(\delta, \mu)}^* \) payoff secure for every \( \delta \in [0, 1) \).

Lemma 2. Suppose that a compact, metric game \( G = (X_i, u_i^{N_i}) \) with \( X_i^* \neq \emptyset \) for each \( i \) satisfies Condition (A). Then there exists \( \mu \in \bar{M} \) such that \( C_{(\delta, \mu)}^* \) payoff secure for every \( \delta \in [0, 1) \).

The following lemmas establish the existence of lap* equilibria (resp. lap equilibrium) in \( G_{\delta\mu} \) (resp. \( G_{\delta\mu}^{a} \)).

Lemma 3. Every member of \( G_{\delta\mu}^{a} \) has a limit admissible perfect equilibrium.

Proof. Fix \( G = (X_i, u_i^{N_i}) \) in \( G_{\delta\mu} \). Take a sequence \( (\delta^n) \) with \( (0, 1) \ni \delta^n \to 0 \). By Lemma 1, there exists \( \mu \in \bar{M} \) such that each \( C_{(\delta^n, \mu)}^* \) is payoff secure. This, together with upper semicontinuity of \( \sum_i u_i \) for each \( n \), a Nash equilibrium \( \sigma^n \) of \( C_{(\delta^n, \mu)}^* \). In fact, upper semicontinuity of \( \sum_i u_i \) (which gives upper semicontinuity of \( \sum_i u_i(E) \) (e.g., Aliprantis and Border, 2006, Theorem 15.5)) and payoff security of \( C_{(\delta^n, \mu)}^* \) imply that \( C_{(\delta^n, \mu)}^* \) is better-reply secure (Reny, 1999, Proposition 3.2) and therefore, \( C_{(\delta^n, \mu)}^* \), being a compact, quasiconcave game, possesses a Nash equilibrium \( \sigma^n \) (Reny, 1999, Theorem 3.1). We claim that \( \sigma^n \) is also a Nash equilibrium of the mixed extension of the game \( (X_i, u_i^{(\delta^n, \mu)}) \). To see this, suppose that \( \sigma^n \) is not an equilibrium of the mixed extension of \( (X_i, u_i^{(\delta^n, \mu)}) \). Then, there exist \( i \) and \( \rho_i \in M \) such that \( \int_{x_i} u_i^{(\delta^n, \mu)}(x_i, \sigma^n_i) dx_i > \int_{x_i} u_i^{(\delta^n, \mu)}(x_i, \rho_i_i) dx_i \), so there is an \( y_i \in X_i \) such that \( \int_{x_i} u_i^{(\delta^n, \mu)}(y_i, \sigma^n_i) dx_i > \int_{x_i} u_i^{(\delta^n, \mu)}(y_i, \rho_i_i) dx_i \). Therefore, since for any \( x_i \in O_i^* \) there exists an action \( z_i \in X_i^* \) that weakly dominates \( x_i \) (Salonen, 1996, Corollary 1), there exists \( z_i \in X_i^* \) such that \( \int_{x_i} u_i^{(\delta^n, \mu)}(z_i, \sigma^n_i) dx_i > \int_{x_i} u_i^{(\delta^n, \mu)}(z_i, \rho_i_i) dx_i \), thereby contradicting that \( \sigma^n \) is a Nash equilibrium of \( C_{(\delta^n, \mu)}^* \).

Now, since \( \sigma^n \) is a Nash equilibrium of the mixed extension of the game \( (X_i, u_i^{(\delta^n, \mu)}) \), \( 1 - \delta^n \sigma^n + \delta^n \mu \) is a Nash equilibrium of \( G_{\delta\mu}^{a} \), and because \( M \) is sequentially compact, we may write (passing to a subsequence if necessary) \( \sigma^n \to \sigma \) for some \( \sigma \in M \). This, together with the fact that \( \sigma^n(\sigma_i) = 0 \) for each \( i \) and all \( n \), gives \( \sigma(\sigma_i) = 0 \). Consequently, \( \sigma \) is a lap* equilibrium of \( G \).

Lemma 4. Every member of \( G_{\delta\mu}^{a} \) has a limit admissible perfect equilibrium.

Proof. Fix \( G = (X_i, u_i^{N_i}) \) in \( G_{\delta\mu} \). Take a sequence \( (\delta^n) \) with \( (0, 1) \ni \delta^n \to 0 \). By Lemma 2, there exists \( \mu \in \bar{M} \) such that each \( C_{(\delta^n, \mu)}^* \) is payoff secure. This, together with upper semicontinuity of \( \sum_i u_i \) for each \( n \), a Nash equilibrium \( \sigma^n \) of \( C_{(\delta^n, \mu)}^* \). In fact, upper semicontinuity of \( \sum_i u_i \) (which gives upper semicontinuity of \( \sum_i u_i(E) \) (e.g., Aliprantis and Border, 2006, Theorem 15.5)) and payoff security of \( C_{(\delta^n, \mu)}^* \) imply that \( C_{(\delta^n, \mu)}^* \) is better-reply secure (Reny, 1999, Proposition 3.2) and therefore, \( C_{(\delta^n, \mu)}^* \), being a compact, quasiconcave game, possesses a Nash equilibrium \( \sigma^n \) (Reny, 1999, Theorem 3.1). We claim that \( \sigma^n \) is also a Nash equilibrium of the mixed extension of the game \( (X_i, u_i^{(\delta^n, \mu)}) \).
Lemma 3 (resp. Lemma 4) of Salonen (1996), there exists a strategy $\varrho_i \in M_i$ that weakly dominates $p_i$ and is not weakly dominated by any member of $M_i$.

For this strategy, we have

$$\varrho_i \in \arg \max_{\nu \in M_i} \int_X u_i^{(\alpha, \mu)} d(\nu, \sigma^\nu_i).$$

and

$$\int_X u_i^{(\alpha, \mu)} d(\nu, \sigma^\nu_i) > \int_X u_i^{(\alpha, \mu)} d\sigma^\nu.$$ 

If $p_i(\varrho_i) = 0$ it is clear that (4) contradicts the fact that $\sigma^\nu$ is a Nash equilibrium of $\overline{p}_i$. If $p_i$ has atoms in $O_i$ (i.e., if $p_i(x_i) > 0$ for some $x_i \in O_i$) it is easy to construct a $p_i^*$ in $M_i$ that weakly dominates $p_i$, a contradiction. It remains to consider the case when $p_i$ has no atoms in $O_i$ and $p_i(O_i) > 0$.

Suppose that $p_i$ has no atoms in $O_i$ and $p_i(O_i) > 0$. Then, because $O_i \subseteq O_i$, it is dense in $O_i$, there exists $z_i \in O_i \subseteq O_i$ and $p_i(z_i) > 0$. Since $z_i \in O_i$, there exists $v_i \in M_i$ such that $U_i(v_i, x_i) \geq U_i(z_i, x_i)$ for all $x_i \in X_i$ and $U_i(v_i, y_i) > U_i(z_i, y_i)$ for some $y_i \in X_i$. Note that if $z_i \in O_i \subseteq O_i$, there is no loss of generality in assuming that there exists a neighborhood $V_{y_i}$ of $y_i$ such that $U_i(v_i, y_i) > U_i(z_i, y_i)$ for all $y_i \in V_{y_i}$. Let $\{\epsilon^n\}$ be a sequence in $(0, 1)$ with $\epsilon^n \downarrow 0$.

Define $p_i^\epsilon(B_i) = p_i(B_i \setminus N_{\epsilon}(z_i)) + v_i(B_i)p_i(N_{\epsilon}(z_i))$

and observe that $p_i^\epsilon$ satisfies (for every Borel set $B_i \subseteq X_i$)

$$p_i^\epsilon(B_i) = p_i(B_i \setminus N_{\epsilon}(z_i)) + v_i(B_i)p_i(N_{\epsilon}(z_i)),$$

where $v_i \in M_i$ is defined by $v_i(B_i) := \frac{p_i(B_i \setminus N_{\epsilon}(z_i))}{\text{supp}(p_i)}$.

Recall that $U_i(v_i, x_i) \geq U_i(z_i, x_i)$ for all $x_i \in X_i$ and $U_i(v_i, y_i) > U_i(z_i, y_i)$. Also recall that if $z_i \in O_i \subseteq O_i$, there exists a neighborhood $V_{y_i}$ of $y_i$ such that $U_i(v_i, y_i) > U_i(z_i, y_i)$ for all $y_i \in V_{y_i}$, and since $\mu_i = \overline{\mu}$, we have $\mu_i(V_{y_i}) > 0$. Finally, observe that if $z_i \in O_i \subseteq O_i$, then we have $\mu_i(V_{y_i}) > 0$. Therefore, we have

$$U_i(v_i, (1 - \delta^n)\sigma^{\nu_i} + \delta^n\mu_{\epsilon,i} - \nu_i) > U_i(z_i, (1 - \delta^n)\sigma^{\nu_i} + \delta^n\mu_{\epsilon,i}).$$

Now, since $\sigma^{\nu_i} + \delta^n\mu_{\epsilon,i}$ is upper semicontinuous at $z_i$, for large enough $k$ we have

$$U_i(v_i, (1 - \delta^n)\sigma^{\nu_i} + \delta^n\mu_{\epsilon,i}) > \eta > U_i(w_i, (1 - \delta^n)\sigma^{\nu_i} + \delta^n\mu_{\epsilon,i})$$

for some $\eta \in \mathbb{R}$ and all $w_i \in N_{\epsilon}(z_i)$, so

$$U_i(v_i, (1 - \delta^n)\sigma^{\nu_i} + \delta^n\mu_{\epsilon,i}) > U_i(v_i^k, (1 - \delta^n)\sigma^{\nu_i} + \delta^n\mu_{\epsilon,i}).$$

Consequently, for large enough $k$ we have

$$U_i(p_i^k, (1 - \delta^n)\sigma^{\nu_i} + \delta^n\mu_{\epsilon,i})$$

$$= \int_{X_i \setminus N_{\epsilon}(z_i)} U_i(\cdot, (1 - \delta^n)\sigma^{\nu_i} + \delta^n\mu_{\epsilon,i}) dp_i$$

$$+ p_i(N_{\epsilon}(z_i))U_i(v_i, (1 - \delta^n)\sigma^{\nu_i} + \delta^n\mu_{\epsilon,i})$$

$$> \int_{X_i \setminus N_{\epsilon}(z_i)} U_i(\cdot, (1 - \delta^n)\sigma^{\nu_i} + \delta^n\mu_{\epsilon,i}) dp_i$$

$$+ p_i(N_{\epsilon}(z_i))U_i(v_i^k, (1 - \delta^n)\sigma^{\nu_i} + \delta^n\mu_{\epsilon,i})$$

$$= U_i(p_i^k, (1 - \delta^n)\sigma^{\nu_i} + \delta^n\mu_{\epsilon,i}),$$

thereby contradicting (5).

Now, since $\sigma^{\nu_i}$ is a Nash equilibrium of the mixed extension of the game $(X_i, u_i^{(\alpha, \mu)})$, $(1 - \delta^n)\sigma^{\nu_i} + \delta^n\mu_{\epsilon,i}$ is a Nash equilibrium of $\overline{p}_i$. Because $M$ is sequentially compact, we may write (passing to a subsequence if necessary) $\sigma^{\nu_i} \to \sigma$ for some $\sigma \in M$. This, together with the fact that $\sigma^{\nu_i}(O_i) = 0$ for each $i$ and all $n$, gives $\sigma(O_i) = 0$. Consequently, $\sigma$ is a Lap equilibrium of $G$.

It remains to show that the map $U_i(\cdot, p_{\epsilon,i})$ is upper semicontinuous for every $p_{\epsilon,i} \subseteq M_i$ if $u_i(x_i)$ is upper semicontinuous on $X_i$ for every $x_i \in X_i$. Because $u_i(x_i)$ is upper semicontinuous on $X_i$ for every $x_i \in X_i$, $u_i(x_i)$ is upper semicontinuous on $M_i$ for every $x_i \in X_i$ (e.g., Aliprantis and Border, 2006, Theorem 15.5), and so, given $v_i \to v_i$,

$$\limsup_{n \to \infty} U_i(v_i, x_i) \leq U_i(v_i, x_i)$$

for all $(x_i, v_i) \in M_i \times X_i$. (6)

Now take $v_i \to v_i$. For each $n$, define $f_i^n : X_i \to \mathbb{R}$ by $f_i^n(x_i) := U_i(v_i, x_i)$. Given $p_{\epsilon,i} \subseteq M_i$, we have, by Fatou's lemma,

$$\limsup_{n \to \infty} \int_{X_i} f_i^n d\mu_{\epsilon,i} \leq \limsup_{n \to \infty} \int_{X_i} f_i^n d\mu_{\epsilon,i}.$$

This, combined with (6), gives

$$\limsup_{n \to \infty} U_i(v_i, p_{\epsilon,i}) \leq U_i(v_i, p_{\epsilon,i}),$$

so $U_i(\cdot, p_{\epsilon,i})$ is upper semicontinuous on $M_i$ for every $p_{\epsilon,i} \subseteq M_i$. □

**Remark 5.** The conditions for existence of $\text{lap}^*$ equilibria (resp. lap equilibria) provided in Lemma 3 (resp. Lemma 4) apply in various economic games. For example, in the examples in Examples 4 and 6 of Carbonell-Nicolau (2011b) are members of $\overline{G}_d$ (resp. $\overline{G}_d$).

Limit admissible perfection (resp. limit admissible perfection*) satisfies $\text{LA}$ (resp. $\text{LA}^*$) and $\text{P}$ (every lap equilibrium (resp. lap* equilibrium) is $\text{thp}$, and every $\text{thp}$ is the limit of Nash equilibria in Selten perturbations). Furthermore, since trembling-hand perfect equilibria are admissible (hence admissible*) in finite games, the set of lap equilibria (resp. lap* equilibria) coincides with the set of trembling-hand perfect equilibria in finite games (i.e., limit admissible perfection (resp. limit admissible perfection*) meets $(\text{R})$).

On the other hand, trembling-hand perfection (and therefore strong perfection) fails $\text{LA}$ (resp. $\text{LA}^*$) within $\overline{G}_d$ (resp. $\overline{G}_d$). In
fact, the game in Example 1 belongs to $\tilde{G}_{\text{bd}} \cap \tilde{G}_{\text{hd}}$, and for this game $O_i = O_i'$ for each $i$.

These observations, along with Lemmas 3 and 4, are summarized by the following statements.

**Theorem 6.** For the family of games $\tilde{G}_{\text{bd}}$, we have the following:

- Limit admissible perfection* satisfies (LA), (P), (R), and (E).
- Strong perfection and trembling-hand perfection satisfy (E), (P), and (R), and fail (LA).
- Weak perfection satisfies (E) and (R), and fails (P) and (LA).

**Theorem 7.** For the family of games $\tilde{G}_{\text{hd}}$, we have the following:

- Limit admissible perfection satisfies (LA), (P), (R), and (E).
- Strong perfection and trembling-hand perfection satisfy (E), (P), and (R), and fail (LA).
- Weak perfection satisfies (E) and (R), and fails (P) and (LA).

**Remark 6.** Observe that properties (S), (W), and (N) do not appear in the statement of Theorems 6 and 7. For the class of games $\tilde{G}_{\text{bd}}$ (resp. $\tilde{G}_{\text{hd}}$), strong perfection (or trembling-hand perfection) and weak perfection need not select Nash equilibria. This is a consequence of the fact that $\tilde{G}_{\text{bd}}$ (resp. $\tilde{G}_{\text{hd}}$) does not sufficiently constrain the players’ payoff functions in the interior of the set of weakly* dominated (resp. weakly dominated) strategies. There is, however, an easy fix to this problem: impose Reny’s (1999) better-reply security of $G$ on the members $G$ of $\tilde{G}_{\text{bd}}$ (resp. $\tilde{G}_{\text{hd}}$).

We now indicate the difficulties that may arise when studying certain strengthenings of trembling-hand perfection. The game in Example 1 has a thp equilibrium $\mu$ that is not limit admissible because, at $\mu$, even though player 1 chooses a strategy, 0, that is weakly dominated by $\frac{1}{2}$, the action $\frac{1}{2}$ outperforms 0 only when the opponent chooses 0, and there are ‘trembles’ (completely mixed strategies) of the opponent that assign measure zero to the action 0. Thus, if player 1 expects player 2 to mischoose according to one such tremble, he finds it optimal to choose the strategy 0, even though 0 is weakly dominated by $\frac{1}{2}$. But if player 1 entertained the possibility that player 2’s imperfect choice might follow one of many possible trembles, some of which put positive mass on the action 0, then player 1 would never choose the weakly dominated action, 0. For example, requiring robustness of Nash equilibria to the players’ choice of any slight tremble within a small neighborhood of completely mixed strategies would guarantee the selection of limit admissible strategy profiles.

While it is well-known that strictly perfect equilibria (Okada, 1981) may fail to exist even in finite games, the above discussion suggests that Kohlberg and Merten’s (1986) set-valued notion of stability might select only limit admissible equilibrium points. Even if this were true, we point out that establishing the existence of strategically stable sets for the classes of games considered in this paper poses difficulties. Our argument is informal and confines attention to the collection $\tilde{G}_{\text{bd}}$.

The existence of strategically stable sets in a game $G$ requires the existence of Nash equilibria in open sets of slight Selten perturbations of $G$. Given $\delta \in (0, 1)^n$, some $\mu \in M$ must exist such that the games in $(\tilde{G}_{\text{bd}} : \phi \in O_i)$ possess a Nash equilibrium, for some neighborhood $O_i$ of $\mu$. However, we have found a game $G$ in $\tilde{G}_{\text{bd}}$ satisfying the following: given any $\mu \in M$ and $\delta \in (0, 1)^n$, one can find many $\phi$ arbitrarily close to $\mu$ such that $\tilde{G}_{\text{bd}}$ fails Reny’s (1999) payoff security and better-reply security. Consequently, Reny’s (1999) results cannot be used to establish the existence of Nash equilibria in open sets of Selten perturbations of $G$.

**6. Concluding remarks**

We have studied trembling-hand perfect equilibria in possibly discontinuous, normal-form games, by comparing the properties of several infinite-game generalizations of Selten’s (1975) notion of perfection. The behavior of the various formulations of trembling-hand perfection in the set $g_0$ of compact, metric, continuous games extends to certain supersets of $g_0$. For larger classes of games, however, the existing solution concepts may select equilibria in the interior of the set of weakly dominated strategies. While this problem might be remedied by using stronger refinement specifications, for the classes of games considered in this paper these specifications are likely to fail conditions that are known to give existence. However, for rich classes of discontinuous games, the notion of limit admissible perfect equilibrium is not subject to this problem, and meets other desiderata.

**References**


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10 For this game, $\tilde{G}_{\text{bd}} = \emptyset$ (so $\mu_1$ in Condition (A) can be freely chosen from $M_1$), $\tilde{G}_{\text{hd}} = \emptyset$, and, letting $Q_0$ be the set of rational numbers in $[0, 1]$, $\phi(x_1) = 0$ for each $x_1 \in Q_0 \cap Q_0$, Hence, for any $\mu = (\mu_1, \mu_2) \in M$ such that $\mu_2(\{0\}) > 0$ we have $\mu \in M$. Given that $X_1 = \{\frac{1}{2}\}$ and $X_2 = \{1\}$, it is easy to verify items (a) and (b) of Condition (A) for any such $\mu$.

11 Better-reply security of $G$ for the members of $\tilde{G}_{\text{bd}}$ (resp. $\tilde{G}_{\text{hd}}$) is satisfied, for example, by the members of $\tilde{G}_{\text{bd}}$ (resp. $\tilde{G}_{\text{hd}}$) that are uniformly payoff secure (cf. Monteiro and Page, 2007).

12 There are at least two solutions to this problem. First, one can further constrain the points of discontinuity of the players’ payoff functions. This is done in Carbonell-Nicolau (2011a), where the existence of stable sets is established for a strict subset of $\tilde{G}_{\text{bd}}$. Second, one can generalize some aspect of the main theorem in Reny (1999) in such a way that open sets of neighboring Selten perturbations can be shown to satisfy an appropriate weakening of better-reply security.