Semicontinuous integrands as jointly measurable maps

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Abstract. Suppose that \((X, \mathcal{A})\) is a measurable space and \(Y\) is a metrizable, Souslin space. Let \(\mathcal{A}^u\) denote the universal completion of \(\mathcal{A}\). For \(x \in X\), let \(\overline{f(x, \cdot)}\) be the lower semicontinuous hull of \(f(x, \cdot)\). If \(f : X \times Y \to \mathbb{R}\) is \((\mathcal{A}^u \otimes \mathcal{B}(Y), \mathcal{B}(\mathbb{R}))\)-measurable, then \(\overline{f}\) is \((\mathcal{A}^u \otimes \mathcal{B}(Y), \mathcal{B}(\mathbb{R}))\)-measurable.

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Let \((X, \mathcal{A})\) be a measurable space. For every bounded measure \(\mu\) on \((X, \mathcal{A})\), let \(\mathcal{A}^\mu\) denote the completion of \(\mathcal{A}\) with respect to \(\mu\). Let

\[ \mathcal{A}^u := \bigcap \{ \mathcal{A}^\mu : \mu \text{ is a bounded measure on } (X, \mathcal{A}) \}. \]

The \(\sigma\)-algebra \(\mathcal{A}^u\) is called the universal completion of \(\mathcal{A}\).

Let \(Y\) be a topological space, and let \(\mathcal{B}(Y)\) represent the \(\sigma\)-algebra of Borel subsets of \(Y\). The space \(Y\) is said to be Souslin if it is Hausdorff and there exist a Polish space \(P\) and a continuous surjection from \(P\) to \(Y\).

Given \(f : X \times Y \to \mathbb{R}\), define the map \(\overline{f} : X \times Y \to \mathbb{R}\) by

\[ \overline{f}(x, y) := \sup_{V_y} \inf_{z \in V_y} f(x, z), \]

where \(V_y\) ranges over all neighborhoods of \(y\). For each \(x \in X\), \(\overline{f}(x, \cdot)\) is the lower semicontinuous hull of \(f(x, \cdot)\). If \(Y\) is metrizable, \(\overline{f}\) can be expressed as

\[ \overline{f}(x, y) = \sup_{n \in \mathbb{N}} \inf_{z \in N_n(y)} f(x, z), \]

where \(N_n(y)\) represents the open \(\frac{1}{n}\)-neighborhood of \(y\).

Theorem. Suppose that \((X, \mathcal{A})\) is a measurable space and \(Y\) is a metrizable, Souslin space. Suppose further that the map \(f : X \times Y \to \mathbb{R}\) is \((\mathcal{A}^u \otimes \mathcal{B}(Y), \mathcal{B}(\mathbb{R}))\)-measurable. Then \(\overline{f}\) is \((\mathcal{A}^u \otimes \mathcal{B}(Y), \mathcal{B}(\mathbb{R}))\)-measurable.

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Proof: Define \( g^n : X \times Y \to \mathbb{R} \) by

\[
g^n(x, y) := \inf_{z \in N^n_1(y)} f(x, z).
\]

We first show that \( g^n \) is \((\mathcal{A}^u \otimes \mathcal{B}(Y), \mathcal{B}(\mathbb{R}))\)-measurable for each \( n \).

Let

\[
D^n := \left\{ (x, y, z) \in X \times Y \times Y : z \in N^n_1(y) \right\}.
\]

The map \( g^n \) is \((\mathcal{A}^u \otimes \mathcal{B}(Y), \mathcal{B}(\mathbb{R}))\)-measurable if for \( a \in \mathbb{R} \),

\[
\{(x, y) \in X \times Y : g^n(x, y) < a\} \in \mathcal{A}^u \otimes \mathcal{B}(Y).
\]

Given \( a \in \mathbb{R} \) we have

\[
\{(x, y) \in X \times Y : g^n(x, y) < a\} = \text{Proj}_{X \times Y}(E^n),
\]

where

\[
E^n := \{(x, y, z) \in D^n : f(x, z) < a\}
\]

and \( \text{Proj}_{X \times Y}(E^n) \) represents the projection of \( E^n \) onto \( X \times Y \). Thus, to establish (1) it suffices to show that \( \text{Proj}_{X \times Y}(E^n) \) belongs to \( \mathcal{A}^u \otimes \mathcal{B}(Y) \).

Because \( Y \) is a Souslin space, \( Y \) is a Lindelöf space, and since \( Y \) is in addition metrizable, \( Y \) is separable. Because \( Y \) is separable, there is a countable, dense subset \( Q \) of \( Y \). Let \( \{y^1, y^2, \ldots\} \) be an enumeration of this set. For \( \alpha > 0 \) and \( y \in Y \), define

\[
A^{(\alpha, y)} := \{(x, z) \in X \times N_\alpha(y) : f(x, z) < a\}.
\]

Let \( \text{Proj}_X(A^{(\alpha, y)}) \) be the projection of \( A^{(\alpha, y)} \) onto \( X \). Let \( \mathbb{Q} \) denote the set of rational numbers in \((0, \frac{1}{n})\). Define

\[
S^n := \{(\alpha, \beta) \in \mathbb{Q} \times \mathbb{Q} : \alpha \beta \leq \frac{1}{n}\}.
\]

We have

\[
\text{Proj}_{X \times Y}(E^n) = \bigcup_{(m, (\alpha, \beta)) \in \mathbb{N} \times S^n} \left[ \text{Proj}_X(A^{(\alpha, y^m)}) \times N_\beta(y^m) \right].
\]

To see this, observe that given \((x, y) \in \text{Proj}_{X \times Y}(E^n)\), there exists \( z \) such that \((x, y, z) \in D^n \) (i.e., \((x, y, z) \in X \times Y \times Y \) and \( z \in N^n_1(y) \)) and \( f(x, z) < a \). Let \( d \) be a compatible metric on \( Y \), and fix

\[
\epsilon \in (0, \frac{1}{3} \left( \frac{1}{n} - d(y, z) \right)).
\]
For \( y' \in N_\varepsilon(y) \) we have
\[
d(y', z) \leq d(y', y) + d(y, z) < \varepsilon + d(y, z) < \frac{1}{3} \left( \frac{1}{n} - d(y, z) \right) + d(y, z),
\]
so there is a rational number
\[
\beta \in \left( \frac{1}{3} \left( \frac{1}{n} - d(y, z) \right), \frac{1}{2} \left( \frac{1}{n} - d(y, z) \right) \right)
\]
such that \( d(y', z) < \beta + d(y, z) \) for all \( y' \in N_\varepsilon(y) \), and hence there is a rational number
\[
\alpha \in \left( \beta + d(y, z), \frac{1}{2} \left( \frac{1}{n} - d(y, z) \right) + d(y, z) \right)
\]
such that \( d(y', z) < \alpha \) for all \( y' \in N_\varepsilon(y) \). Consequently, since by denseness of \( Q \) in \( Y \) one may choose \( m \) such that \( y^m \in N_\varepsilon(y) \), we have \( z \in N_\alpha(y^m) \). It follows that \( (x, z) \in X \times N_\alpha(y^m) \) and \( f(x, z) < a \) (so that \( x \in \text{Proj}_X(A^{(\alpha,y^m)}) \)) and, since
\[
d(y, y^m) < \varepsilon \leq \frac{1}{3} \left( \frac{1}{n} - d(y, z) \right) < \beta,
\]
we have \( y \in N_\beta(y^m) \). We conclude that \( (x, y) \in \text{Proj}_X(A^{(\alpha,y^m)}) \times N_\beta(y^m) \) with \( (\alpha, \beta) \in \mathbb{Q} \times \mathbb{Q} \) and
\[
\alpha + \beta \leq \frac{1}{3} \left( \frac{1}{n} - d(y, z) \right) + d(y, z) + \frac{1}{2} \left( \frac{1}{n} - d(y, z) \right) \leq \frac{1}{n}.
\]
Conversely, if \((x, y) \in \text{Proj}_X(A^{(\alpha,y^m)}) \times N_\beta(y^m) \) for some \((m, (\alpha, \beta)) \in \mathbb{N} \times S^n\), then there exists \( z \) such that \((x, z) \in X \times N_\alpha(y^m) \) and \( f(x, z) < a \). In addition,
\[
d(y, z) \leq d(y, y^m) + d(y^m, z) < \beta + \alpha \leq \frac{1}{n}.
\]
Consequently, \((x, y, z) \in X \times Y \times Y \) and \( z \in N_{\frac{1}{n}}(y) \) (so that \((x, y, z) \in D^n\)) and \( f(x, z) < a \), which implies that \((x, y) \in \text{Proj}_{X \times Y}(E^n)\).

Because \( f \) is \( (A^u \otimes B(Y), B(\mathbb{R})) \)-measurable, we have \( A^{(\alpha,y)} \in A^u \otimes B(Y) \) for every \( \alpha > 0 \) and \( y \in Y \). Therefore, because \( Y \) is a Souslin space, the measurable projection theorem (e.g., Sainte-Beuve [6, Theorem 4]) gives \( \text{Proj}_X(A^{(\alpha,y)}) \in A^u \) for \( \alpha > 0 \) and \( y \in Y \).\(^1\) In light of (3), therefore, we conclude that \( \text{Proj}_{X \times Y}(E^n) \in A^u \otimes B(Y) \).

Because \( \text{Proj}_{X \times Y}(E^n) \in A^u \otimes B(Y), g^n \) is \( (A^u \otimes B(Y), B(\mathbb{R})) \)-measurable (recall (2) and (1)). It only remains to observe that
\[
\bar{f}(x, y) = \sup_{n \in \mathbb{N}} \inf_{z \in N_{\frac{1}{n}}(y)} f(x, z) = \sup_{n \in \mathbb{N}} g^n(x, y),
\]
so \( \bar{f} \) is \( (A^u \otimes B(Y), B(\mathbb{R})) \)-measurable. \(\square\)

In the remainder of the paper we present an application of the above result. Let \((X, \mathcal{A}, \mu)\) be a finite measure space with \( \mathcal{A} = A^u \). Let \( Y \) be a metrizable Lusin space (i.e., a metrizable topological space which is homeomorphic to a Borel subset

\(^1\)For the case when \( Y \) is Polish, the measurable projection theorem can also be found in Cohn [5, Proposition 8.4.4].
of a compact metrizable space). A transition probability with respect to \((X, A)\) and \((Y, B(Y))\) is a function \(\sigma : B(Y) \times X \to [0, 1]\) satisfying the following:

- \(\sigma(\cdot | x)\) is a probability measure on \((Y, B(Y))\) for every \(x \in X\);
- \(\sigma(B|\cdot)\) is \((A, B([0, 1]))\)-measurable for every \(B \in B(Y)\).

The set of transition probabilities with respect to \((X, A)\) and \((Y, B(Y))\) is denoted by \(S\).

A normal integrand on \(X \times Y\) is a map \(f : X \times Y \to \mathbb{R}\) satisfying the following:

- \(f(x, \cdot)\) is lower semicontinuous on \(Y\) for every \(x \in X\);
- \(f\) is \((A \otimes B(Y), B(\mathbb{R}))\)-measurable.

Let \(L_1(X, A, \mu)\) represent the set of \((A, B(\mathbb{R}))\)-measurable functions \(\xi : X \to \mathbb{R}\) such that

\[
\int_X |\xi(x)|\mu(dx) < \infty.
\]

The set of all normal integrands \(f\) on \(X \times Y\) for which there exists \(\xi \in L_1(X, A, \mu)\) such that \(\xi(x) \leq f(x, y)\) for all \((x, y) \in X \times Y\) is denoted by \(\mathcal{F}\).

For \(f \in \mathcal{F}\), the functional \(I_f : S \to \mathbb{R}\) is defined by

\[
I_f(\sigma) := \int_X \int_Y f(x, y)\sigma(dy|x)\mu(dx).
\]

The narrow topology on \(S\) is the coarsest topology that makes the functionals in \(\{I_f : f \in \mathcal{F}\}\) lower semicontinuous. This topology has been studied by Balder [1], [2], [3] and applied to the theory of games with incomplete information (e.g., Balder [2] and Carbonell-Nicolau and McLean [4]).

Suppose that the map \(f : X \times Y \to \mathbb{R}\) is \((A \otimes B(Y), B(\mathbb{R}))\)-measurable. Suppose further that there exists \(\xi \in L_1(X, A, \mu)\) such that \(\xi(x) \leq f(x, y)\) for all \((x, y) \in X \times Y\). Then \(f\) satisfies \(\varphi(x) \leq f(x, y)\) for all \((x, y) \in X \times Y\) and for some \(\varphi \in L_1(X, A, \mu)\). In addition, \(f(x, \cdot)\) is lower semicontinuous on \(Y\) for every \(x \in X\), and, by virtue of Theorem, \(f\) is \((A \otimes B(Y), B(\mathbb{R}))\)-measurable. Consequently, \(f \in \mathcal{F}\). It follows that if \(S\) is endowed with the narrow topology, for each \(\epsilon > 0\) and every \(\sigma \in S\) there exists an open set \(V\) in \(S\) containing \(\sigma\) such that

\[
I_{\overline{f}}(\nu) \geq I_f(\sigma) - \epsilon, \quad \text{for all } \nu \in V.
\]

REFERENCES


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